Fraïssé Theory

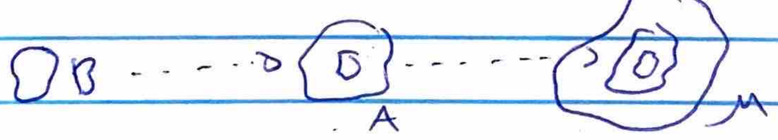
- We generalize the construction of the random graph.
- Sps $\mathcal{L}$ a relat'l lang. (not necessary — just so we can deal w/ finite instead of finitely generated substructures)
- Given $\mathcal{L}$-struct $\mathcal{M}$, define:

$$\mathrm{Age}(\mathcal{M}) = \{A : A \text{ a finite } \mathcal{L}\text{-struct embeddable in } \mathcal{M}\}$$

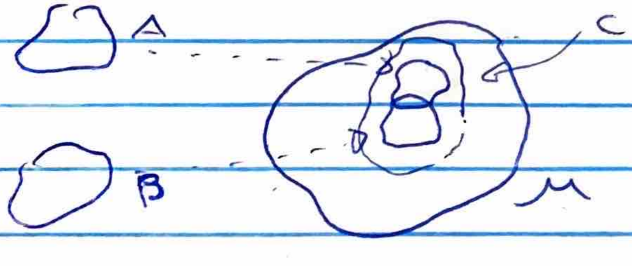
- Given structs A, B, write $A \leq B$ if there is an embedding of A into B
- So $\mathrm{Age}(\mathcal{M}) = \{A : A \text{ finite and } A \leq \mathcal{M}\}$

Observe: $\mathrm{Age}(\mathcal{M})$ has following properties:

① hereditary property (HP): if $A \in \mathrm{Age}(\mathcal{M})$ and $B \leq A$ then $B \in \mathrm{Age}(\mathcal{M})$



② joint-embedding property (JEP): if $A, B \in \mathrm{Age}(\mathcal{M})$ there is $C \in \mathrm{Age}(\mathcal{M})$ s.t. $A \leq C$ and $B \leq C$



e.g. if A', B' are embedded images of A, B resp. in M can take $C = A' \cup B'$

↳ we usually distinguish structs in $Age(M)$ only up to isomorphism
 i.e. think of $Age(M)$ as a class of isomorphism types

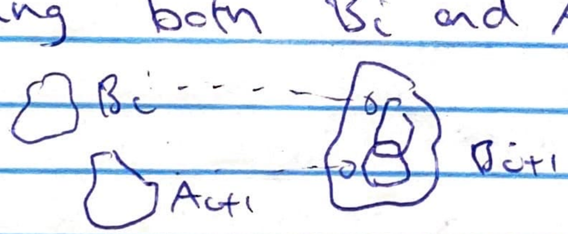
↳ Then: if M is ctbl, $Age(M)$ is ctbl too, in sense that only ctbly many non-iso structs in $Age(M)$.

<u>Theorem</u> Sps K is a class of finite $\mathcal{L}$-structures that contains only ctbly many structures up to isomorphism,

and is closed under isomorphism

and K satisfies HP, JEP. Then there is a ctbl $\mathcal{L}$-struct M st. $K = Age(M)$

<u>Pf.</u> – let $A_0, A_1, \ldots$ list every struct in K (up to isomorphism)
– we build a sequence $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$
– let $B_0 = A_0$
 given B_i, let B_{i+1} = any $C \in K$ embedding both B_i and A_{i+1} (using JEP)



– may in fact assume $B_i, A_{i+1} \subseteq B_{i+1}$

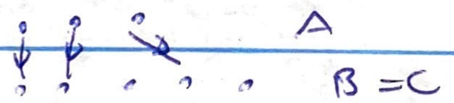
(using K closed under $\cong$)

- Let $M = \bigcup_{i \in \mathbb{N}} B_i$ be limit structure
- By construction, if $A \in K$ then $A \leq M$
  i.e. $K \subseteq \mathrm{Age}(M)$
- and if $C \subseteq M$ is finite have $C \subseteq B_i$ for large enough i
- follows (by HP) that $C \in K$ (since $C \subseteq B_i$)

$\Rightarrow \mathrm{Age}(M) = K$ ✓

<u>Note:</u> lim struct M from theorem needn't be unique!

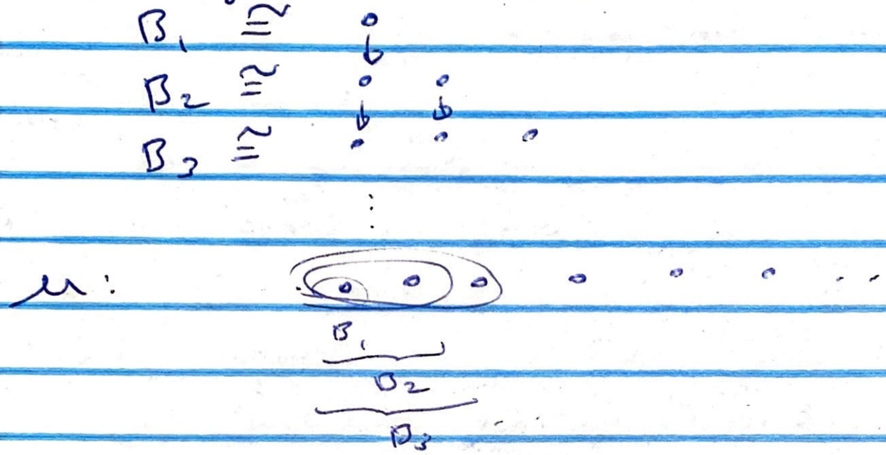
- e.g. consider K = class of finite linear orders
- Let $A_n = (A_n, <)$ be any fixed l.o. w/ n points
- then $A_1, A_2, \ldots$ represents every order in K up to iso.
- K clearly has HP
  JEP too: if $(A,<), (B,<) \in K$ have sizes $n \leq m$ resp. then both embed in $(C,<) = (B,<)$



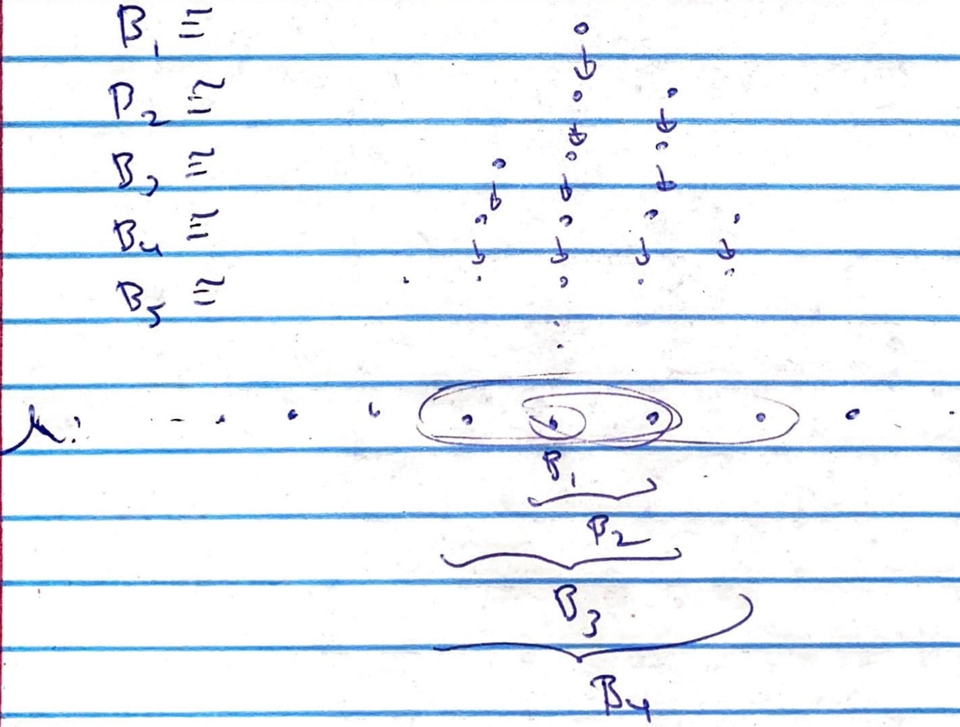
- So: for sequence $B_1 \subseteq B_2 \subseteq \cdots$ from theorem, can take the B_i's to be incr. sequence of l.o.'s s.t. B_n has size n

– point: can get different $\mathcal{M}$'s depending on how each B_i sits in B_{i+1}

↳ can get $\mathcal{M} \cong (\mathbb{N}, <)$ as follows:



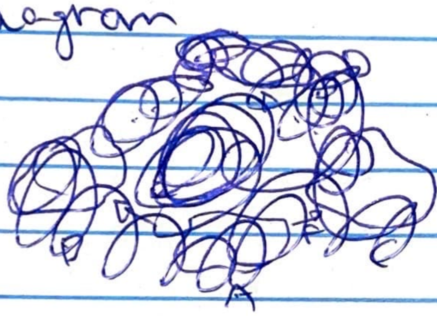
or $\mathcal{M} \cong (\mathbb{Z}, <)$:

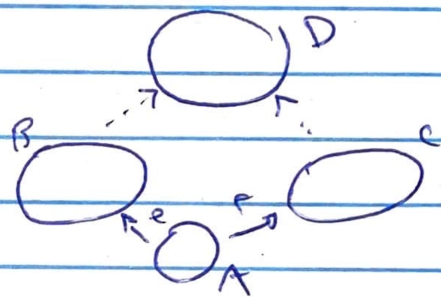


<u>In fact</u>: can get <u>any</u> ctbl l.o. as outcome of this process! (exercise)

↪ to get <u>uniqueness</u> of lim structs, need a new property:

- Say K (class of finite $\mathcal{L}$-structs) has the <u>amalgamation property</u> (AP) if when $A, B, C \in K$ and there are embeddings $e: A \to B$, $f: A \to C$ the diagram





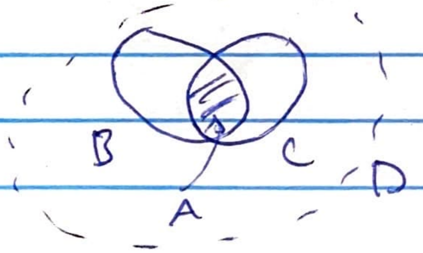
can be realized in K, i.e. there is $D \in K$ and embeddings $g: B \to D$, $h: C \to D$ s.t. diagram

$$\begin{tikzcd} & D & \\ B \arrow[ur, "g"] & & C \arrow[ul, "h"'] \\ & A \arrow[ul, "e"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes

i.e. $g \circ e = h \circ f$

Int'n: think of A as substruct of B and C



AP says can find $D \in K$ containing "$B \cup C$"

not formally defined

Ex

let K = class of finite L.O's again

- Claim K has AP
- pf: by two examples:

① if A = · · = 2

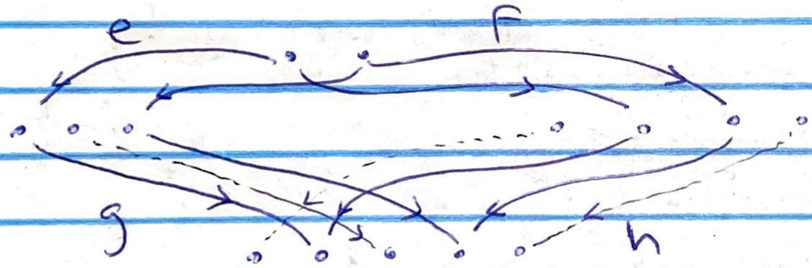
B = · · · = 3

C = · · · · = 4

and e is:  f is 

let D = · · · · · = 5

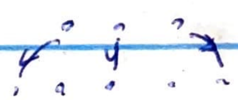
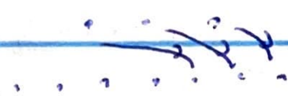
Can complete the diagram:



② if A = · · · = 3

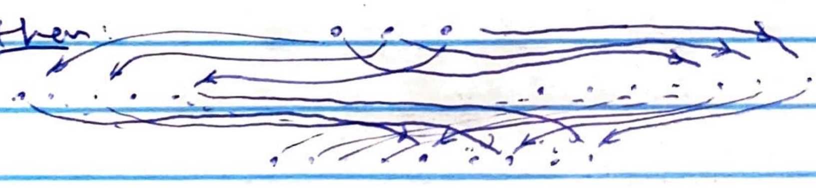
B = · · · · · = 5

C = · · · · · · · = 7

e is:  f is: 

take D = · · · · · · · · · = 9

then:



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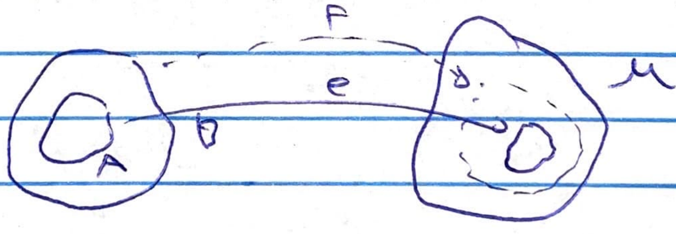
Theorem Sps $\mathcal{M}$ is a ctbl, ultrahomogeneous $\mathcal{L}$-struct. Then $Age(\mathcal{M})$ has HP, JEP, AP.

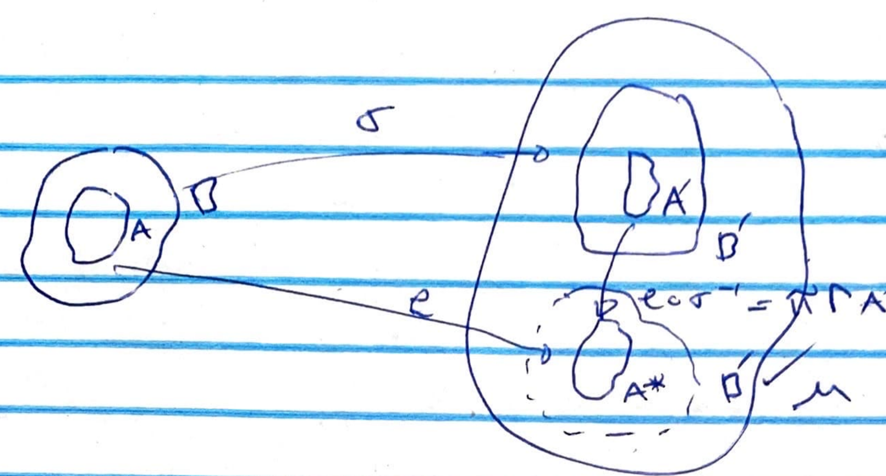
(recall: whenever $A, B \subseteq \mathcal{M}$ are finite wo substructs, any wo $\pi : A \to B$ extends to an automorphism $\tilde{\pi} : A \to B$.)

Pf: need following

Lemma $Age(\mathcal{M})$ has extension property over $\mathcal{M}$, i.e. if $A, B \in Age(\mathcal{M})$ and $A \subseteq B$ then for any $e : A \to \mathcal{M}$ an embedding there is $f : B \to \mathcal{M}$ extending e.

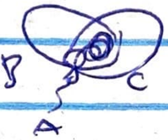


Pf: Since $B \in Age(\mathcal{M})$ there is $B' \subseteq \mathcal{M}$ s.t. $B' \cong B$. Fix $\sigma : B \to B'$ an iso.
- Let $A' = \sigma[A]$
- Let $A^* = e[A]$
- then $e \circ \sigma^{-1} : A' \to A$ is an iso
- by AP can be extended to auto $\pi : \mathcal{M} \to \mathcal{M}$
- Let $f = \pi \circ \sigma$ works (check!)

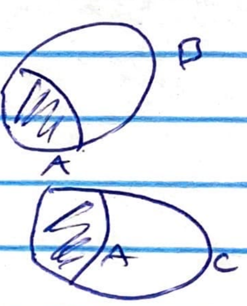


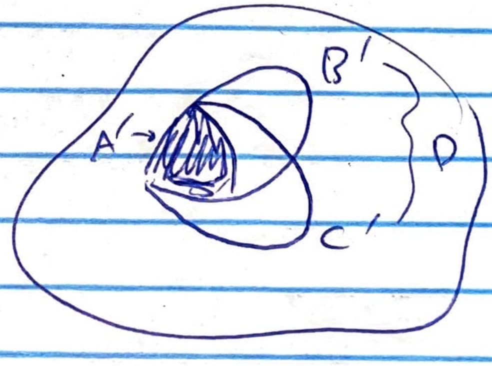
Pf of thm: Need only check AP

- Sps $A, B, C \in \mathrm{Age}(M)$ and $e: A \to B$ embedding
  $f: A \to C$
- by replacing B, C w/ isomorphic structures in which $e[A] = f[A] = A$, may assume $A \subseteq B, C$



- need to find $D \in \mathrm{Age}(M)$ and $g: B \to D$, $h: C \to D$ that agree on A (i.e. $g \upharpoonright A = h \upharpoonright A$)
- Fix $A' \subseteq M$ iso to A and fix $i: A \to A'$ an isomorphism
- by lemma can find $B', C' \subseteq M$ iso to B, C and iso's $d_0: B \to B'$, $d_1: C \to C'$ both extending i
- take $D = B' \cup C'$ ✓





<u>Int'n</u>: M allows us to formally define the amalgamated structure "$B \cup C$"

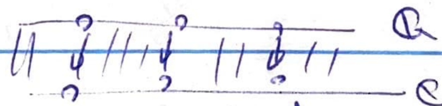
<u>Theorem</u> (Fraïssé) Sps K is class of finite $\mathcal{L}$-structs (closed under $\cong$) and containing ctbly many structs up to $\cong$, and satisfies HP, JEP, AP.

Then there is a <u>unique</u> (up to $\cong$) ctbl, ultrahomogeneous M s.t. $Age(M) = K$

- M is called the Fraïssé <u>limit</u> of K
  write $M = Flim(K)$
- Before pf, two ex's:

① We justified K = finite l.o.'s has HP, JEP, AP. What is $Flim(K)$?
- By Fraïssé's thm: must be $(\mathbb{Q}, <)$! because
  - $Age(\mathbb{Q}) = K$
  - $\mathbb{Q}$ is ctbl and
  - <u>Claim</u> $(\mathbb{Q}, <)$ is ultrahomogeneous ✓

<u>Pic</u>:



extends to auto

② Let K = class of finite graphs.
- satisfies HP ✓
- JEP too (can take disjoint union of two graphs)

- What about AP?
- Given finite graph A and finite B, C containing a copy of A, may assume $A \subseteq B, C$.
- take $D = B \cup C$ where $B \cup C$ means graph on $V(B) \cup V(C)$ in which we add no new edges

(B∪C defined alright in this case since edge relations on graphs are local) ✓

What is Flim(K) in this case?
By Fraïssé: must be the random graph (R, E):

- is cbl, ultrahomogeneous
- and $Age(R) = K$ since R universal ✓