- Omitting types theorem says: converse is true! ... at least for T in a ctbl lang.

Theorem (omitting types) Sps $\mathcal{L}$ is ctbl, T is an $\mathcal{L}$-theory, and p is a non-isolated n-type. Then there is a countable model $\mathcal{M} \models T$ in which p is omitted.

Pf. - we modify our proof of the compactness theorem to build a Henkin model (i.e. elements are equiv. classes of constants) omitting p.

Recall: ① **Def'n** an $\mathcal{L}^*$-theory T^* has the *witness property* if for every $\mathcal{L}^*$-fmla $\varphi(v)$ there is $c \in \mathcal{L}^*$ s.t. $T^* \models (\exists v\, \varphi(v)) \Rightarrow \varphi(c)$

② **Prop'n** if T^* is maximal ($\varphi \in T^*$ or $\neg\varphi \in T^*$ $\forall$ sentence φ), fin. sat., and has witness prop. then T^* is satisfiable; namely the model $\mathcal{M}$ w/ universe $M = C/\sim$ consisting of equiv classes of constant symbols is a model of T^* (where $c \sim d$ iff $c = d \in T^*$)

- call $\mathcal{M}$ the Henkin model of T^*
- we'll build $T^* \supseteq T$ satisfying the hypotheses in the prop'n ... s.t. the Henkin model omits p.

- let $C = \{c_0, c_1, \dots\}$ be ctbly many new constants (diff. from those in $\mathcal{L}$)
- let $\mathcal{L}^* = \mathcal{L} \cup C$ (still ctbl)

[...]

- along the way: arrange that if $d_1, \dots, d_n \in \mathcal{L}^*$ are constants, there is $\varphi(\vec{v}) \in p$ s.t. $T^* \models \neg\varphi(d_1, \dots, d_n)$
  - ↳ since every el't of $\mathcal{M}$ is (interp. of) a constant symbol, this ensures $\mathcal{M}$ omits p.

- we build a seq $\theta_0, \theta_1, \theta_2, \dots$ of $\mathcal{L}^*$-sentences and let $T^* = T \cup \{\theta_0, \theta_1, \theta_2, \dots\}$
- each θ_{s+1} will be of form $\theta_s \wedge \varphi$ for some sentence φ
  - ↳ θ_s's are longer + longer conjunctions

$$\theta_0 = \theta_0$$
$$\theta_1 = \theta_0 \wedge \varphi$$
$$\theta_2 = \theta_0 \wedge \varphi \wedge \varphi'$$
$$\vdots$$

- in particular will have ~~[illegible]~~ $T \models \theta_t \to \theta_s$ for all $t > s$.
- build θ_i's to accomplish 3 goals:
  1. completeness (actually: maximality) of T^*
  2. witness property
  3. final model omits p.
- work toward these goals @ stages $3i+1$, $3i+2$, $3i+3$ resp.

- let $\varphi_0, \varphi_1, \ldots$ list all $\mathcal{L}^*$-sentences
- let $\bar{d}_0, \bar{d}_1, \ldots$ list all n-tuples of constants from $\mathcal{L}^*$

possible since $\mathcal{L}^*$ ctbl.

Stage 0: let θ_0 be $\forall x\,(x = x)$
(or any other fixed, always true sentence)

- Sps we've constructed $\theta_0, \ldots, \theta_s$ s.t. $T \cup \{\theta_0, \ldots, \theta_s\}$ is sat'ble.

  equiv: $T \cup \{\theta_s\}$ is sat'ble
- three cases:

① $s+1 = 3i+1$ (completeness stage)
- either $T \cup \{\theta_s\} \cup \{\varphi_i\}$ is sat'ble, in which case let $\theta_{s+1} = \theta_s \wedge \varphi_i$
or not, in which case let $\theta_{s+1} = \theta_s \wedge \neg\varphi_i$ (then $T \cup \{\theta_{s+1}\}$ is sat'ble)

② $s+1 = 3i+2$ (witness prop. stage)

- sps φ_i is of form ~~[scribbled]~~ $\exists v \psi(v)$ and $T \cup \{\theta_s\} \vDash \varphi_i$

(notice: then by prev. stage $\varphi_i := \exists v \psi(v)$ is actually a conjunct in θ_s, so in particular $T \vdash \theta_s \Rightarrow \exists v \psi(v)$)

- want to find a witness for ψ, i.e. arrange so that at end $T^* \vdash \psi(c)$ for ~~[scribbled]~~ some constant c.
- pick $c \in C$ not appearing in θ_s
- let $\theta_{s+1} = \theta_s \wedge \psi(c)$
- then $T \cup \{\theta_{s+1}\}$ is sat'ble: if $\mathcal{N} \vDash T \cup \{\theta_s\}$ there is $a \in N$ s.t. $\mathcal{N} \vDash \psi(a)$, so by interpreting c as a, get $\mathcal{N}' \vDash \psi(c)$ hence $\mathcal{N}' \vDash T \cup \{\theta_{s+1}\}$)

↳ if φ_i not of form $\exists v \psi(v)$, or $T \cup \{\theta_s\} \nvDash \varphi_i$, do nothing, i.e. let $\theta_{s+1} = \theta_s$

③ $s+1 = 3i+3$ (Completing P stage)

- write $\vec{d_i} = (e_1, .., e_n)$
- idea is: the constants e_i may appear.

in Θ_s, so may view Θ_s as an assertion $\Theta_s(\vec{d_i})$ about the tuple $\vec{d_i} = (e_1, .., e_n)$

- since p is not violated: Θ_s does not violate p, i.e. $\Theta_s(\vec{d_i})$ does not entail $\varphi(\vec{d_i})$ for every $\varphi(\vec{v}) \in p$
- so: for some $\varphi(\vec{v}) \in p$, $T \cup \{\Theta_s \wedge \neg\varphi(\vec{d_i})\}$ is satisfiable, and we let $\Theta_{s+1} = \Theta_s \wedge \neg\varphi(\vec{d_i})$

↳ in more detail:

- let $\psi(\vec{v})$ be fmla obtained by: repl. every instance of e_i in Θ_s by v_i, and repl. every other constant c by some variable v_c, adding $\exists v_c$ to front.
- e.g. if Θ_s is $e_1 < e_2 \wedge c + d = b$ then $\psi(\vec{v})$ is $\exists v_c \exists v_d \exists v_b (v_1 < v_2 \wedge v_c + v_d = v_b)$
- point is: $\psi(\vec{v})$ is fmla in orig. lang. $\mathcal{L}$ in free vars $v_1, .., v_n$
- since p non-iso, there is some $\varphi(\vec{v}) \in p$ s.t. $T \nvDash \forall \vec{v}(\psi(\vec{v}) \Rightarrow \varphi(\vec{v}))$ i.e. $T \cup \{\psi(\vec{v}) \wedge \neg\varphi(\vec{v})\}$ is sat.
- let $\mathcal{N} \vDash T \cup \{\psi(\vec{v}) \wedge \neg\varphi(\vec{v})\}$ i.e. $\mathcal{N} \vDash T$ and $\exists \vec{a} \in N^n$ s.t. $\mathcal{N} \vDash \psi(\vec{a}) \wedge \neg\varphi(\vec{a})$
- can view $\mathcal{N}$ as a model of Θ_s: interpret each e_i as a_i and each c not among e_i as witness to v_c
- let $\Theta_{s+1} = \Theta_s \wedge \neg\varphi(\vec{d_i})$

– then $T \models \theta_{s+1}$ so $T \cup \{\theta_{s+1}\}$ is sat. ✓

This completes construction, we check it works, i.e. letting $T^* = T \cup \{\theta_0, \theta_1, \dots\}$ check T^* is max'l, fin. sat., has witness prop. + Henkin model omits p.

– T^* clearly complete, to get maximality should really define $T^* = T \cup \{$every conjunct appearing in some $\theta_s\} \cup \{\theta_i : i \in \mathbb{N}\}$ ... but they're equiv.

every $\mathcal{L}^*$ sentence is φ_i for some i

– T^* is fin. sat since we arranged $T \cup \{\theta_0, \dots, \theta_s\}$ satisfiable $\forall s \in \mathbb{N}$.

– if $\psi(v)$ is an $\mathcal{L}^*$-formula and $T^* \models \exists v \psi(v)$ then $\exists v \psi(v)$ is φ_i for some i, so @ stage $3i+2$ arranged there is c st. $T \models \psi(c)$ ...

$\Rightarrow T^*$ has witness prop

– if $\mathcal{M}$ Henkin model for T^*, we claim $\mathcal{M}$ omits p

– since every el't of $\mathcal{M}$ is interp of some constant symbol, every n-tuple of M^n is $\bar{d}_i^{\mathcal{M}}$ for some i.

– but then @ stage $3i+3$ we arranged $\mathcal{M} \models \neg\varphi(\bar{d})$ for some $\varphi \in p$

$\Rightarrow$ $\vec{a}$ does not realize p
$\Rightarrow$ $\mathcal{M}$ omits p ✓