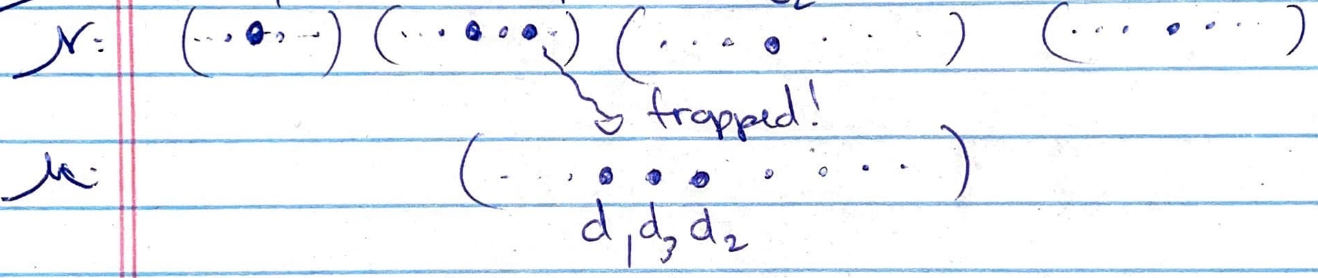
<u>problem</u> I can play el'ts of $L\times\mathbb{Z} = \mathcal{N}$ that are inf. far apart, but II must respond w/ el'ts in $\mathbb{Z}$ finitely far apart.
<u>Sol'n</u> since II knows length of game can play el'ts "sufficiently far apart" to avoid losing (i.e. getting non-<-preserving f)

<u>Problem</u>



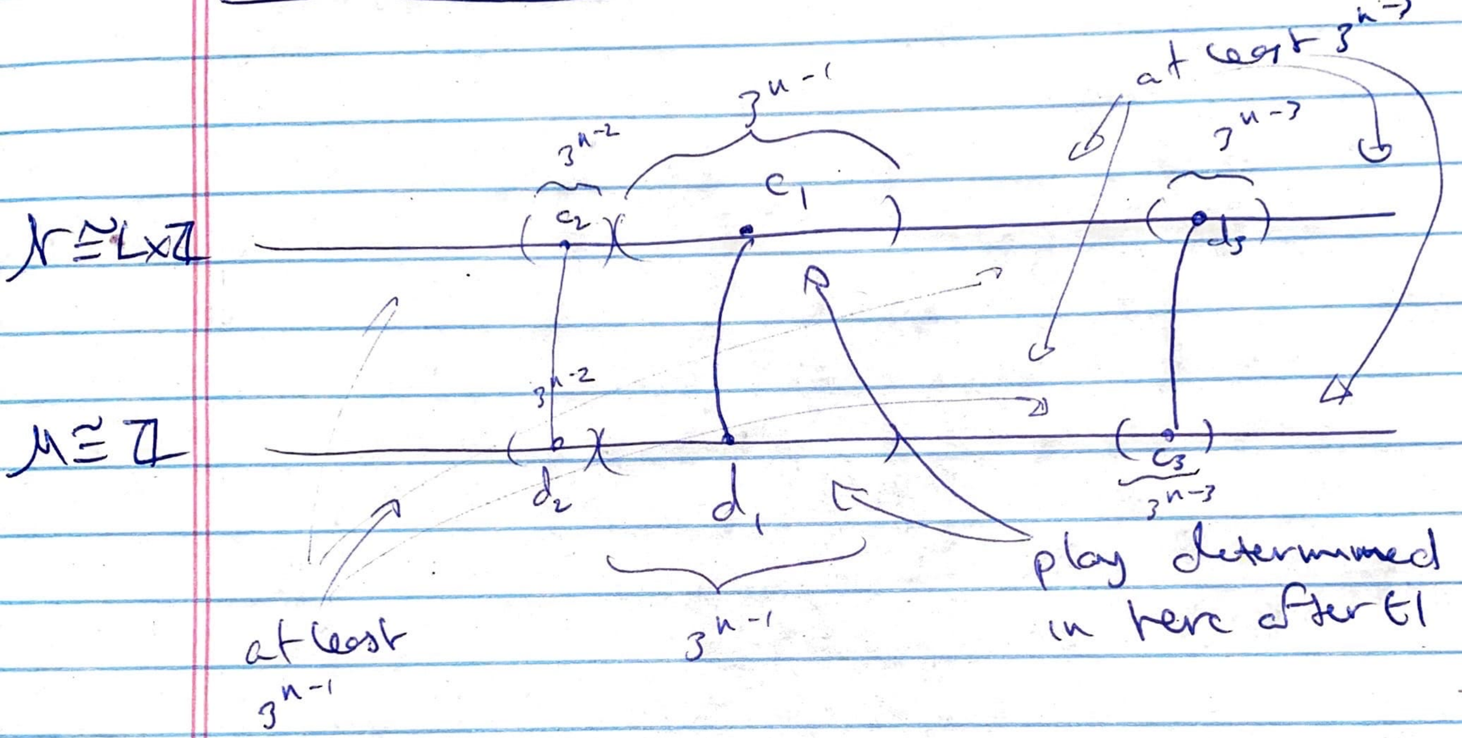
<u>Claim</u>: at every stage $m \leq n$ of $G_n(\mathcal{M},\mathcal{N})$ II can inductively ensure:

(*) if $a_1 < a_2 < \dots < a_m$ are el'ts played in $\mathcal{N} \cong L\times\mathbb{Z}$ and $b_1 < b_2 < \dots < b_m$ are el'ts played in $\mathcal{M} \cong \mathbb{Z}$, and $a_i \mapsto b_i$ is partial iso, then $\forall i \leq m$ we have:

- if $\operatorname{dist}(a_i, a_{i+1}) > 3^{n-m}$ then $\operatorname{dist}(b_i, b_{i+1}) > 3^{n-m}$
- if $\operatorname{dist}(a_i, a_{i+1}) \leq 3^{n-m}$ then $\operatorname{dist}(b_i, b_{i+1}) = \operatorname{dist}(a_i, a_{i+1})$

- Thm follows since then final map $a_i \mapsto b_i$ after n stages is partial iso and II wins

<u>Pf of claim</u> (by picture)



The random graph

- Sps $V = \{v_1, v_2, \ldots\}$ is a ctbl set of vertices
- imagine we construct a graph $G = (V, E)$ by going thru each pair $\{v_i, v_j\}$ of vertices and drawing an edge w/ probability $1/2$
- ofc every graph on V is a possible outcome of this process
- but can show: there is a graph R on V ("the random graph") s.t. almost surely (i.e. w/ prob 1) outcome is a G that is isomorphic to R.
- we'll approach R a different way...

Finite graphs:

- Fix a vertex set $\{v_1, \ldots, v_n\}$
- there are $2^{\binom{n}{2}} = 2^{n(n-1)/2}$ graphs on these vertices
- call set of such graphs $\mathcal{G}_n$
- for a sentence φ in lang of graphs $\mathcal{L} = \{E\}$, define

$$pr_n(\varphi) = \frac{|G \in \mathcal{G}_n : G \models \varphi|}{\underbrace{2^{n(n-1)/2}}_{|\mathcal{G}_n|}}$$

= prob. that φ is true in a graph of size n.

Ex's – if φ is "there are no edges"
then $pr_n(\varphi) = 1/2^{n(n-1)/2}$
which $\to 0$ as $n \to \infty$

all these φ expressible in $\mathcal{L} = \{E\}$

– if φ is "there is at least one edge"
then $pr_n(\varphi) = 1 - 1/2^{n(n-1)/2}$
which $\to 1$ as $n \to \infty$

– if φ is "there is a unique edge"
then $pr_n(\varphi) = \binom{n}{2} / 2^{\binom{n}{2}} = \frac{n(n-1)/2}{2^{n(n-1)/2}}$
$\to 0$ as $n \to \infty$

(notice: $pr_n(\varphi) \to 0$ or 1 for these φ)

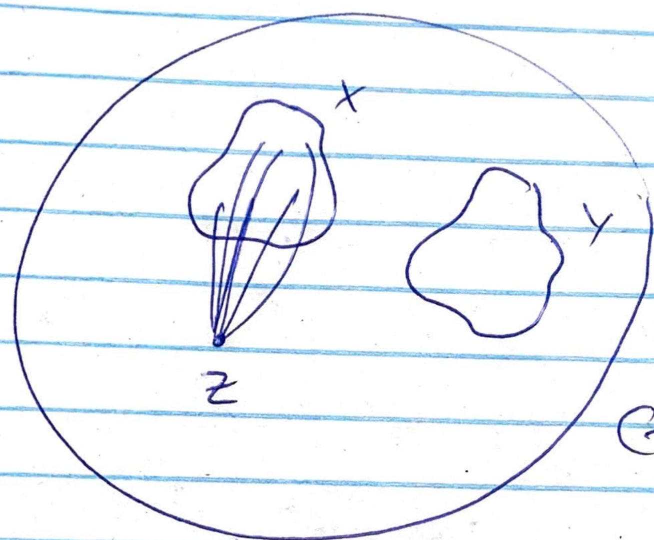
– For $m = 1, 2, 3, \dots$ let ρ_m be sentence "for every pair of disjoint subsets of m-many vertices X and Y, there is a z connected to every $x \in X$ and no $y \in Y$"

↳ we also add clause asserting $\exists$ at least $2m$ distinct vertices

i.e. ρ_m is:

$$\exists z_1 \ldots \exists z_{2m} \left(\bigwedge_{i<j\le 2m} z_i \neq z_j \right) \wedge$$

$$\forall x_1 \ldots \forall x_m \forall y_1 \ldots \forall y_m \Big[\Big(\bigwedge_{i<j\le m} x_i \neq x_j \wedge \bigwedge_{i\le j\le m} y_i \neq y_j \wedge \bigwedge_{i,j\le m} x_i \neq x_j \Big) \rightarrow \exists z \Big(\bigwedge_{i\le m} E(x_i, z) \wedge \bigwedge_{i\le m} \neg E(y_i, z) \Big) \Big]$$



By an elementary probability arg, one can show:

<u>Claim</u> For every fixed $m \in \mathbb{N}$, $pr_n(\varphi_m) \to 1$ as $n \to \infty$.

<u>Pf</u>: you try!

- we now describe random graph R
- int'n: if φ a sentence in lang of graphs, then $R \models \varphi$ iff $pr_n(\varphi) \to 1$
- in particular, φ_m's true in R

- In fact: ρ_m's characterize R.
- Let T be theory
  {axioms for graphs} $\cup \{\rho_1, \rho_2, \rho_3, \ldots\}$
- Say: T is "theory of the random graph"

<u>Theorem</u> T is satisfiable and $\aleph_0$-categorical (hence complete - since no finite models)

<u>Pf</u>. - we'll show sat'ability later, prove cat'g'c'y
- WTS if $G, H \models T$ and G, H ctbl then $G \cong H$
- sufficient to show (by prev. work): if $G, H \models T$ (any cardinality) then $G \cong_p H$.
- i.e. can find family of partial iso's w/ b-f prop.

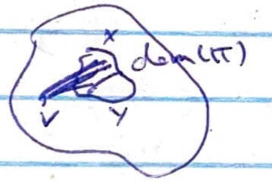
- observe: if $G \models T$ then for any pair X, Y of finite subsets w/ $X \cap Y = \emptyset$, there is $z \in G$ connected to every $x \in X$ and no $y \in Y$
- because: if $|X| = |Y| = m$ ~~[illegible]~~ get from $G \models \rho_m$, but if $|X| < |Y| = m$ say,

Can extend X to X' of size m + still disjoint from Y (G is infinite) then use pm.

- Fix $G, H \models T$
- Let I = family of finite partial iso's from G to H.

(note: $\pi : G \to H$ partial iso if $\forall v_1, v_2 \in \mathrm{dom}(\pi)$
$E(v_1, v_2) \Leftrightarrow E(\pi(v_1), \pi(v_2))$
i.e. π preserves adjacency and non-adjacency)

- we show I has b-f prop.
- Fix $\pi \in I$ and $v \in G$.
- Let $X = \{x \in \mathrm{dom}(\pi) : E(x,v)\}$
  $Y = \{y \in \mathrm{dom}(\pi) : \neg E(y,v)\}$



G

H

- Since $H \models T$ there is $w \in H$ connected to every vertex in $\pi[X]$ none in $\pi[Y]$
- $\pi' = \pi \cup \{(v,w)\}$ is a partial iso $\Rightarrow \pi' \in I$. And $v \in \mathrm{dom}(\pi')$ ✓

- symmetric if given $w \in H$ ✓

- Let R denote unique (up to isomorphism) ctbl model of T.

<u>Theorem</u> R is

(i) <u>universal</u>: i.e. for every ctbl G there is $G' \subseteq R$ s.t. $G \cong G'$

(ii) <u>ultrahomogeneous</u>: i.e. if $X, Y \subseteq R$ are finite isomorphic subgraphs, then any isomorphism $\pi: X \to Y$ can be extended to an automorphism $\tilde{\pi}: R \to R$.

Conversely, any ctbl graph w/ these properties is iso to R.

<u>Pf</u> (i) (sketch): Given G ctbl graph construct a chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ of ctbl graphs:

- to get G_1 enumerate all pairs $\{X_i, Y_i\}_{i \in \mathbb{N}}$ of disjoint finite subsets of $G = G_0$ (since G ctbl, only ctbly many such pairs)
- for each, add a vertex v_i to G_0, connect to each $x \in X_i$, none in Y_i
- Let G_1 be result
- repeat to get G_2, etc.
- then $R' = \bigcup G_i$ satisfies the P_n's

Hence $R' \models T$ Hence $R' \cong R$
Hence R has subgraph iso to G
(btw: this shows T is satisfiable!)

(ii) – prove something slightly more general.
– Sps $G, H \models T$, both ctbl (hence $\cong R$)
– follows from above: given $X \subseteq G$, $Y \subseteq H$ both finite and an iso $\pi : X \to Y$, can be extended to iso $\bar{\pi} : G \to H$
(follow pf that $M \cong_p N \Rightarrow M \cong N$ for M, N ctbl, starting from $\pi_0 = \pi$) ✓

– Conversely sps G, H both ctbl and satisfy (i), (ii)
– we show $G \cong H$ (follows $G \cong H \cong R$)
(since R also sats (ii (iii))
– suffices to show $G \cong_p H$
– Let I be all finite part'l iso's
we show has b-f prop.
– given $\pi \in I$, let $G_1 = dom(\pi)$ and $H_1 = ran(\pi) = \pi[G_1]$
– given $a \in G$, let $G_1' = G_1 \cup \{a\}$
– By universality, there is $K_1' \subseteq H$ s.t. $G_1' \cong K_1'$
– Fix $\tau : G_1' \to K_1'$ an isomorphism
– Let $K_1 = \tau[G_1]$ be subgraph of

K_1' iso to G_1, H_1

- let $c = \tau(a)$, so $K_1 = K_1' \setminus \{c\}$
- notice: $\tau \circ \pi^{-1} : H_1 \to K_1$ is an isomorphism, by ultrahomogeneity there is σ extending $\tau \circ \pi^{-1}$, $\sigma : H \to H$ automorphism
- let $b = \sigma^{-1}(c)$
- notice: $\sigma^{-1} \circ \tau [G_1'] = H_1'$
- but σ^{-1} on $\underset{\tau[G_1]}{\overset{}{K_1}}$ is $\pi \circ \tau^{-1}$

So $\sigma^{-1} \circ \tau \restriction G_1 = \pi \circ \tau^{-1} \circ \tau = \pi$

- thus $\sigma^{-1} \circ \tau$ extends π and sends a to b
- i.e. $\pi \cup \{(a,b)\}$ is a partial iso ✓