- using prop'n, can show if $\mathcal{M} \equiv \mathcal{N}$ there is $\mathcal{N}' \succ \mathcal{N}$ embedding elem copy of $\mathcal{M}$
- more generally:

<u>Corollary</u> Given $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$, $B \subseteq M$, and $f: B \to \mathcal{N}$ partial elementary, there is an elem ext'n $\mathcal{N}' \succ \mathcal{N}$ and $g: \mathcal{M} \to \mathcal{N}'$ extending f.

<u>Pf.</u> - use prop'n to keep adding el'ts to $\mathrm{dom}(f)$ one at a time, until domain $= M$.
- more explicitly, sps $\mathcal{M}$ is ctbl, for simplicity
- enumerate $M \setminus B$ as $m_0, m_1, \ldots$
- by prop'n we can find a sequence of maps $f = f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$

and seq. of structures $\mathcal{N} = \mathcal{N}_0 \prec \mathcal{N}_1 \prec \mathcal{N}_2 \prec \ldots$ s.t. $f_i : B \cup \{m_0, m_1, \ldots, m_{i-1}\} \to \mathcal{N}_i$ is partial elementary $\forall i$
- let $g = \bigcup f_i$ $\mathcal{N}' = \bigcup \mathcal{N}_i$
- then $\mathrm{dom}(g) = \mathcal{M}$ and since g is an inc. union of part'l elem maps, is elem. (why?)
- case when $\mathcal{M}$ is unctbl is similar (just longer induction) ✓

Now we can prove:

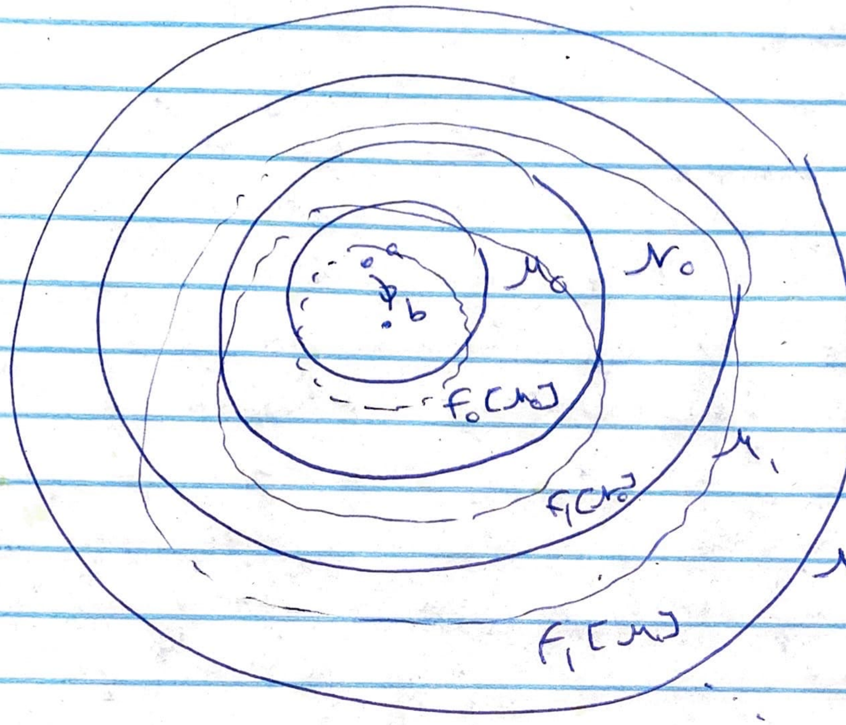
<u>Theorem</u> given $\mathcal{L}$-structs $\mathcal{M}, \mathcal{N}$ and $A \subseteq M$ and $a, b \in M$ s.t. $tp^{\mathcal{M}}(a/A) = tp^{\mathcal{M}}(b/A)$, there is elem ext'n $\mathcal{N} \succ \mathcal{M}$ and automorphism $\sigma: \mathcal{N} \to \mathcal{N}$ s.t. $\sigma(a) = b$. and σ fixes A pointwise

<u>Pf</u>: - Let $f: A \cup \{a\} \to A \cup \{b\}$ be the identity on A and $f(a) = b$
- since $tp(a/A) = tp(b/A)$, follows f is part'l elem.

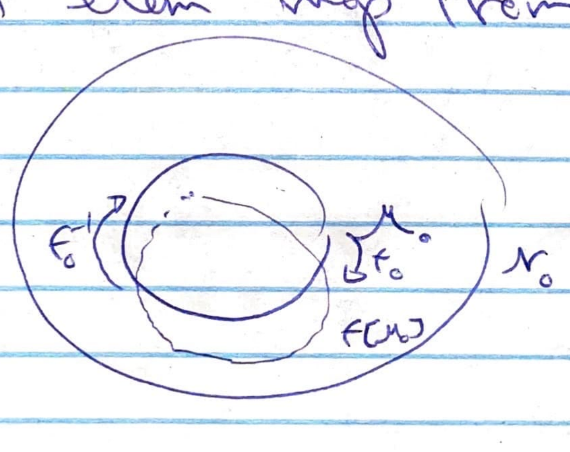
  <u>why</u>: $\mathcal{M} \models \varphi(a, a_1, \ldots, a_n)$ iff $\mathcal{M} \models \varphi(b, a_1, \ldots, a_n)$ $\forall$ fmla $\mathcal{L}_A$, $a_1, \ldots, a_n \in A$.
- By corollary, there is $\mathcal{N}_0 \succeq \mathcal{M}$ an elem. ext'n and an elem embedding $f: \mathcal{M} \to \mathcal{N}_0$ extending f
- we build an elem. chain
$$\mathcal{M} = \mathcal{M}_0 < \mathcal{N}_0 < \mathcal{M}_1 < \mathcal{N}_1 < \ldots$$
along w/ elem. embeddings $f_i: \mathcal{M}_i \to \mathcal{N}_i$ s.t. $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$ and s.t. $\mathcal{N}_i \subseteq$ image of f_{i+1}.

- then we let $\mathcal{N} = \bigcup \mathcal{M}_i = \bigcup \mathcal{N}_i$ and $\sigma = \bigcup f_i$
- then $\mathcal{M} \prec \mathcal{N}$ since lim of elem chain and $\sigma: \mathcal{N} \to \mathcal{N}$ is elem map s.t. $\sigma \restriction A = id$

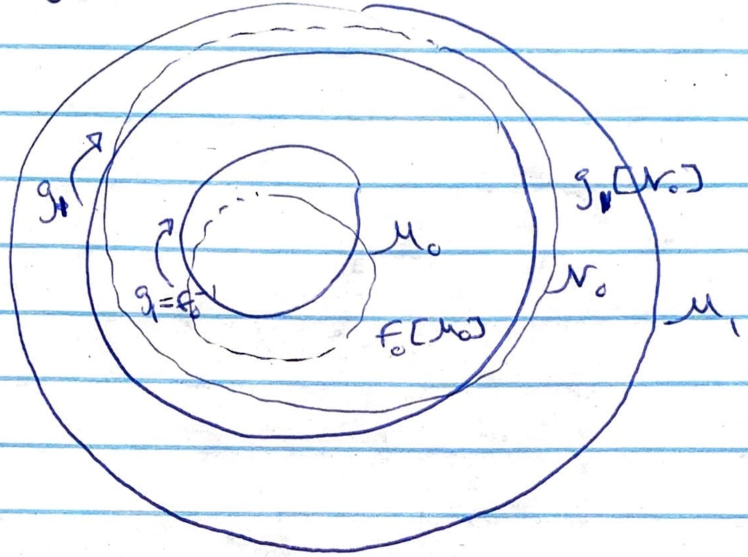
- Since $N_i \subseteq f[M_{i+1}]$, σ is surjective, hence an automorphism



- So how to get $M_0 < N_0 < M_1 < N_1 < \dots$?
- already have $f_0 : M_0 \to N_0$, key insight is how to get N_1 from M_0
- can view the inverse map f_0^{-1} as a partial elem map from N_0 (on)to M_0



- so: by corollary can find $\mathcal{M}_1 \succ \mathcal{N}_0$ and elem $g_1 : \mathcal{N}_0 \to \mathcal{M}_1$ extending f_0^{-1} (so: $g_1^{-1} = f_0$ on $\mathcal{M}_0$)



- now iterate: can view g_1^{-1} ($= f_0$ on $\mathcal{M}_0$) as part'l elem map of $\mathcal{M}_1$ (on) to $\mathcal{N}_1$
- find $\mathcal{N}_1 \succ \mathcal{M}_1$ and $f_1 : \mathcal{M}_1 \to \mathcal{N}_1$ extending g_1^{-1} (hence f_0)

... etc.

$\mathcal{L} = \{<\}$

<u>Ex</u> consider T = axioms for linear orders + axiom that every pt has unique successor and predecessor

"theory of discrete linear orders"

- we saw before: $\mathcal{M} = (M, <) \models T$ iff $\mathcal{M} \cong (L \times \mathbb{Z}, <_{lex})$ for some linear order L.
- consider $\mathcal{M} = (2 \times \mathbb{Z}, <_{lex})$ = "$\mathbb{Z} + \mathbb{Z}$"

$\mathcal{M}$: $(\cdots \underset{a}{\bullet} \cdots) \, (\cdots \underset{b}{\bullet} \cdots)$

$\mathbb{Z} + \mathbb{Z}$

- Fix $a \in M$ in left copy of $\mathbb{Z}$ and $b \in M$ in right copy

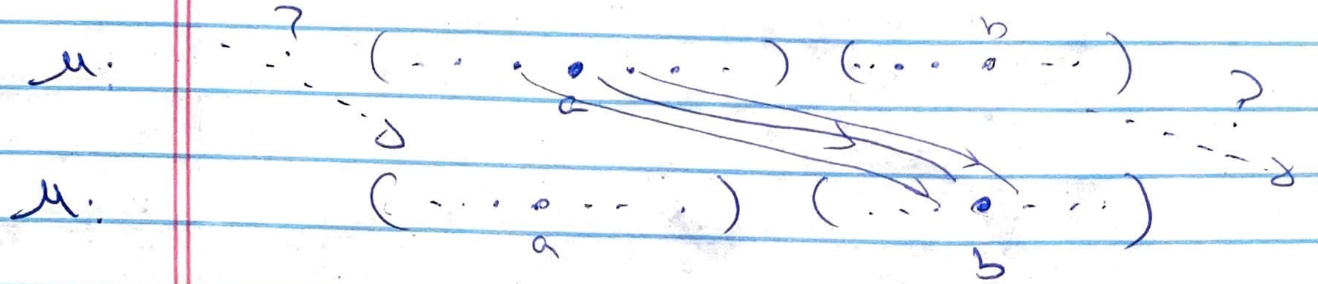
<u>Claim</u> $tp^{\mathcal{M}}(a/\emptyset) = tp^{\mathcal{M}}(b/\emptyset)$
(write: $tp^{\mathcal{M}}(a) = tp^{\mathcal{M}}(b)$)

<u>Pf sketch</u>: "$tp^{\mathcal{M}}(a) = tp^{\mathcal{M}}(b)$" is equiv. to assertion "$\mathcal{N}_a \equiv \mathcal{N}_b$" where
- $\mathcal{N}_a, \mathcal{N}_b$ are both copies of $(2 \times \mathbb{Z}, <_{lex})$ in expanded lang w/ new constant c
  - c is interpreted as a in $\mathcal{N}_a$, as b in $\mathcal{N}_b$.

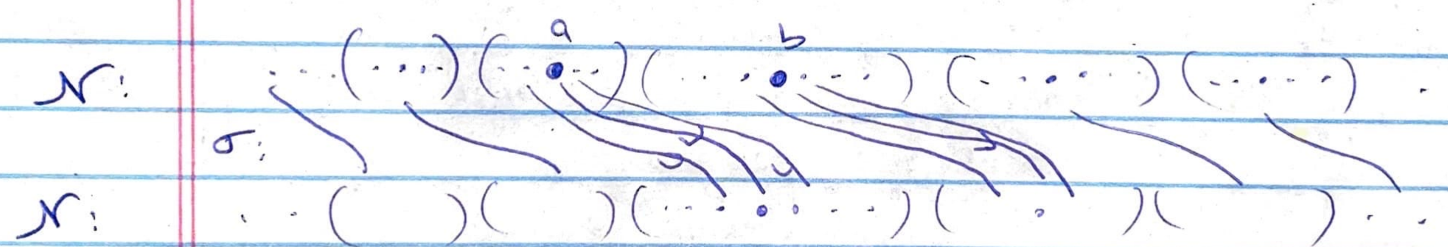
- we showed: any two models $\mathcal{N}, \mathcal{N}' \models T$ are elem equiv'nt by showing II has winning strat in $G_n(\mathcal{N}, \mathcal{N}')$ for all n
- in that game, first move of I, II could be arbitrary
- It follows: any two models $\mathcal{N}, \mathcal{N}'$ of T in <u>expanded lang</u> w/ c are $\equiv$
  (II still wins: imagine game starts w/ $c^{\mathcal{N}}, c^{\mathcal{N}'}$ already played)

- so $\mathcal{N}_a \equiv \mathcal{N}_b$, i.e. $tp^{\mathcal{M}}(a) = tp^{\mathcal{M}}(b)$ ✓

<u>Observe</u> there is no automorphism $\sigma: \mathcal{M} \to \mathcal{M}$ s.t. $\sigma(a) = b$

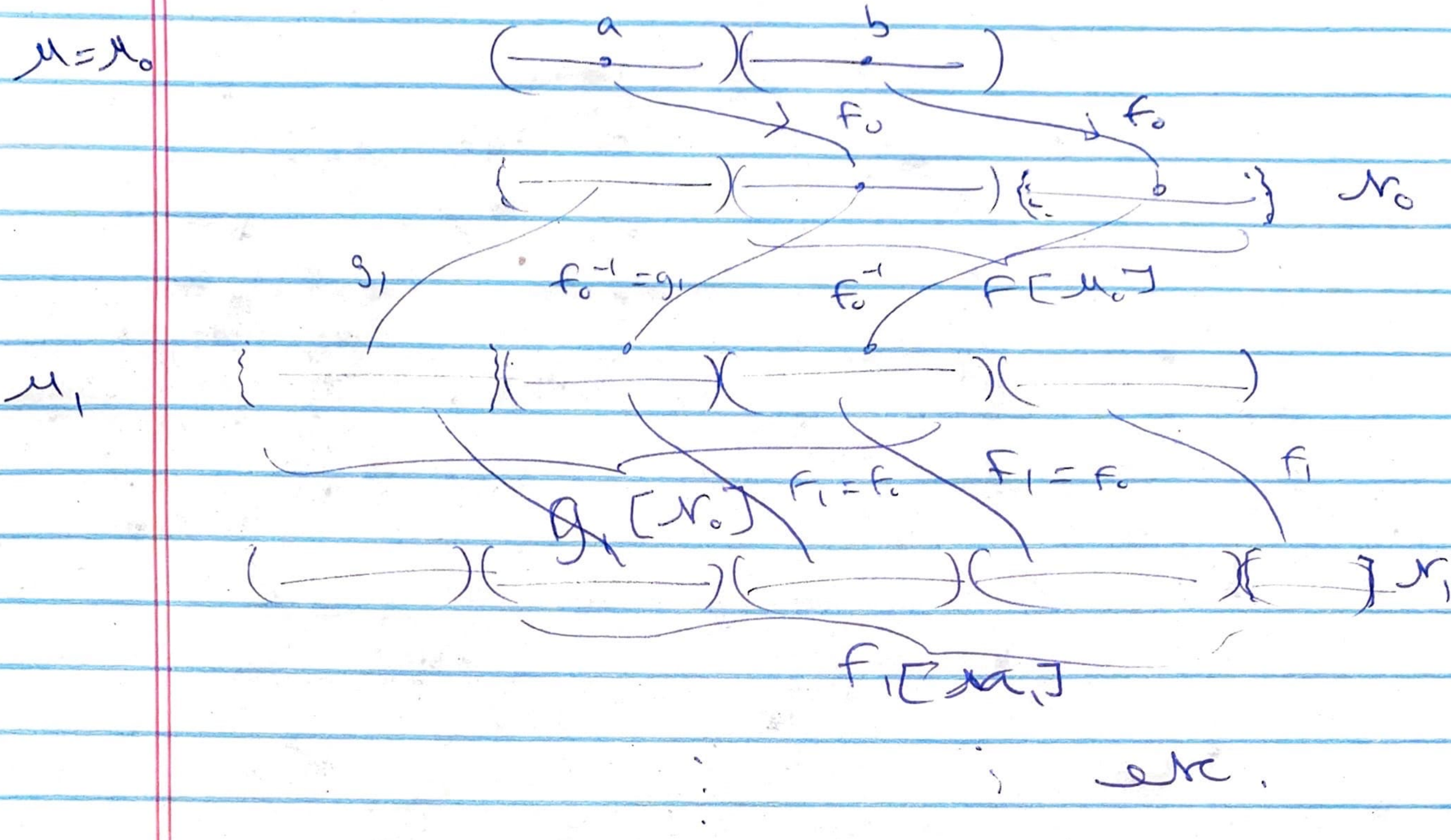


- by thm: there is elem ext'n $\mathcal{N} \succ \mathcal{M}$ and auto $\sigma: \mathcal{N} \to \mathcal{N}$ s.t. $\sigma(a) = b$
- indeed: consider $\mathcal{N} = (\mathbb{Z} \times \mathbb{Z}, <_{lex})$ w/ a, b in adj. copies of $\mathbb{Z}$:
- ez to see there is $\sigma: \mathcal{N} \to \mathcal{N}$ $\sigma(a) = b$:



- How does proof of theorem "construct" σ and $\mathcal{N}$? Like this:

over



Topologizing $S_n^{\mathcal{M}}(A)$

- we've shown: we can realize types (over a base model $\mathcal{M}$)

→ can we omit types?

i.e. given a theory T and a type p, can we find $\mathcal{M}\models T$ that omits p?

- to answer: need to put a topology on space of (complete) types S_n

(no background w/ topology? no worries: ultimately it's just some extra terminology that allows us to prove the omitting types theorem)

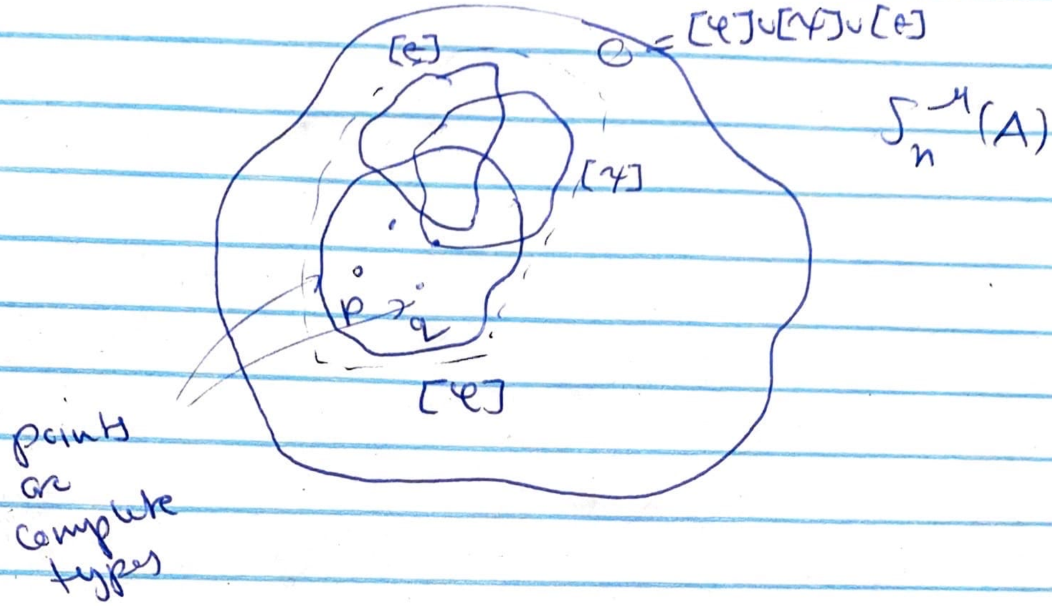
- Given $\mathcal{M}$, $A \subseteq M$ we consider space of complete n-types $S_n^{\mathcal{M}}(A)$
- given $\varphi(v_1, \ldots, v_n)$ an $\mathcal{L}_A$-fmla in n free vars $v_1, \ldots, v_n$ define:

$$[\varphi] = \{p \in S_n^{\mathcal{M}}(A) : \varphi \in p\}$$

- Observe: since any $p \in S_n^{\mathcal{M}}(A)$ is complete, if $\varphi \vee \psi \in p$ then either $\varphi \in p$ or $\psi \in p$.
- follows $[\varphi \vee \psi] = [\varphi] \cup [\psi]$
- similarly: $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$

- We view the sets $[\varphi]$ as the basic open sets of $S_n^{\mathcal{M}}(A)$
- the Stone topology on $S_n^{\mathcal{M}}$ is the topology generated by the sets $[\varphi]$
- i.e. $O \subseteq S_n^{\mathcal{M}}(A)$ is open iff

$$O = \bigcup_{i \in I} [\varphi_i] \quad \text{is a union of basic open sets}$$



<u>Fact</u> under the Stone topology, $S_n^{\mu}(A)$ is a compact, totally disconnected space

– omit proof, won't need this (just culture)

<u>Def'n</u> $p \in S_n^{\mu}(A)$ is <u>isolated</u> if $\{p\}$ is an open set.

– Int'n: isolated types p are determined by a <u>single</u> fmla $\varphi \in p$

<u>Prop'n</u> Fix $p \in S_n^{\mu}(A)$. TFAE:

(i) p is isolated

(ii) $\{p\} = [\varphi]$ for some $\mathcal{L}_A$-fmla $\varphi = \varphi(\vec{v}) \in p$ (say: φ <u>isolates</u> p)

(iii) there is an $\mathcal{L}_A$-fmla $\varphi(\vec{v}) \in p$

s.t. for all fmlas $\psi(\vec{v})$ we have:
$\psi \in p$ iff $Th_A(\mathcal{M}) \models \varphi(\vec{v}) \Rightarrow \psi(\vec{v})$

Pf (i) $\Rightarrow$ (ii) $\{p\}$ open $\Rightarrow$ $\{p\} = \bigcup_{i \in I} [\varphi_i]$
$\Rightarrow \{p\} = [\varphi] = [\varphi_i]$ for all i s.t. $[\varphi_i] \neq \emptyset$.
(ii) $\Rightarrow$ (iii): Sps $\{p\} = [\varphi(\vec{v})]$ and fix $\psi(\vec{v})$
- Sps first $\psi(\vec{v})$, we prove $Th_A(\mathcal{M}) \models \varphi \Rightarrow \psi$
- if not $Th_A(\mathcal{M}) \cup \{\varphi \wedge \neg\psi\}$ is sat'ble
i.e. $\{\varphi \wedge \neg \psi\}$ is a (incomplete) type.
  ↳ can find $\mathcal{N} \succ \mathcal{M}$ s.t. $\exists \vec{a} \in N^n$ and $\mathcal{N} \models \varphi(\vec{a}) \wedge \neg\psi(\vec{a})$
- consider $q = tp^{\mathcal{N}}(\vec{a}/A)$ $\in S_n^{\mathcal{N}}(A)$ (which $= S_n^{\mathcal{M}}(A)$)
- q is complete + contains φ, so $q \in [\varphi] = \{p\}$, so $q = p$
- but $\psi \in p$, contradiction ✓

- Now sps $\psi \notin p$
- then $\neg\psi \in p$, since p complete
- by arg above $Th_A(\mathcal{M}) \models \varphi \Rightarrow \neg\psi$
- which gives $Th_A(\mathcal{M}) \not\models \varphi \Rightarrow \psi$
(since $Th_A(\mathcal{M}) \cup \{\varphi\}$ is satisfiable) ✓

(iii) $\Rightarrow$ (i): we check $\{p\} = [\varphi]$
- have $\varphi \in p$ since $Th_A \models \varphi \Rightarrow \varphi$, hence $p \in [\varphi]$
- fix $q \in [\varphi]$, we show $q = p$

- given $\psi \in p$ hence $Th_A(M) \models \varphi \to \psi$
  hence $\psi \in q$ (since $q \cup Th_A(M)$ satisfiable)
- if $\neg\psi \in p$, then $\neg\psi \in q$ by same arg
  hence $p = q$ ✓

i.e. $\psi \notin p$ by completeness