Ex Sps $\overset{=(M,<)}{\mathcal{M}} \models DLO^*$ is a dense linear order w/o endpoints and $A \subseteq M$.

- want to understand space $S_1^{\mathcal{M}}(A)$ of complete 1-types /A
- given $p(v) \in S_1^{\mathcal{M}}(A)$ and $a \in A$, exactly one of the fmlas $v<a$, $v>a$, $v=a$ $\in p$. (why?) ($p = tp(c/A)$ in some ext'n)

Case 1: For some $a \in A$, $v=a \in p$

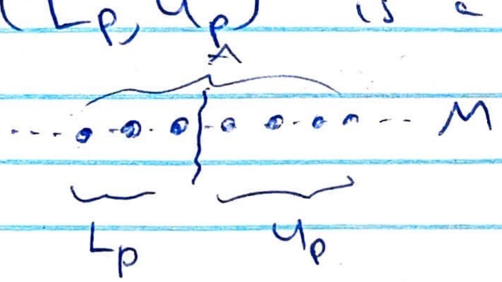
↳ then $p = \{\varphi(v) : \mathcal{M} \models \varphi(a)\} = tp^{\mathcal{M}}(a/A)$

$\Rightarrow$ p is isolated by $v=a$, i.e. $p = [v=a]$

(why? only one complete type /A containing $v=a$!)

Case 2 $v=a \notin p$ $\forall a \in A$

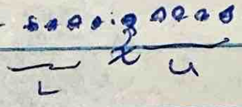
- Define $L_p = \{a \in A : v>a \in p\}$
  $U_p = \{a \in A : v<a \in p\}$
- then by assumption $A = L_p \cup U_p$, and since p sat'ble every pt in L_p < every pt in U_p
- i.e. (L_p, U_p) is a **cut** in A.



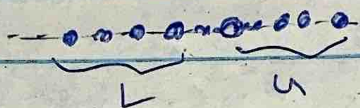
- conversely, sps (L,U) is a cut in A (i.e. $A = L \cup U$ and $L<U$)

- let $p = \{v > a : a \in L\} \cup \{v < a : a \in U\}$
- **claim**: p is a type, i.e. $p \cup Th_A(M)$ is sat'ble

pf: if $\exists x \in M$ s.t. $L < x < U$ then x realizes p



- if not, A must be infinite



- but for any finite $\Delta \subseteq p \cup Th_A(M)$ can find (since M dense) $x \in M$ sat'ing the fin. many rel'ns $v < a_i$, $v > a_j \in \Delta$
- thus $M \models \Delta$ (interp'ing v as x) and so $p \cup Th_A(M)$ is fin. sat ✓

↳ we claim this p determines a complete type, i.e.

- **Claim** if q, q' are complete types extending p, then $q = q'$

pf: sufficient to show $q \subseteq q'$, since these types are complete.

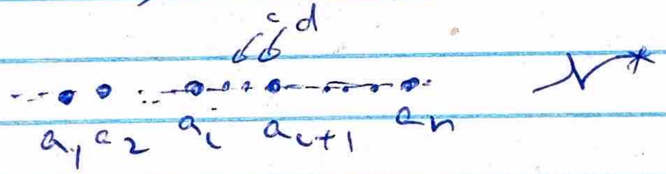
- fix $\varphi(v) \in q$.
- then φ involves only fin. many constants from A, say $a_1 < a_2 < \dots < a_n$ written in ascending order
- let $N \succ M$ be elem ext'n s.t.

$$N \models q \cup q' \cup Th_A(M)$$

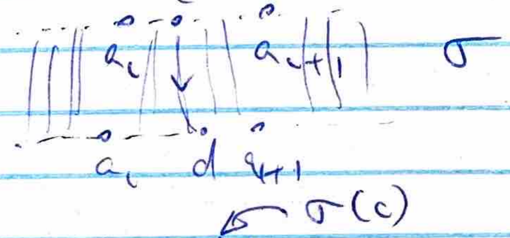
(can find $\mathcal{N}$ in two steps: find $\mathcal{N}_0 > \mathcal{M}$ realizing q, then $\mathcal{N} > \mathcal{N}_0$ realizing q')

- let $c, d \in \mathcal{N}$ realize q, q' resp.
- then $\mathcal{N} \models \varphi(c)$; wts $\mathcal{N} \models \varphi(d)$ (as this gives $\varphi \in q'$)

- view $\mathcal{N}$ as struct in lang $[a_1, .., a_n, <]$
- By downward LS theorem, can find a <u>ctbl</u> $\mathcal{N}^* < \mathcal{N}$ s.t. $c, d \in \mathcal{N}^*$
- since c, d both satisfy p (as $q, q' \supseteq p$) have $a_i < c$ iff $a_i < d$, i.e. c, d fall in same cut in $[a_1, .., a_n]$
- sps $a_i < c, d < a_{i+1}$:



- since interval $(a_i, a_{i+1}) \subseteq \mathcal{N}^*$ is ctbl DLO
$\exists$ an automorphism $\sigma : \mathcal{N}^* \to \mathcal{N}^*$ s.t. $\sigma(c) = d$



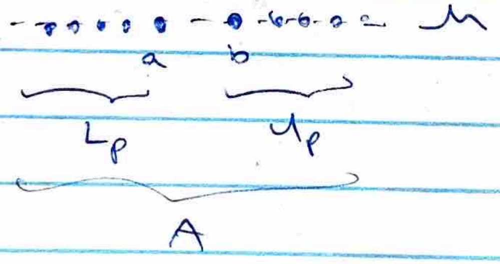
$\Rightarrow \mathcal{N}^* \models \varphi(d)$ since $\mathcal{N}^* \models \varphi(c)$
but then $\mathcal{N} \models \varphi(d)$ too (since $\mathcal{N}^* < \mathcal{N}$)

✓

Summary: given $M \models DLO^*$ and $A \subseteq M$, types $p \in S_1^M(A)$ are determined either by
(i) el'ts $a \in A$, or
(ii) cuts (L,U) in A

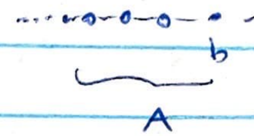
Q: which types $p \in S_1^M(A)$ are isolated?

- fix $p \in S_1^M(A)$
- if $v = a \in p$ for some $a \in A$, then $v = a$ isolates p
- if not, consider cut (L_p, U_p) in A determined by p
- Sps L_p has a greatest el't a and U_p has a least el't b



- then $a < v \wedge v < b \in p$
- this fmla isolates p: if $q \in S_1^M(A)$ and $a < v < b \in q$, then q determines same cut in A as p, hence $q = p$ by above.
- Similarly:
  - if $A = L_p$ (and $U_p = \emptyset$) (i.e. $a < v \in p$ for all $a \in A$) and A has a max el't a

b, then p is isolated (by $v > b$)

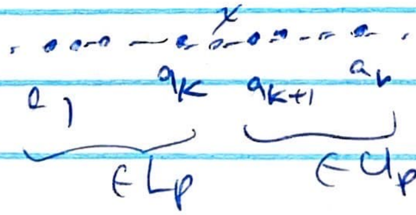


— or if $U_p = A$, $L_p = \emptyset$ and U_p has least el't c, then p is isolated (by $v < c$)

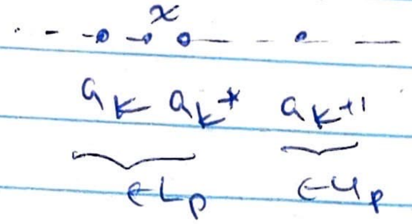
↳ claim: these are the only isolated types! i.e. if (L_p, U_p) is cut in A determined by $p \in S_1^M(A)$ and either L_p has no greatest el't (and $\neq \emptyset$) or U_p has no least el't (and $\neq \emptyset$) then p non-isolated

Why: e.g. sps L_p has no max el't

- fix $\varphi(v) \in p$, we claim $\varphi(v)$ does not isolate p, i.e. there is $q \in S_1^M(A)$ $q \neq p$ s.t. $\varphi(v) \in q$
- $\varphi(v)$ contains only fin many pts in A, say $a_1 < a_2 < \dots < a_n$
- sps $\{a_1, \dots, a_k\} \subseteq L_p$ and $\{a_{k+1}, \dots, a_n\} \subseteq U_p$
- by sim arg to above, any $x \in M$ s.t. $a_k < x < a_{k+1}$ satisfies φ



- Since L_p has no max, $\exists a_k^* \in L_p$ above a_k



- Fix x, $a_k < x < a_k^*$

- then $M \models \varphi(x)$ but x does not realize p (as $x < a_k^*$ but $v > a_k^* \in p$)

- Let $q = tp^M(x/A)$. then $q \neq p$ but $\varphi(v) \in q$ ✓

- Case when U_p has no least el't similar

Summary: - Given $M \models DLO^*$, $A \subseteq M$, the types $p \in S_1^M(A)$ not realized by el'ts in A correspond to cuts (L_p, U_p) in A
- p is non-isolated iff either L_p has no greatest el't (and is $\neq \emptyset$) or U_p has no least el't (and is $\neq \emptyset$)

Omitting types

- we slightly modify our notion of type then show we can build models in which certain types are omitted

- given $\mathcal{L}$ + an $\mathcal{L}$-theory T, say that a collect'n of fmlas $p = \{\varphi(v_1, \dots, v_n)\}$ is an n-type if $p \cup T$ is satisfiable.
- p is complete if either $\varphi \in p$ or $\neg\varphi \in p$ for all fmlas $\varphi(\vec{v})$ w/ n-free vars $v_1, \dots, v_n$
- $S_n(T)$ = space of complete n-types $/T$

Observe: if T is complete + satisfiable, then for any $\mathcal{M} \models T$ have $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$
(why: cause $Th_{\emptyset}(\mathcal{M}) = Th(\mathcal{M}) = T$)

- in general tho: if $p \in S_n(T)$ for T incomplete can only say $p \in S_n^{\mathcal{M}}(\emptyset)$ for some $\mathcal{M} \models T$
- e.g. if $T = \emptyset$ then $S_n(T)$ = all n-free-var fmla collections p that are true in some $\mathcal{L}$-structure!

- topology defined sim'ly on $S_n(T)$: basic open sets are of form $[\varphi] = \{p \in S_n(T) : \varphi \in p\}$
- p is isolated if $\{p\} = [\varphi]$ for some $\varphi \in p$

we slightly extend this so "isolated" can apply to incomplete types too.

Recall: if T complete (so $T = Th(\mathcal{M})$ for any $\mathcal{M} \models T$) then $p \in S_n^{\mathcal{M}}(\emptyset) = S_n(T)$ is isolated iff $p = [\varphi]$ iff $\forall \psi \in p$, $T \models \forall \vec{v}(\varphi(\vec{v}) \Rightarrow \psi(\vec{v}))$

This motivates:

Def'n Sps $\varphi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula s.t. $T \cup \{\varphi(\vec{v})\}$ is sat'ble, and p is an n-type (possibly incomplete, possibly not containing φ)

Say: φ *isolates* p if $\forall \psi \in p$
$$T \models \forall \vec{v}\,(\varphi(\vec{v}) \Rightarrow \psi(\vec{v}))$$

- isolated types are always realized, in following sense:

Prop'n (i) if $\varphi(\vec{v})$ isolates p, then p is realized in any model of $T \cup \{\exists \vec{v}\, \varphi(\vec{v})\}$
(ii) if T is complete, every isolated type is realized in every model $\mathcal{M} \models T$

Pf. (i) immediate: if $\mathcal{M} \models T$ there is $\vec{c} \in M^n$ s.t. $\mathcal{M} \models \varphi(\vec{c})$ and then $\vec{c}$ realizes p
(ii) follows from (i) since if T complete and $T \cup \{\exists \vec{v}\, \varphi(\vec{v})\}$ is satisfiabl then $\exists \vec{v}\, \varphi(\vec{v}) \in T$ ✓