Proof of Fraïssé's theorem

- Assume $\mathcal{L}$ is relational lang. throughout.

Thm (Fraïssé) Sps K is a class of finite structures (closed under $\cong$) containing only ctbly many structures (up to $\cong$) satisfying HP, JEP, AP.
There is a unique (up to $\cong$) ctbl ultrahomogeneous $\mathcal{M}$ s.t. $Age(\mathcal{M}) = K$.

Pf: ① Uniqueness
- Sps $\mathcal{M}, \mathcal{N}$ are ctbl, u.h. (ultrahomogeneous) and $Age(\mathcal{M}) = Age(\mathcal{N}) = K$
- We prove $\mathcal{M} \cong_p \mathcal{N}$, which gives $\mathcal{M} \cong \mathcal{N}$ since both ctbl.
- Let I = family of all finite partial iso's $\pi: \mathcal{M} \to \mathcal{N}$. Show I has b-f prop.

- given $\pi \in I$ and $a \in \mathcal{M}$, let $A = dom(\pi)$, $B = dom(\pi) \cup \{a\}$.
- Want π' w/ $dom(\pi') \supseteq B$ extending π.
- Since $A, B \subseteq \mathcal{M}$ finite have $A, B \in Age(\mathcal{M})$ hence $A, B \in Age(\mathcal{N})$
- Showed last time: $\mathcal{N}$ u.h $\Rightarrow$ $Age(\mathcal{N})$ has extension property
- Since $\pi: A \to \mathcal{N}$ is an embedding and $A \subseteq B$ are in $Age(\mathcal{N})$ there is

$\pi': B \to N$ extending π, by extension prop ✓
- cfc $\pi' \in I$ ✓
- symmetric given $\pi \in I$, $b \in N$ ✓

② Existence

- want to construct ctbl u.h. M s.t. $Age(M) = K$

Lemma: If M is ctbl and satisfies extension prop. then M is ultrahomogeneous

Pf: - Let I = all finite partial iso's $\pi: M \to M$
- as above (taking $M = N$), extension prop $\Rightarrow I$ has b-f prop.
- but then: can extend any $\pi \in I$ to an automorphism $\hat{\pi}: M \to M$ (just follow pf that $M \equiv_p N \Rightarrow M \cong N$ for ctbl M, N, beginning w/ $\pi_0 = \pi$) ✓

- last time we showed: M u.h. $\Rightarrow M$ has ext. prop
- so Lemma shows ext. prop $\Leftrightarrow$ u.h
- so: to finish, suffices to find ctbl M w/ $Age(M) = K$ s.t. M has ext. prop.

- We construct a sequence
$M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$
w/ $M_i \in K$ for all i

(*) s.t.: if $A \subseteq B$ are in K and $f: A \to M_i$ is an embedding, there is $j \geq i$ and an embedding $g: B \to M_j$ extending f

We'll then be done. Why:
- Let $M = \bigcup_{i \in \mathbb{N}} M_i$ be limit
- if $A \subseteq M$ finite, $A \subseteq M_i$ for some i hence $A \in K$ (HP)
- follows $Age(M) \subseteq K$
- Conversely, if $A \in K$, can find $B \in K$ embedding A and M_0 (JEP)
- may assume $M_0 \subseteq B$.
- using (*) wrt inclusion embedding $f: M_0 \to M_0$ can find j and $g: B \to M_j$ extending f.
- so $A \leq M_j$ so $A \leq M$ so $A \in Age(M)$
$\Rightarrow K = Age(M)$
- So because $K = Age(M)$, extension prop for M follows from (*) ✓

How to get (*)?

- write down ctbl list $\{(A_i, B_i)\}_{i \in \mathbb{N}}$ s.t. $\forall i$, $A_i \subseteq B_i$ are in K, and for all $A \subseteq B$ in K there is i s.t. "$(A,B) \cong (A_i, B_i)$", i.e. $\exists$ iso $\pi : B \to B_i$ sending A to A_i

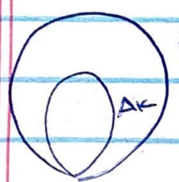
  ↳ possible since only ctbly many structs in K up to iso.

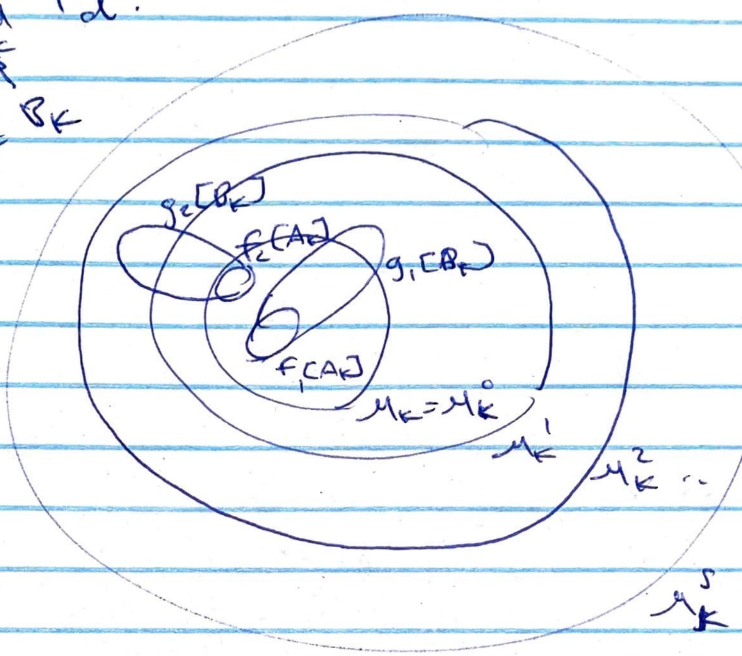
- We build $M_0 \subseteq M_1 \subseteq \cdots$ inductively
- Let $M_0 = A_0$
- Spse have $M_0 \subseteq \cdots \subseteq M_k$ s.t. $\forall i \leq k$ have A_i embeds in M_i and $\forall i < k$ have every embedding $f : A_i \to M_i$ extends to some $g : B_i \to M_{i+1}$
- We build M_{k+1} to preserve this condition.
- enumerate all embeddings $f : A_k \to M_k$ (at least one exists by ind. hyp) as $f_1, \dots, f_s$ (only finitely many!)
- Let $M_k^0 = M_k$ and iteratively apply AP to the diagrams

$$\begin{tikzcd} M_k^{d-1} \arrow[rr, dotted, dash] & & B_k \\ & A_k \arrow[ul, "f_d"] \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

to get M_k^d and $g_d : B_k \to M_k^d$ extending f_d.

$$\begin{array}{ccc} M_k^{d-1} & \xrightarrow{g_d} & M_k^d \\ \uparrow f_d & & \uparrow \\ A_k & \subseteq & B_k \end{array}$$





- may assume $M_k^d \subseteq M_k^{d+1}$ $\forall d < s$
- Let M_{k+1} be any struct in K containing M_k^s and embedding A_{k+1}
- then $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_k \subseteq M_{k+1}$ ✓ as desired ✓

To check $(*)$: Fix $A \subseteq B$ in K and $f : A \to M$.
- For some k have $(A, B) \cong (A_k, B_k)$
- View $f : A \to M_k$ as being into M_k
- then f is among the f_d at kth stage of construction
- so by above $\exists\, g : B \to M_{k+1}$ extending f. Take $j = k+1$ ✓

The theory of Flim(K)

- Now sps $\mathcal{L}$ is finite and relational.
- Fix a Frasse class K (i.e. satisfies hypotheses of theorem: HP, JEP, AP, etc)
- Goal: find concrete T axiomatizing Th($\mathcal{M}$) where $\mathcal{M}$ = Flim(K)
  (think: DLO* axiomatizing Th($\mathbb{Q}$, <) or how we axiomatized Th(random graph))

- We'll find such a T and moreover get $\mathcal{N}$ ctbl and $\mathcal{N} \models T \Rightarrow \mathcal{N} \cong \mathcal{M}$
- i.e. T is $\aleph_0$-cat'g'l (hence complete)

Idea: find T that expresses "any model $\mathcal{N}$ has extension prop. and age($\mathcal{N}$) = K"

- suffices, by Frasse's theorem.

- Main point: since $\mathcal{L}$ finite rel'n'l any finite $\mathcal{L}$-struct A of a given size n can be described by a fmla $\varphi_A(x_1, \dots, x_n)$ in sense that:
  if $\mathcal{N} \models \varphi(a_1, \dots, a_n)$
then $\{a_1, \dots, a_n\} \subseteq \mathcal{N}$ is a finite substruct $\cong$ A.

- So, for each n, can list all (up to $\cong$) $\mathcal{L}$-structs $A_1, \ldots, A_N$ of size n in K and let Φ_n be sentence:

$$\forall x_1, \ldots, x_n \left(\text{if the } x_i \text{ distinct} \Rightarrow \bigvee_{j \le N} \varphi_{A_j}(\vec{x}) \right)$$

"any finite substruct of size n is iso to some A_j"

- If we include Φ_n's in T, get $Age(\mathcal{N}) \subseteq K$ for any $\mathcal{N} \models T$
- to get $K \subseteq Age(\mathcal{N})$ include the sentences $\exists x_1, \ldots, x_n (\varphi_A(\vec{x}))$ for every $A \in K$ of size n.

- how to get extension prop? Think about how to formalize the following:
  - given $A \subseteq B$ in K w/ say $A = \{a_1, \ldots a_n\}$ and $B = \{a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}\}$ can write sentence expressing:
  "$\forall x_1, \ldots, x_n$ (if $\{x_1, \ldots, x_n\} \cong A$ under embedding $a_i \to x_i$ $i \le n$ then $\exists x_{n+1} \ldots x_{n+k}$ s.t. $\{x_1, \ldots, x_{n+k}\} \cong B$ under embedding $a_i \to x_i$ $i \le n+k$)"

- Let T = collection of these sentences
- if $N \models T$ then N has extension property and $Age(N) = K$

$\Rightarrow$ if N ctbl is $\cong M$ by Fraisse. ✓

<u>Ex</u>: - Given finite rel'l $\mathcal{L}$, can consider K = <u>all</u> finite $\mathcal{L}$-structures
- Will be a Fraisse class:

HP (clear) ✓

JEP (take disjoint union) ✓

AP (if $A \subseteq B \cup C$, then $B \cup C$ is $\mathcal{L}$-struct in natural way so can amalgamate) ✓

$\Rightarrow$ unique ctbl u.h. limit $M = Flim(K)$
- think of K as the "random" ctbl $\mathcal{L}$-struct
- e.g. if $\mathcal{L} = \{R\}$ w/ R binary can think of an $\mathcal{L}$-struct M as a directed graph (w/ loops, no repeated edges): R^M = just a collection of pairs (a, b) i.e... directed edges!

- think of $M = Flim(K)$ as "random ctbl digraph" - is structure obtained (a.s.) by starting w/ vertices $v_0, v_1, \ldots$ and adding a directed edge (v_i, v_j) over each pair w/ prob. $1/2$.

Why?

- sufficient to show such M has (c.s.) extension property and every finite directed graph appears as a subgraph

- but this is "clearly" true for M
- think why!