Since can always rewrite such as "equiv." fmla, e.g. $(x = c \wedge \forall z(f(z) = y))$

- formulas w/ no free vars are called <u>sentences</u>

  e.g. $\exists y \forall x (f(x) = y)$

- intu'n: sentences are T or F

  formulas don't have T/F value until their vars are specified
- Let's formalise this ↑!

<u>Def'n</u> Given $\mathcal{L}$, and $\mathcal{M}$ an $\mathcal{L}$-structure, and φ a fmla w/ variables among $\bar{v} = (v_{i_1}, \ldots, v_{i_n})$,

(... we write $\varphi(\bar{v})$ to indicate this)

and $\bar{a} = (a_1, \ldots, a_n) \in M^n$

we inductively define $\mathcal{M} \models \varphi(\bar{a})$

(read: "$\mathcal{M}$ models $\varphi(\bar{a})$" or "$\varphi(\bar{a})$ is true in $\mathcal{M}$")

① if φ is $t_1 = t_2$

then $\mathcal{M} \models \varphi(\bar{a})$ iff $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$

② if φ is $R(t_1, \ldots, t_{n_R})$

then $\mathcal{M} \models \varphi(\bar{a})$ iff $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$

③ if φ is $\neg \psi$

then $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{M} \not\models \psi(\bar{a})$

(4) if φ is $\chi_1 \wedge \chi_2$
then $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{M} \models \chi_1(\bar{a})$ and $\mathcal{M} \models \chi_2(\bar{a})$
(5) if φ is $\chi_1 \vee \chi_2$
then $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{M} \models \chi_1(\bar{a})$ or $\mathcal{M} \models \chi_2(\bar{a})$
(6) if φ is $\exists v_i \chi$
then $\mathcal{M} \models \varphi(\bar{a})$ iff there is $b \in M$ s.t. $\mathcal{M} \models \chi(\bar{a}, b)$
(7) if φ is $\forall v_i \chi$
then $\mathcal{M} \models \varphi(\bar{a})$ iff for all $b \in M$, $\mathcal{M} \models \chi(\bar{a}, b)$

- we'll also use connectives $\Rightarrow$ and $\Leftrightarrow$ in fmlas ("implies" "iff")
- formally: take them as abbreviations:
  $\varphi \Rightarrow \chi$ is really $\neg\varphi \vee \chi$
  $\varphi \Leftrightarrow \chi$ " " $(\varphi \wedge \chi) \vee (\neg\varphi \wedge \neg\chi)$
  which is equiv to $(\varphi \Rightarrow \chi) \wedge (\chi \Rightarrow \varphi)$
- sometimes useful to think of $\vee$ and $\forall$ as abbrev's too:
  $\varphi \vee \chi$ is $\neg(\neg\varphi \wedge \neg\chi)$
  $\forall x\, \varphi$ is $\neg\exists x \neg\varphi$

Ex's - Sps $\mathcal{L} = \{f, g, R, c, d\}$
f, g, R binary
- Consider $\mathcal{L}$-structs:
  $\mathcal{M} = (\mathbb{Q}, +, \cdot, <, 0, 1)$
  $\mathcal{N} = (\mathbb{R}, +, \cdot, <, 0, 1)$

- notice: $\mathcal{M}$ a substructure of $\mathcal{N}$

(Notation: write $\mathcal{M} \subseteq \mathcal{N}$ for "$\mathcal{M}$ is a substruct of $\mathcal{N}$")

- Consider fmlas:
  $\varphi(x,y)$: $x \cdot x = y$
  ψ : $\exists x \, (x \cdot x = 1+1)$

- Then $\mathcal{M} \models \varphi(2,4)$ (and $\mathcal{N}$ too) since $2 \cdot 2 = 4$
  but $\mathcal{M} \not\models \varphi(1,2)$ since $1 \cdot 1 \neq 2$

  $\mathcal{N} \models \psi$ but $\mathcal{M} \not\models \psi$ since $\sqrt{2} \notin \mathbb{Q}$

- Both $\mathcal{M}, \mathcal{N} \models \forall x \forall y \, (x<y \Rightarrow x+1<y+1)$
  $\neg(x<y) \vee (x+1<y+1)$
- $\mathcal{N} \models \forall x \, [\exists y \, (y \cdot y = x) \Leftrightarrow (x=0 \vee 0<x)]$
  but $\mathcal{M} \not\models$ " " since there are rationals $q>0$ w/o rational sqrts.

- Ex's suggest question: if $\mathcal{M} \subseteq \mathcal{N}$ when is truth value of a given fmla φ the same in both?
- Not always, by the ex's.
  ... but:

Theorem Sps $\mathcal{M} \subseteq \mathcal{N}$, $\vec{a} \in M$, and $\varphi(\vec{v})$ is a quantifier free fmla then: $\mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{N} \models \varphi(\vec{a})$

Intuition is: quantifiers only way to destroy preservation of truth between substructure / superstructure

$\exists x(\ldots)$ could be true in superstructure but not in substructure

$\forall x(\ldots)$ " " substructure

" " superstructure

PF of thm: by "induction on terms" then "induction on fmlas"

Claim. if $t(\vec{v})$ is a term then
$$t^{\mathcal{M}}(\vec{a}) = t^{\mathcal{N}}(\vec{a})$$

PF: (Base cases) - if t is c, then $t^{\mathcal{M}}(\vec{a}) = c^{\mathcal{M}}$

smaller $\longrightarrow = c^{\mathcal{N}} = t^{\mathcal{N}}(\vec{a})$

- if t is v_k, then $t^{\mathcal{M}}(\vec{a}) = a_k = t^{\mathcal{N}}(\vec{a})$

(Induction step) - if t is $f(t_1, \ldots, t_{n_f})$ and claim true for $t_1, \ldots, t_{n_f}$ then

$$t^{\mathcal{M}}(\vec{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\vec{a}), \ldots, t_{n_f}^{\mathcal{M}}(\vec{a}))$$

smaller $\longrightarrow = f^{\mathcal{N}}(t_1^{\mathcal{M}}(\vec{a}), \ldots, t_{n_f}^{\mathcal{M}}(\vec{a}))$

induction hyp $\longrightarrow = f^{\mathcal{N}}(t_1^{\mathcal{N}}(\vec{a}), \ldots, t_{n_f}^{\mathcal{N}}(\vec{a})) = t^{\mathcal{N}}(\vec{a})$

Claim proved ✓

now prove theorem by induction on fmlas quantifier free

(Base Cases) - if $\varphi(\vec{v})$ is $t_1 = t_2$
then $\mathcal{M} \models \varphi(\vec{a})$ iff $t_1^{\mathcal{M}}(\vec{a}) = t_2^{\mathcal{M}}(\vec{a})$
claim $\to$ iff $t_1^{\mathcal{N}}(\vec{a}) = t_2^{\mathcal{N}}(\vec{a})$
iff $\mathcal{N} \models \varphi(\vec{a})$ ✓

- if φ is $R(t_1, \ldots, t_{n_R})$ then
$\mathcal{M} \models \varphi(\vec{v})$ iff $(t_1^{\mathcal{M}}(\vec{a}), \ldots, t_{n_R}^{\mathcal{M}}(\vec{a})) \in R^{\mathcal{M}}$
claim $\to$ iff $(t_1^{\mathcal{N}}(\vec{c}), \ldots, t_{n_R}^{\mathcal{N}}(\vec{c})) \in R^{\mathcal{M}}$
$\mathcal{M} \subseteq \mathcal{N} \to$ iff $(t_1^{\mathcal{N}}(\vec{c}), \ldots, t_{n_R}^{\mathcal{N}}(\vec{c})) \in R^{\mathcal{N}}$
iff $\mathcal{N} \models \varphi(\vec{c})$ ✓

(IS's) Sps thm true for φ, χ
Then:
$\mathcal{M} \models \neg\varphi(\vec{a})$ iff $\mathcal{M} \not\models \varphi(\vec{a})$
IH $\to$ iff $\mathcal{N} \not\models \varphi(\vec{c})$
iff $\mathcal{N} \models \neg\varphi(\vec{c})$

and: $\mathcal{M} \models (\varphi \wedge \chi)(\vec{a})$ iff $\mathcal{M} \models \varphi(\vec{a})$ and $\mathcal{M} \models \chi(\vec{a})$
iff $\mathcal{N} \models \varphi(\vec{c})$ and $\mathcal{N} \models \chi(\vec{a})$
iff $\mathcal{N} \models (\varphi \wedge \chi)(\vec{a})$ ✓

Since all q.f. fmlas built from atomic fmlas using $\neg$ and $\wedge$ (viewing $\vee$ as abbreviation), theorem is proved ✓

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e.g. if $\varphi(x)$ is $(1 = (1+1)\cdot x) \vee (1 < (1+1)\cdot x)$

(quantifier free!)

For any $r \in \mathbb{Q}$

then $\mathcal{M} = (\mathbb{Q}, +, \cdot, <, 0, 1) \models \varphi(r)$
iff $\mathcal{N} = (\mathbb{R}, +, \cdot, <, 0, 1) \models \varphi(r)$
iff $r \geq 1/2$.

Again: ex's show thm not true in general when φ is not q.f.

<u>Def'n</u> Sps $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. Say $\mathcal{M}$ is an <u>elementary substructure</u> of $\mathcal{N}$ (write: $\mathcal{M} \prec \mathcal{N}$) if for every fmla $\varphi(\vec{v})$ and $\vec{a} \in M$ we have:

$$\mathcal{M} \models \varphi(\vec{a}) \text{ iff } \mathcal{N} \models \varphi(\vec{a})$$

- examples above show $\mathcal{M} = (\mathbb{Q}, +, \cdot, <, 0, 1)$ is <u>not</u> elem substruct of $\mathcal{N} = (\mathbb{R}, +, \cdot, <, 0, 1)$

- we'll show later: $\mathcal{M} = (\mathbb{Q}, <)$ <u>is</u> elem substruct of $\mathcal{N} = (\mathbb{R}, <)$
- so: while these structures are very different, e.g. $(\mathbb{R}, <)$ is complete, $(\mathbb{Q}, <)$ is not
  $\therefore$ can't distinguish difference in 1st-order way!

<u>Def'n</u> Given $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ say $\mathcal{M}$ is <u>elementarily equivalent</u> to $\mathcal{N}$ (write: $\mathcal{M} \equiv \mathcal{N}$) if for all sentences φ we have $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$

17

<u>Observe</u>: if $\mathcal{M} \prec \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$

so e.g. $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$

(but in general not true that $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \subseteq \mathcal{N}$ implies $\mathcal{M} \prec \mathcal{N}$)

(insert here)

<u>Defn</u> the (first-order) <u>theory</u> of $\mathcal{M}$, written $Th(\mathcal{M})$, is set of all sentences φ s.t. $\mathcal{M} \models \varphi$

so: $\mathcal{M} \equiv \mathcal{N}$ iff $Th(\mathcal{M}) = Th(\mathcal{N})$

- Next theorem says: if two structures are structurally indistinguishable then they're logically indistinguishable (as we would hope!)

isomorphic to

<u>Thm</u> if $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$

<u>Pf</u>: We show something stronger.

Fix $j : \mathcal{M} \to \mathcal{N}$ an isomorphism

Fix $\vec{a} \in M$, $\vec{a} = (a_1, \ldots, a_n)$

Write $j(\vec{a})$ for $(j(a_1), \ldots, j(a_n))$

<u>Claim</u> Given any fmla $\varphi(\vec{v})$ (w/ at most n free vars) we have

$$\mathcal{M} \models \varphi(\vec{a}) \text{ iff } \mathcal{N} \models \varphi(j(\vec{a}))$$

Pf: induction on fmlas. Deal w/ terms first

<u>Subclaim</u> for t a term, $j(t^{\mathcal{M}}(\vec{a})) = t^{\mathcal{N}}(j(\vec{a}$

j on wo

18

Pf. – if t is c: $j(t^{\mu}(\vec{a})) = j(c^{\mu}) \overset{j}{=} c^{\nu} = t^{\nu}(j(\vec{a}))$

– if t is v_{i_k}: $j(t^{\mu}(\vec{a})) = j(a_k) = t^{\nu}(j(\vec{a}))$

– if t is $f(t_1, \ldots, t_{n_f})$, subclaim true for $t_1, \ldots, t_{n_f}$

then $j(t^{\mu}(\vec{a})) = j(f^{\mu}(t_1^{\mu}(\vec{a}), \ldots, t_{n_f}^{\mu}(\vec{a})))$

j iso $\longrightarrow$ $= f^{\nu}(j(t_1^{\mu}(\vec{a})), \ldots, j(t_{n_f}^{\mu}(\vec{a})))$

IH $\longrightarrow$ $= f^{\nu}(t_1^{\nu}(j(\vec{a})), \ldots, t_{n_f}^{\nu}(j(\vec{a})))$

$= t^{\nu}(j(\vec{a}))$ ✓

subclaim proved ✓

back to fmlas:

– if $\varphi(\vec{v})$ is $t_1(\vec{v}) = t_2(\vec{v})$ then

$\mu \models \varphi(\vec{a})$ iff $t_1^{\mu}(\vec{a}) = t_2^{\mu}(\vec{a})$ iff

$j(t_1^{\mu}(\vec{a})) = j(t_2^{\mu}(\vec{a}))$ iff $t_1^{\nu}(j(\vec{a})) = t_2^{\nu}(j(\vec{a}))$

iff $\nu \models \varphi(j(\vec{a}))$ ✓

– if $\varphi(\vec{v})$ is $R(t_1, \ldots, t_{n_R})$:

$\mu \models \varphi(\vec{a})$ iff $(t_1^{\mu}(\vec{a}), \ldots, t_{n_R}^{\mu}(\vec{a})) \in R^{\mu}$

iff $(j(\quad), \ldots, j(\quad)) \in R^{\nu}$ (j on wo)

iff $(t_1^{\nu}(j(\vec{a})), \ldots, t_{n_R}^{\nu}(j(\vec{a}))) \in R^{\nu}$ by subcl.

iff $\nu \models \varphi(j(\vec{a}))$

– Sps thm true for ψ, ψ'

– if $\varphi(\vec{v})$ is $\neg\psi(\vec{v})$:

$\mu \models \varphi(\vec{a})$ iff $\mu \not\models \psi(\vec{a})$ iff

iff $\nu \not\models \psi(j(\vec{a}))$ (IH) iff $\nu \models \varphi(j(\vec{a}))$

– if $\varphi(\vec{v})$ is $\psi \wedge \psi'(\vec{v})$

... you try

if $\varphi(\vec{v})$ is $\exists x\, \psi(\vec{v}, x)$ then:

$\mathcal{M} \models \varphi(\vec{a})$ iff $\mathcal{M} \models \exists x\, \psi(\vec{a}, x)$ iff

there is $b \in M$ $\mathcal{M} \models \psi(\vec{a}, b)$ iff (IH)

$\mathcal{N} \models \psi(j(\vec{a}), j(b))$ iff there is $c \in N$

s.t. $\mathcal{N} \models \psi(j(\vec{a}), c)$ ← since j is onto!

iff $\mathcal{N} \models \exists x\, \psi(j(\vec{a}), x)$

iff $\mathcal{N} \models \varphi(j(\vec{a}))$ ✓

Claim ~~Thm~~ proved by induction on formulas (viewing $\vee, \forall$ as abbrev's) ✓

Original theorem follows since if φ has no free variables, claim gives $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$. ✓

<u>Ex</u> - Consider $\mathcal{M} = (\mathbb{N}, <)$ and $\mathcal{N} = (\mathbb{N}+1, <)$ where $\mathbb{N}+1 = \{1, 2, 3, \ldots\}$

$\mathbb{N} = \{0, 1, 2, \ldots\}$

- ~~[illegible]~~ $j : \mathcal{M} \to \mathcal{N}$, $j(n) = n+1$ is isomorphism: 1-1/onto and $n < m$ iff $n+1 < m+1$ i.e. $j(n) < j(m)$

- Consider $\varphi(x)$: $\forall y\, (y \not< x)$ "x is minimal" ← $\neg(y < x)$

- then $\mathcal{M} \models \varphi(0)$ and $\mathcal{N} \models \varphi(1)$ i.e. $\varphi(j(0))$

- Also $\mathcal{M}, \mathcal{N} \models \exists x \forall y \neg(y < x)$