Realizing + omitting types

- we're skipping ch. 3 in Marker (ch. 4 is more fun)

Types

- Given $\mathcal{L}$, and a set of $\mathcal{L}$-formulas $p = \{\varphi(v)\}$, can think of p as a set of properties an el't v in an $\mathcal{L}$-struct $\mathcal{M}$ might possess
- Given $\mathcal{M}$, can ask:
  ① is there such an el't in $\mathcal{M}$, i.e. an $a \in \mathcal{M}$ s.t. $\mathcal{M} \models \varphi(v)$ for all $\varphi \in p$?
  ② if not, is there an elementary extension $\mathcal{N} \succeq \mathcal{M}$ in which there is such an el't?

- Can be a bit more refined.
- Given $\mathcal{L}$-struct $\mathcal{M}$ and $A \subseteq M$
  Let $\mathcal{L}_A$ = lang. obtained by adding a constant to $\mathcal{L}$ for every $a \in A$.
- View $\mathcal{M}$ as $\mathcal{L}_A$-struct by interpreting each new symbol as el't it stands for.
- Let $Th_A(\mathcal{M})$ = set of $\mathcal{L}_A$ sentences φ s.t. $\mathcal{M} \models \varphi$

<u>Note</u>: $Th_A(M) \subseteq Diag_{el}(M)$ and is $=$ iff $A = M$.

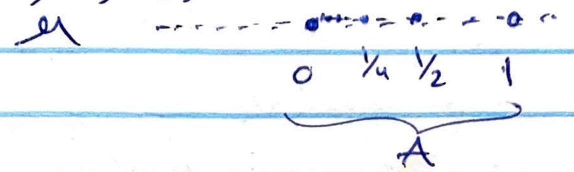
<u>**Def'n**</u> Let p be a set of $\mathcal{L}_A$-fmlas $\varphi(v_1, \ldots, v_n)$ in the free vars $v_1, \ldots, v_n$. If $p \cup Th_A(M)$ is satisfiable, say p is an <u>n-type</u> (over A). It is a <u>complete n-type</u> if either $\varphi(\vec{v}) \in p$ or $\neg\varphi(\vec{v}) \in p$ for every $\varphi(\vec{v})$ in the free vars $\vec{v} = (v_1, \ldots, v_n)$.

Denote the set of complete n-types over A by $S_n^{M}(A)$

- we write $p(v_1, \ldots, v_n)$ to indicate p is n-type in vars $v_1, \ldots, v_n$
- <u>note</u>: "$p(\vec{v}) \cup Th_A(M)$ is satisfiable" means there is $N \models Th_A(M)$ and $\vec{a} \in N$ s.t. $N \models \varphi(\vec{a})$ for all $\varphi \in p$
- can formalize this by adding new constants $\vec{c} = (c_1, \ldots, c_n)$ and saying $Th_A(M) \cup p(\vec{c})$ is satisfiable
  - $\hookrightarrow$ but we'll use "$p(\vec{v})$ is satisfiable" and think of the vars $\vec{v}$ as constants.

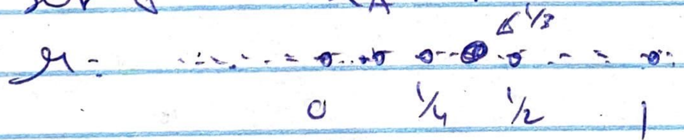
- only ever need to check <u>finite</u> satisfiability (by compactness)

Ex's ① Sps $\mathcal{M} = (\mathbb{Q}, <)$ and $A = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$



- let $p(v)$ be fmlas $\{0 < v\} \cup \{v < \frac{1}{2^n} : n \in \mathbb{N}\}$
  ↪ p asserts "v is larger than 0 but $< \frac{1}{2^n}$ $\forall n$"
- we check p is a 1-type over A, i.e. $p \cup Th_A(\mathcal{M})$ is (fin) sat.
- given finite $\Delta \subseteq$ $p \cup Th_A(\mathcal{M})$ there is largest N s.t. "$v < \frac{1}{2^N}$" is in Δ
- get $(\mathbb{Q}, <) \models \Delta$ by interpreting v as any rational q s.t. $0 < q < \frac{1}{2^N}$
- hence $p \cup Th_A(\mathcal{M})$ is finsat, hence sat ✓
- notice: there is no $q \in \mathcal{M}$ satisfying p!

② For same $\mathcal{M}$, let $p' = \{\varphi(v) :$ φ an $\mathcal{L}_A$-fmla and $\mathcal{M} \models \varphi(\frac{1}{3})\}$
$=$ set of all $\mathcal{L}_A$-fmlas true of $\frac{1}{3}$ in $\mathbb{Q}$



eg. $v < \frac{1}{2}$ and $v > \frac{1}{4}$, $v > \frac{1}{8}$, ... all in p'

- notice: p' a complete 1-type since for every $\varphi(v)$, either $\mathcal{M} \models \varphi(\frac{1}{3})$ or $\mathcal{M} \models \neg\varphi(\frac{1}{3})$

- note: ofc there is $q \in \mathcal{M}$ satisfying p namely $q = 1/3$.
- are there others?

↳ Ex. 2 shows a general way of getting complete types.
- Given $\mathcal{M}$ and $A \subseteq M$ and $\vec{a} = (a_1, .., a_n) \in M^n$
let $tp^{\mathcal{M}}(\vec{a}/A) = \{\varphi(\vec{v}) : \varphi$ an $\mathcal{L}_A$-fmla and $\mathcal{M} \models \varphi(\vec{a})\}$ = fmlas true of $\vec{a}$ in $\mathcal{M}$ (viewed as $\mathcal{L}_A$-struct)
- by some reasoning, $tp^{\mathcal{M}}(\vec{a}/A)$ always a complete n-type.
- we write $tp^{\mathcal{M}}(\vec{a})$ when $A = \emptyset$.

↳ the p' of ex. 2 was $tp^{\mathcal{M}}(1/3/A)$ where $\mathcal{M} = (\mathbb{Q}, <)$ and $A = \{0\} \cup \{1/2, 1/4, 1/8, ..\}$

<u>Def'n</u> if p is an n-type over A in $\mathcal{M}$, say that $\vec{a} = (a_1, .., a_n) \in M^n$ <u>realizes</u> p if $\mathcal{M} \models \varphi(\vec{a})$ for all $\varphi \in p$.
if there is no such $\vec{a}$ in $\mathcal{M}$, say $\mathcal{M}$ <u>omits</u> p.

↳ will write $\mathcal{M} \models p(\vec{a})$ to mean $\vec{a}$ realizes p.

<u>Ex's</u> (i) $p(v)$ from (c) above is omitted in $\mathcal{M} = (\mathbb{Q}, <)$ over $A = \{0, 1/2, 1/4, \ldots\}$ (since there is no rational > 0 but $< 1/2^n$ $\forall n$)

(ii) $1/3$ realizes $p'(v)$ in $\mathbb{Q}$

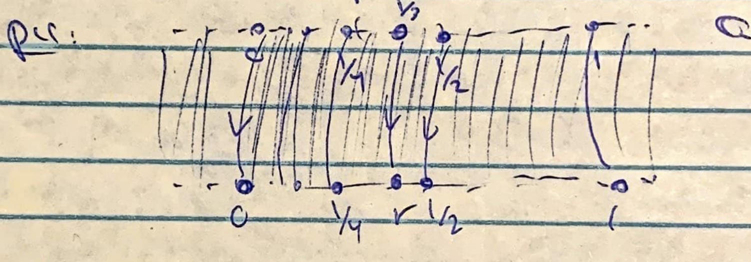
... but not the only realization in $\mathbb{Q}$!

↳ Let r be any rational s.t. $1/4 < r < 1/2$

– <u>claim</u>: $\mathcal{M} \models p'(r)$

– <u>why</u>: there is an order-automorphism $\sigma: \mathcal{M} \to \mathcal{M}$ that fixes every el't of A (i.e. σ is an automorphism of $\mathcal{M}$ viewed as $\mathcal{L}_A$-struct)

pic:



(get σ using ultrahomogeneity of Ctbl DLO's)

– we proved before: if $\sigma: \mathcal{M} \to \mathcal{M}$ an auto then $\mathcal{M} \models \varphi(a)$ iff $\mathcal{M} \models \varphi(\sigma(a))$ $\forall \varphi$

$\Longrightarrow \mathcal{M} \models \varphi(r)$ $\forall \varphi \in p'$

– so r realizes p', as claimed

– OTOH: no rational r s.t. $r \leq 1/4$ or $r \geq 1/2$ realizes p' (since $1/4 < v$ and $v < 1/2$ are in p')

Following prop'n says every type can be realized in an elementary extension

<u>Prop'n</u> Given $\mathcal{M}$, and p an n-type over $A \subseteq M$, there is an elementary extension $\mathcal{N} \succ \mathcal{M}$ and $\vec{a} \in N^n$ realizing p.

<u>Pf</u>: compactness!

- Let $\Gamma =$ $p \cup \text{Diagel}(\mathcal{M})$
(notice: $p \cup Th_A(\mathcal{M}) \subseteq \Gamma$)
- Since p a type there is a model $\mathcal{N}_0 \models p \cup Th_A(\mathcal{M})$
- we use $\mathcal{N}_0$ to show Γ is fin. sat. (hence sat.)
- fix $\Delta \subseteq \Gamma$ finite
- by taking a conjunction, may assume Δ is a single fmla
so has form:

$$\varphi(v_1, \dots, v_n, a_1, \dots, a_m) \wedge \psi(a_1, \dots, a_m, b_1, \dots, b_e)$$

where φ is conj'n of fmlas from p and ψ is conj'n of fmlas from Diagel; $a_1, \dots, a_m$ constants from A, $b_1, \dots, b_e$ constants from $M \setminus A$ that appear.

- know $\mathcal{N}_0 \models \varphi(v_1, \dots, v_n, a_1, \dots, a_m)$
$\mathcal{L}_A$-struct   i.e. $\exists$ elts in $\mathcal{N}_0$ making true
- how to get $\mathcal{N}_0 \models \psi(a_1, \dots, a_m, b_1, \dots, b_e)$?
have to interpret in $\mathcal{N}_0$
- know $\exists w_1, \dots, w_e\, \psi(a_1, \dots, a_m, w_1, \dots, w_e)$ is in $Th_A(\mathcal{M})$ since true in $\mathcal{M}$, since $\mathcal{M} \models \psi(a_1, \dots, a_m, b_1, \dots, b_e)$

- Hence $N_0 \models \exists \vec{w}\, \chi(\vec{a}, \vec{w})$
- if we interpret $b_1, .., b_\ell$ as any witnesses get $N_0 \models \chi(a_1, .., a_m, b_1, .., b_\ell)$
  $\Rightarrow N_0' \models \Delta$ ✓

- Hence Γ is finsat hence sat
- if $N \models \Gamma$ then $N \models \mathrm{Diagel}(M)$
  so (may assume) $M \prec N$
- if $(c_1, .., c_n)$ is interp of constants $v_1, .., v_n$ in N_0 have $\vec{c}$ realizes p ✓

Observe: since $M \prec N$, have $N \models \mathrm{Diagel}(M)$ and in particular $N \models \mathrm{Th}_A(M)$
So follows $\mathrm{Th}_A(N) = \mathrm{Th}_A(M)$
($\supseteq$ immediate & since $\mathrm{Th}_A(M)$ complete $\mathcal{L}_A$-theory)
- thus $S_n^M(A) = S_n^N(A)$
- i.e. the complete n types are same in M as in any elem extension
  So we get:

$\bigcirc$ i.e. p a complete n-type/$A \subseteq M$

Corollary $p \in S_n^M(A)$ iff there is $N \succ M$ and $\vec{a} \in N$ s.t. $tp^N(\vec{a}/A) = p$

Pf. ($\Leftarrow$) follows since $tp^N(\vec{a}/A)$ is complete n type + observation above
($\Rightarrow$) follows from prop'n.

<u>Summary</u>:

- types p can always be realized in some $N \succ M$
- <u>Complete</u> types are always of form $tp(\vec{a}/A)$ ... possibly for $\vec{a}$ in some extension $N \succ M$.

<u>Question</u>: given $a, b \in M$, what if $tp^M(a/A) = tp^M(b/A)$?

↳ we saw for $M = (\mathbb{Q}, <)$ and $A = \{0, 1/2, 1/4, ...\}$ we have $tp^M(r/A) = tp^M(1/3/A)$ iff r is a rational s.t. $1/4 < r < 1/2$.

↳ and in this case there is automorphism $\sigma: M \to M$ fixing A pointwise s.t. $\sigma(1/3) = r$

... is converse true in general?
... yes! ... if we pass to an extension:

<u>Theorem</u> Given M and $A \subseteq M$ sps $\vec{a}, \vec{b} \in M^n$ are n-tuples s.t. $tp^M(\vec{a}/A) = tp^M(\vec{b}/A)$. Then there is an elem. extension $N \succ M$ and an automorphism $\sigma: N \to N$, fixing A pointwise, s.t. $\sigma(\vec{a}) = \vec{b}$.

- will need some prelim results first.

<u>Def'n</u> Given $\mathcal{L}$-structs $\mathcal{M}, \mathcal{N}$ and $B \subseteq M$, we say that $f: B \to N$ is a <u>partial elementary map</u> if

$$\mathcal{M} \models \varphi(\vec{b}) \text{ iff } \mathcal{N} \models \varphi(f(\vec{b}))$$

← $(f(b_1), \dots, f(b_n))$

for all tuples $\vec{b} \in B$

↳ <u>Note</u>: if $B = \emptyset$ then the existence of an partial elementary map (the empty map) is just the condition $\mathcal{M} \equiv \mathcal{N}$.

so this has to be true for <u>any</u> partial elem map to exist.

- we know: not true in general that $\mathcal{M} \equiv \mathcal{N}$ implies $\mathcal{N} \prec \mathcal{M}$ or $\mathcal{M} \prec \mathcal{N}$, but following prop'n will help show: if $\mathcal{M} \equiv \mathcal{N}$ there is $\mathcal{M}'$ s.t. $\mathcal{M}, \mathcal{N} \prec \mathcal{M}'$ (or: at least embed)

<u>Prop'n</u> Sps $\mathcal{M}, \mathcal{N}, B$ as in theorem statement above and $f: B \to N$ is a partial elementary map.

If $b \in M$, there is an elem extension $\mathcal{N}_1 \succ \mathcal{N}$ and a partial elem map $f_1 : B \cup \{b\}$ extending f.

fmlas we want $f(b)$ to satisfy in $\mathcal{N}_1$

Pf – Let $\Gamma = \{\varphi(v, f(a_1), \dots, f(a_n)) : \mathcal{M} \models \varphi(b, a_1, \dots, a_n)$ and $a_1, \dots, a_n \in B\} \cup \mathrm{Diag}_{el}(\mathcal{N})$

- Sps we find $\mathcal{N}_1 \models \Gamma$
- Since $\mathcal{N}_1 \models \mathrm{Diag}_{el}(\mathcal{N})$ may assume $\mathcal{N} \preceq \mathcal{N}_1$
- if $c \in N_1$ is el't satisfying the fmlas $\varphi(v, f(a_1), \dots, f(a_n))$ then can extend f to $f_1 : B \cup \{b\} \to \mathcal{N}_1$ by setting $f_1(b) = c$
- will be partial elementary too

- So suffices to prove Γ is satisfiable ($\Leftrightarrow$ finsat)
- Fix $\Delta \subseteq \Gamma$ finite; we show $\mathcal{N} \models \Delta$
  ↳ need to interpret free var. v appearing in Δ in $\mathcal{N}$
- may assume Δ consists of a single fmla $\varphi(v, f(a_1), \dots, f(a_n))$ + some fmlas (from $\mathrm{Diag}_{el}(\mathcal{N})$) that are true in $\mathcal{N}$
- Know $\mathcal{M} \models \varphi(b, a_1, \dots, a_n)$

$\Rightarrow \mathcal{M} \models \exists v\, \varphi(v, a_1, \dots, a_n)$

$\Rightarrow \mathcal{N} \models \exists v\, \varphi(v, f(a_1), \dots, f(a_n))$ since f partial elem

$\Rightarrow$ there is $c \in \mathcal{N}$, $\mathcal{N} \models \varphi(c, f(a_1), \dots, f(a_n))$

interpreting v as c shows $\mathcal{N} \models \Delta$ ✓

Hence Γ finsat, hence sat ✓