Prime and atomic models

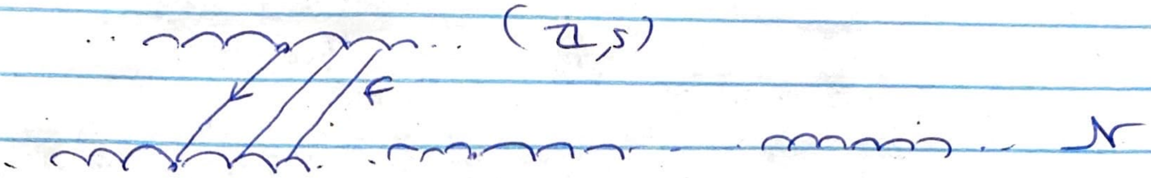
- Throughout: $\mathcal{L}$ is ctbl and T a complete $\mathcal{L}$-theory w/ infinite models
- We consider two concepts of what it means for $\mathcal{M} \models T$ to be "as small as possible"
  ... and show they are the same for countable $\mathcal{M}$

Def'n a model $\mathcal{M} \models T$ is called a prime model of T if whenever $\mathcal{N} \models T$ there is an elem embedding of $\mathcal{M}$ into $\mathcal{N}$.

Ex's ① - Sps $\mathcal{M}$ is a ctbl structure and $\mathcal{L}$ includes a constant symbol m for every el't of $\mathcal{M}$ (i.e. $\mathcal{L} = \mathcal{L}_M$)
- Let $T = Th(\mathcal{M}) = Diag_{el}(\mathcal{M})$
- then $\mathcal{M}$ is itself a prime model of T: we showed if $\mathcal{N} \models Diag_{el}(\mathcal{M})$ then there is elem embedding of $\mathcal{M}$ into $\mathcal{N}$.

② - Sometimes $\mathcal{M}$ is a prime model of $T = Th(\mathcal{M})$ even if $\mathcal{L}$ does not include constant symbols $\forall m \in M$.
- e.g. if $\mathcal{M} = (\mathbb{Z}, s)$ as on HW8 we claim $\mathcal{M}$ is a prime model of $Th(\mathcal{M})$

- Let $\mathcal{N} \models Th(\mathbb{Z}, s)$; you showed on HW 8: $\mathcal{N}$ is a disjoint union of copies of $(\mathbb{Z}, s)$
- Let $f : (\mathbb{Z}, s) \to \mathcal{N}$ be <u>any</u> embedding



- claim f is elem.
- if $\vec{z} = (z_1, \dots, z_n)$ any n-tuple in $\mathbb{Z}$, HW 8 $\Rightarrow$ $tp(\vec{z}) = tp(f(\vec{z}))$ (since these tuples have same configuration)

but then $\underset{(\mathbb{Z}, s)}{\mathcal{M}} \models \varphi(\vec{z})$ iff $\mathcal{N} \models \varphi(f(\vec{z}))$

- since $\vec{z}$ was arb, embedding is elem ✓

(3) - Consider $\mathcal{N}$ = two copies of $(\mathbb{Z}, s)$

- then $\mathcal{N}$ is <u>not</u> a prime model of $T = Th(\mathcal{N})$
- reason is: $\mathcal{N} \models Th(\mathbb{Z}, s)$ hence $Th(\mathcal{N}) = Th(\mathbb{Z}, s)$ (these are complete theories)
- hence $(\mathbb{Z}, s) \models Th(\mathcal{N})$, but there is no embedding of $\mathcal{N}$ into $(\mathbb{Z}, s)$ ✓

↳ in general, prime models cannot realize non-isolated types!

Prop'n if $\mathcal{M}$ is a prime model of T and $\vec{a} \in M^n$, then $tp(\vec{a})$ is isolated.

Pf: if $p = tp(\vec{a})$ is not isolated, then by omitting types theorem there is there is $\mathcal{N} \models T$ that omits p
- since $\mathcal{M}$ prime, there is $j: \mathcal{M} \to \mathcal{N}$ elem.
- since j elem, $tp(\vec{a}) = tp(j(\vec{a}))$
- but then $j(\vec{a}) \in N^n$ realizes p, a contradiction ✓

Def'n $\mathcal{M} \models T$ is **atomic** if $tp(\vec{a})$ is isolated for every $\vec{a} \in M^n$

- Prop'n says: prime $\Rightarrow$ atomic
- converse also true! ... for ctbl models

Theorem Sps $\mathcal{L}$ is ctbl and T is a complete $\mathcal{L}$-theory w/ infinite models. Then $\mathcal{M} \models T$ is prime iff $\mathcal{M}$ is ctbl and atomic.

Pf: ($\Rightarrow$) if $\mathcal{M}$ is prime, must be ctbl since by L-S T has ctbl models and $\mathcal{M}$ must embed in them

and by prop'n $\mathcal{M}$ must be atomic.

($\Leftarrow$) Sps $\mathcal{M}$ is ctbl + atomic and $\mathcal{N} \models T$
- we build elem $f: \mathcal{M} \to \mathcal{N}$.
- enumerate $\mathcal{M} = \{m_0, m_1, \dots\}$
- for each i, $tp(m_0, \dots, m_i)$ is isolated so fix an isolating fmla $\theta_i(v_0, \dots, v_i)$
- we build a seq $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ of partial elem maps s.t. $dom(f_i) = \{m_0, \dots, m_{i-1}\}$
- then $f = \bigcup f_i$ is elem embedding.

- let $f_0 = \emptyset$; is partial elem since $\mathcal{M} \equiv \mathcal{N}$.
- given f_k, let $n_i = f_k(m_i)$ $\forall i < k$
- because $\mathcal{M} \models \theta_k(m_0, \dots, m_k)$ we have $\mathcal{M} \models \exists v\, \theta_k(m_0, \dots, m_{k-1}, v)$
- since f_k is partial elem, this gives $\mathcal{N} \models \exists v\, \theta_k(n_0, \dots, n_{k-1}, v)$
- pick $n_k \in N$ s.t. $\mathcal{N} \models \theta_k(n_0, \dots, n_{k-1}, n_k)$
- <u>because θ_k isolates</u> $tp(m_0, \dots, m_k)$, we have $tp(m_0, \dots, m_k) = tp(n_0, \dots, n_k)$
$\Rightarrow f_{k+1} = f_k \cup \{(m_k, n_k)\}$ is partial elem

($\Rightarrow$) follows from theorem that if $\mathcal{M}$ is ctbl and $tp(\vec{a})$ is isolated $\forall \vec{a} \in M^n$, then $\mathcal{M}$ is a prime model for $T = Th(\mathcal{M})$.

Homogeneous models

Goal: show prime models are unique.

Def'n A ctbl $M \models T$ is homogeneous if whenever $\vec{a}, \vec{b} \in M^n$ and $tp(\vec{a}) = tp(\vec{b})$ then for any $c \in M$ there is $d \in M$ s.t. $tp(\vec{a}, c) = tp(\vec{b}, d)$

- Another way to read def'n: if the map $\vec{a} \to \vec{b}$ is partial elem and $c \in M$, then $\exists d \in M$ s.t. $\vec{a}, c \to \vec{b}, d$ is partial elem.
- Since $\vec{a} \to \vec{b}$ is partial elem iff $\vec{b} \to \vec{a}$ is, this is also same as saying that collection I of all finite partial elem maps from M to M has b-f property.

Prop'n if M is ctbl + homogeneous and $\vec{a}, \vec{b} \in M^n$, $tp(\vec{a}) = tp(\vec{b})$, then there is an automorphism $\sigma: M \to M$ s.t. $\sigma(\vec{a}) = \vec{b}$

pf: - follow proof that if M, N ctbl and $M \equiv_p N$ then $M \cong N$ using as starting map the partial elem map $\vec{a} \to \vec{b}$

Lemma if M is ctbl + atomic (hence prime) then M is homogeneous.

Pf: - sps $\vec{a} \to \vec{b}$ is partial elem (i.e. $tp(\vec{a}) = tp(\vec{b})$)
- fix $c \in M$
- let $\phi(\vec{v}, w)$ isolate $tp(\vec{a}, c)$ (M atomic)
- then $M \models \exists w\, \phi(\vec{a}, w)$
- hence $M \models \exists w\, \phi(\vec{b}, w)$ (tps the same)
- pick any $d \in M$ s.t. $M \models \phi(\vec{b}, d)$
- since ϕ isolates $tp(\vec{a}, c)$ must have $tp(\vec{b}, d) = tp(\vec{a}, c)$
- i.e. $\vec{a}, c \to \vec{b}, d$ is partial elem. ✓

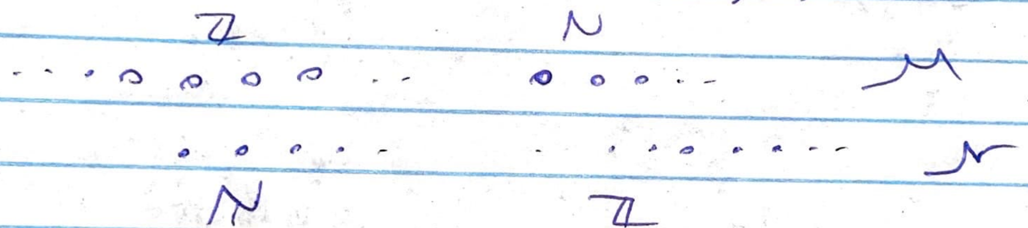
Theorem Sps T is a complete theory in a ctbl language $\mathcal{L}$. Sps $M, N \models T$ are ctbl and homogeneous. If M, N realize the same types, then $M \cong N$.

Pf: - mimic proof (from section on Fraïssé theory) that any two ctbl structures that are ultrahomogeneous + universal are isomorphic.

Corollary of T if $\mathcal{M}, \mathcal{N}$ are prime models then $\mathcal{M} \cong \mathcal{N}$.

Pf: $\mathcal{M}, \mathcal{N}$ prime $\Rightarrow$ ctbl + atomic
$\Rightarrow \mathcal{M}, \mathcal{N}$ realize same types (the isolated types in $S_n(T)$)
– moreover, $\mathcal{M}, \mathcal{N}$ homogeneous by lemma
$\Rightarrow \mathcal{M} \cong \mathcal{N}$ by theorem ✓

Ex Consider $(\mathbb{Z}+\mathbb{N}, <) = \mathcal{M}$
and $(\mathbb{N}+\mathbb{Z}, <) = \mathcal{N}$



Claim ① $\mathcal{M}$ is the prime model of $T_1 = Th(\mathcal{M})$
② $\mathcal{N}$ is not the prime model of $T_2 = Th(\mathcal{N})$ (in fact $(\mathbb{N}, <)$ is)