- Claim I has b-f prop
- (1) given $\pi \in I$ and $a \in M$
  - π corresponds to some run of game, say w/ n stages in which I plays $c_1, \ldots, c_n$
  - consider game in which I then plays $c_{n+1} = a$
  - let $b = \tau(c_1, \ldots, c_n, a) \in N$
  - since II wins this game (no matter how I plays from here) if they play by τ, $\pi' = \pi \cup \{(a,b)\}$ is a partial iso (in I) extending π w/ $a \in \mathrm{dom}(\pi')$

- Symmetric given $b \in N$. ✓

- Next we'll show that finite length versions of the game can be used to characterize elementary equivalence
- Write $G_n(M, N)$ for the game w/ only n rounds o/w played like $G_\omega(M,N)$
- i.e. II wins if $f = \{\langle c_i, d_i \rangle : i \leq n\} \cup \{(c^M, c^N) : c \in \mathcal{L}\}$ is a partial iso from M to N, o/w I wins
- Goal is to show:

<u>Thm</u> Sps $\mathcal{L}$ is a finite relational lang. Given $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ we have $\mathcal{M} \equiv \mathcal{N}$ iff II has a winning strat. in $G_n(\mathcal{M}, \mathcal{N})$ for all $n \in \mathbb{N}$.

- Before can prove, need to define <u>quantifier depth</u> of a fmla
- given $\mathcal{L}$-fmla φ, inductively define depth(φ) by:

$$\text{depth}(\varphi) = 0 \text{ if } \varphi \text{ is quantifier free}$$
$$\text{depth}(\neg\varphi) = \text{depth}(\varphi)$$
$$\text{depth}(\varphi \wedge \chi) = \max\{\text{depth}(\varphi), \text{depth}(\chi)\}$$
$$\text{depth}(\exists v \varphi) = \text{depth}(\varphi) + 1$$

e.g. $\text{depth}(c_1 < c_2) = 0$
$\text{depth}(\forall x \exists y (y > x) \wedge \exists z (z = z)) = 2$

- ~~[illegible]~~ Given $\mathcal{M}, \mathcal{N}$ write $\mathcal{M} \equiv_n \mathcal{N}$ if for all sentences φ w/ depth(φ) $\leq n$ we have $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$.
- So $\mathcal{M} \equiv \mathcal{N}$ iff $\mathcal{M} \equiv_n \mathcal{N}$ for all n
- So thm will be proved if we show:

<u>Claim</u> II has a winning strat in $G_n(\mathcal{M}, \mathcal{N})$ iff $\mathcal{M} \equiv_n \mathcal{N}$.

- need first:

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<u>Lemma</u> fix $n, \ell \in \mathbb{N}$. There is a <u>finite</u> list of fmlas $\varphi_1(\vec{x}), \ldots, \varphi_k(\vec{x})$ in the vars $\vec{x} = (x_1, \ldots, x_\ell)$ s.t. every fmla $\varphi(\vec{x})$ of depth $\leq n$ is equivalent to one of them.

~~(...)~~ (i.e. can find $i \leq k$ s.t. for any $\mathcal{M}$ and $\vec{c} \in \mathcal{M}$, $\mathcal{M} \models \varphi(\vec{c})$ iff $\mathcal{M} \models \varphi_i(\vec{c})$)

<u>Pf</u> - induction on the depth n
- if $n = 0$, $\varphi(\vec{x})$ is quant'r free, i.e. a boolean combo (i.e. combo using $\wedge, \vee, \neg$) of atomic fmlas
- only finitely many c's/R's + no f's in $\mathcal{L}$ $\Rightarrow$ only finitely many terms.
  $\Rightarrow$ only finitely many <u>atomic</u> fmlas $\tau_1(\vec{x}), \ldots, \tau_j(\vec{x})$ in vars $\vec{x}$
- literally speaking there are inf. many boolean combos of the τ_i
  ... but only finitely many are inequivalent! (think about it.)

(one way to make precise: "disjunctive normal form" - see pf in book for more details)

- done for $n = 0$ ✓

- assume claim true for fmlas of depth $\leq n$, for every l.
- fmlas of ~~[scribbled out]~~ depth $n+1$ look like $\bigwedge \exists x_{l+1}\ \varphi(\vec{x}, x_{l+1})$ where depth$(\varphi) \leq n$ — boolean combos of fmlas of form
- by ind. hyp. only finitely many inequiv fmlas of depth n so similar reasoning $\Rightarrow$ only finitely many inequiv of depth $n+1$ ✓

We prove claim

Pf: - induction on n (length of game)
- if $n=0$, $M \equiv_0 N$ iff $\{(c^M, c^N) : c \in \alpha\}$ is a partial iso
  iff II wins empty game $G_0(M, N)$ ✓
- sps claim true up to $n > 0$
- consider n:
($\Rightarrow$) sps $M \equiv_n N$
- must find winning strat for II in G_n
- will be: "figure out turn 1, then follow my strat for G_{n-1}"
- for turn 1: sps I plays $a \in M$ (if plays $b \in N$, symmetric arg)
- we find $b \in N$ s.t.
  $M \models \varphi(a)$ iff $N \models \varphi(b)$

for any fmla φ of depth $< n$

- sps $\varphi_0(v), \ldots, \varphi_m(v)$ list all fmlas of depth at most $n-1$ in variable v, up to equivalence.
- let $X = \{i \leq m : \mathcal{M} \models \varphi_i(a)\}$
- let $\Phi(v)$ be fmla
$$\bigwedge_{i \in X} \varphi_i(v) \wedge \bigwedge_{i \notin X} \neg \varphi_i(v)$$
- then depth $(\exists v\, \Phi(v)) = n$ and since $\mathcal{M} \models \Phi(a)$ have $\mathcal{M} \models \exists v\, \Phi(v)$
- since $\mathcal{M} \equiv_n \mathcal{N}$ we have $\mathcal{N} \models \exists v\, \Phi(v)$ so there is $b \in N$ s.t. $\mathcal{N} \models \Phi(b)$
- II plays b on turn 1.

- Consider new lang $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where c is a new constant.
- Write $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ for the $\mathcal{L}^*$ expansions of $\mathcal{M}, \mathcal{N}$ where we interpret $c^{\mathcal{M}}$ as a and $c^{\mathcal{N}}$ as b, resp.
- by above, for any $\varphi(v)$ of depth $\leq n-1$ have $\mathcal{M} \models \varphi(a)$ iff $\mathcal{N} \models \varphi(b)$
- follows (since $\mathcal{M} \equiv_{n-1} \mathcal{N}$ also) that $(\mathcal{M}, a) \equiv_{n-1} (\mathcal{N}, b)$
- by induction, II has a winning strat in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))$
- for remaining $n-1$ turns of game $G_n(\mathcal{M}, \mathcal{N})$

II plays according to this strat.
- can check: if $f:(\mathcal{M},a) \to (\mathcal{N},b)$ is partial iso from G_{n-1} game,
- then $f^*: \mathcal{M} \to \mathcal{N}$ defined by $f^* = f \cup \{(a,b)\}$ is partial iso $\Rightarrow$ II wins $G_n(\mathcal{M},\mathcal{N})$ ✓

($\Leftarrow$) Sps $\mathcal{M} \not\equiv_n \mathcal{N}$
- there is sentence χ of depth $\leq n$ s.t. $\mathcal{M} \models \chi$ and $\mathcal{N} \not\models \chi$
- χ a bool'n combo of fmlas $\exists v\,\varphi(v)$ w/ depth$(\varphi) \leq n-1$

$\Rightarrow$ must have fmla of this form about which $\mathcal{M},\mathcal{N}$ disagree (o/w would agree on bool'n combos too)
- wlog say $\mathcal{M} \models \exists v\,\varphi(v)$ and $\mathcal{N} \models \neg\exists v\,\varphi(v)$ i.e. $\mathcal{N} \models \forall v \neg\varphi(v)$
- describe a winning strat for I in $G_n(\mathcal{M},\mathcal{N})$
  - on turn 1, play $a \in M$ s.t. $\mathcal{M} \models \varphi(a)$ (exists by hypothesis)
  - if II plays b, let $(\mathcal{M},a)$, $(\mathcal{N},b)$ be defined as above
  - then since $\mathcal{N} \not\models \varphi(b)$ have $(\mathcal{M},a) \not\equiv_{n-1} (\mathcal{N},b)$
  - by induction, I has winning strat in $G_{n-1}((\mathcal{M},a),(\mathcal{N},a))$

check: if I plays according to this strat for remaining $n-1$ turns, wins G_n ✓

Ex – We use theorem to find ax'm'n for $Th(\mathbb{Z}, <)$

– Let $L = \{<\}$. Let T be axioms for linear order + the axiom

$$\forall x \exists y \exists z \left(y < x < z \wedge \neg \exists w \left(y < w < x \vee x < w < z \right) \right)$$

"every x has unique predecessor and successor"

– what does a model $\mathcal{N} \models T$ look like?

– given $x \in N$, is contained in a copy of $\mathbb{Z}$:

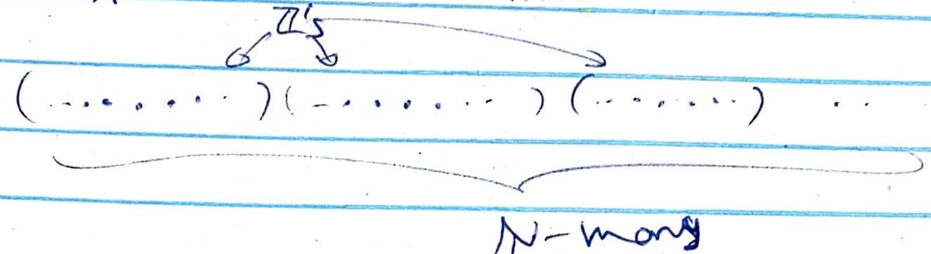
$$\mathcal{N}: \quad \underbrace{(\cdots \cdot \cdot \underset{x}{\cdot} \cdot \cdot \cdot \cdots)}_{\cong \mathbb{Z}}$$

– if there is $y \notin$ x's copy of $\mathbb{Z}$, contained in its own copy

$$\mathcal{N}: \quad (\cdots \cdot \underset{x}{\cdot} \cdot \cdot \cdots) \quad (\cdots \cdot \underset{y}{\cdot} \cdot \cdot \cdots)$$

– copies of $\mathbb{Z}$ are ordered wrt one another.

- if "L-many" copies (L a linear order) then $\mathcal{N}$ isomorphic to $(L\times\mathbb{Z}, <_{lex})$
- e.g. $\mathbb{N}\times\mathbb{Z}$ looks like:



- Conversely, if $\mathcal{N} \cong (L\times\mathbb{Z}, <_{lex})$ for some L, then $\mathcal{N} \models T$.

- <u>Goal</u>: prove T is complete
- can check: T <u>not</u> κ-cat'g'l for <u>any</u> infinite κ
- so need new technique to prove completeness
will use theorem.

<u>Thm</u> T axiomatizes $Th(\mathbb{Z},<)$
(follows: T is complete)

<u>Pf</u>: - let $\mathcal{M} = (\mathbb{Z},<)$ and fix $\mathcal{N} \models T$
- then $\mathcal{N} \cong (L\times\mathbb{Z}, <_{lex})$ for some L.
- will show $\mathcal{M} \equiv \mathcal{N}$ by finding winning strat for II in $G_n(\mathcal{M},\mathcal{N})$ for every $n\in\mathbb{N}$.

↳ i.e. II can ensure outcome of G_n is partial iso $f: \mathbb{Z} \to L\times\mathbb{Z}$

must preserve $<$.