More completeness

<u>Def'n</u> Sps κ is an infinite cardinal and T is a theory w/ at least one model of size κ. Say T is <u>κ-categorical</u> if any two models of T of size κ are isomorphic.

<u>Ex's</u> ① last time showed DLO* is $\aleph_0$-categorical, i.e. if $\mathcal{M}, \mathcal{N}$ ctbl models of DLO* then $\mathcal{M} \cong \mathcal{N}$
② if $T = Th(\mathbb{N}, +, \cdot, \leq, 0, 1)$ then T is <u>not</u> $\aleph_0$-cat'g'l: if $\mathcal{M} \models T$ is ctbl and has an infinite element, then $\mathcal{M} \not\cong (\mathbb{N}, +, \cdot, \leq, 0, 1)$ (why?)

One reason to care about categoricity:

<u>Theorem (Vaught's test)</u> Sps T is a satisfiable $\mathcal{L}$-theory w/o finite models. If T is κ-cat'g'l for some $\kappa \geq |\mathcal{L}|$ then T is complete.

<u>Pf</u>: - Sps T isn't complete
- so there is φ s.t. $T \not\models \varphi$ and $T \not\models \neg\varphi$
- i.e. $T \cup \{\varphi\} =: T_0$ and $T \cup \{\neg\varphi\} =: T_1$ both have models.

- Since no finite models of T, there are infinite models $\mathcal{M}_0 \models T_0$ and $\mathcal{M}_1 \models T_1$
- Since $\kappa \geq |\mathcal{L}|$, by upward LS can find elem extensions $\mathcal{M}_0' > \mathcal{M}_0$ and $\mathcal{M}_1' > \mathcal{M}_1$ of size κ
  - $\hookrightarrow$ in particular $\mathcal{M}_0' \models T_0$, $\mathcal{M}_1' \models T_1$
- but then $\mathcal{M}_0' \models \varphi$ and $\mathcal{M}_1' \models \neg\varphi$
- hence $\mathcal{M}_0' \not\cong \mathcal{M}_1'$, contradicting κ-categoricity of T. ✓
  $\Rightarrow T$ is complete. ✓

<u>Ex's</u> ① DLO* is $\aleph_0$-categorical and hence complete (we already showed this last time using b-f prop)

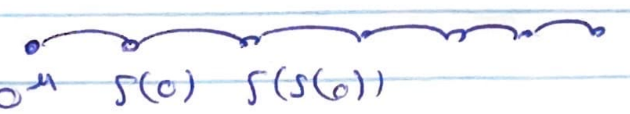
What about theories that are κ-cat'g'e for some unctbl κ?

② - Consider $\mathcal{N} = (\mathbb{N}, S, 0)$ where S is the successor function: $S(n) = n+1$.
- we find an axiom'tn of $Th(\mathcal{N})$
- let T be the axioms
  1. $\forall x\ (S(x) \neq 0)$
  2. $\forall x \forall y\ (x \neq y \Rightarrow S(x) \neq S(y))$
  3. $\forall y\ (y \neq 0 \Rightarrow \exists x (S(x) = y))$
  4n. (for a fixed $n \in \mathbb{N}$):

  $\forall x\ (\underbrace{S(S(\cdots(S(x))\cdots))}_{n\text{-times}} \neq x)$

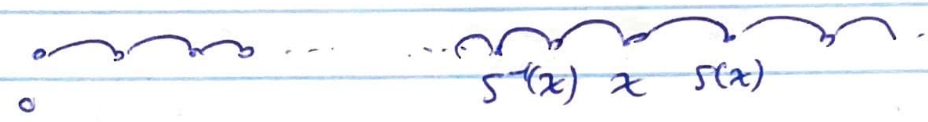
- Clearly $\mathcal{N} \models T$
- What do models $\mathcal{M} \models T$ look like in general?
  - every $\mathcal{M} \models T$ has a copy of $\mathbb{N}$:

$\mathcal{M}$:



  - every $x \in M$ not in $\mathcal{M}$'s copy of $\mathbb{N}$, contained in a copy of $\mathbb{Z}$.

$\mathcal{M}$:



  - no order structure! These $\mathbb{Z}$'s are just "floating around" $\mathbb{N}$:

$\Rightarrow T$ not $\aleph_0$-categ'l : eg $\mathbb{N}$
$\mathbb{N}$ + one $\mathbb{Z}$
$\mathbb{N}$ + two $\mathbb{Z}$'s
$\vdots$

pairwise non-iso ctbl models of T.

but, if $\mathcal{M}, \mathcal{N} \models T$ and $|M| = |N| = \kappa$ for some $\kappa > \aleph_0$ (e.g. $\kappa = \aleph_1$)
then $\mathcal{M} \cong \mathcal{N}$!

Idea is: both $\mathcal{M}$ and $\mathcal{N}$ look like $\mathbb{N}$ + κ-many copies of $\mathbb{Z}$

hence $\mathcal{M} \cong \mathcal{N}$ (think about how to formalize!)

$\Rightarrow$ T is κ-cat'g'l for all unctbl κ hence complete by Vaught (no finite models of T)

$\Rightarrow$ T axiomatizes $Th(\mathbb{N}, S, 0)$!

(3) ACF_p := axioms for alg. closed fields of characteristic p (e.g. $\mathbb{F}_p^{alg}$)

ACF_0 := " " 0 (e.g. $\mathbb{C}$)

<u>Prop'n</u> ACF_p (for p a prime or 0) is κ-categ'l for all $\kappa > \aleph_0$.

<u>Pf (sketch)</u> - Two a.c.f's are iso iff they have same characteristic and transcendence degree (algebra fact)

- if $\mathcal{M}$ is a.c.f. of size $\kappa > \aleph_0$ then its trans deg is κ

$\Rightarrow$ if $\mathcal{M}, \mathcal{N}$ have same char p and $|M| = |N| = \kappa$ then $\mathcal{M} \cong \mathcal{N}$ ✓

<u>Corollary</u> ACF_p is complete:

Pf: by Vaught — no finite a.c.f.'s

- In particular ACF_0 is an axiomatization of $Th(\mathbb{C}, +, \cdot, 0, 1)$

<u>Prop'n</u> Sps φ is a sentence in lang of rings. TFAE

1. φ is true in $\mathbb{C}$
2. φ is true in every a.c.f. of char. 0
3. φ is true in <u>some</u> a.c.f. of char. 0
4. There is m s.t. for all primes $p > m$ φ is true in all a.c.f.'s of char p.

<u>Pf</u> - ①②③ are equiv since ACF_0 complete.
- you showed ② ⇒ ④ on HW3
- for ④ ⇒ ②: Sps ④, but ② fails
- then actually $ACF_0 \models \neg\varphi$, by completeness

<u>Lemma</u> Sps T a theory. If $T \models \psi$ then there is a finite $\Delta \subseteq T$ s.t. $\Delta \models \psi$.

Pf. If no such Δ, then by compactness there is a model of $T \cup \{\neg\psi\}$ a contradiction (spell this out!)

So: by Lemma there is a finite $\Delta \subseteq ACF_0$ s.t. $\Delta \models \neg\varphi$
- Δ contains only finitely many of the char. 0 axioms $1+1+\ldots+1 \neq 0$

$\Rightarrow$ for all large enough p there is a.c.f. $\mathbb{F}$ of char p s.t. $\mathbb{F} \models \Delta$ (and hence $\mathbb{F} \models \neg\varphi$), contradiction ✓

Prop'n can be used to prove:

<u>Theorem</u> (Ax-Grothendieck) if $f: \mathbb{C}^n \to \mathbb{C}^n$ is an injective polynomial map, then it is surjective

<u>Pf</u>: see book. Uses prop'n above + fact that if $f: k \to k$ is an injective map on a finite field, must be surjective ✓

<u>Ehrenfeucht - Fraïssé games</u>

- we explore back-and-forth constructions using 2-player games.

- Sps $\mathcal{L}$ is a "relational language" i.e. no function symbols (constants allowed)
- Sps $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structures w/ $M \cap N = \emptyset$.
- <u>recall</u>: a partial iso $\pi: \mathcal{M} \to \mathcal{N}$ is an injective map w/ $\mathrm{dom}(\pi) \subseteq M$ that preserves R's and c's.

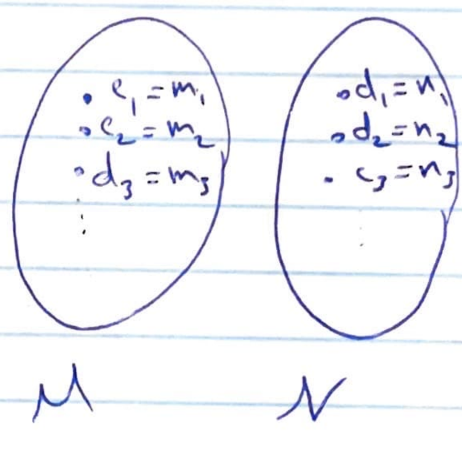
- we define a <u>two-player game</u> $G_\omega(\mathcal{M}, \mathcal{N})$

  ω-many stages

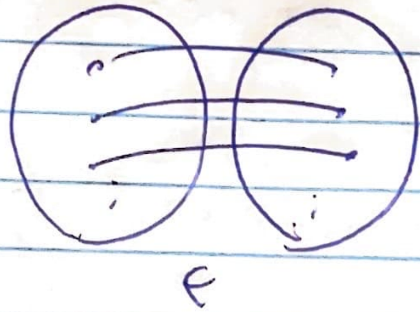
- the players are I and II
- at stage i, I either plays some $m_i \in M$
  in which case II responds w/ $n_i \in N$
  or I plays some $n_i \in N$
  then II plays some $m_i \in M$

- let c_i be I's play at stage i, d_i be II's play. Game looks like:

I	II
c_1	
	d_1
c_2	
	d_2
⋮	



- players can repeat elements but usually we imagine they don't.
- let $\langle c_i, d_i \rangle$ denote pair (c_i, d_i) if $c_i \in M$, $d_i \in N$, or pair (d_i, c_i) if $c_i \in N$, $d_i \in M$.
- say II <u>wins</u> if $f = \{\langle c_i, d_i \rangle : n \in \omega\}$ is a partial iso from $\mathcal{M}$ to $\mathcal{N}$.
- o/w I wins.



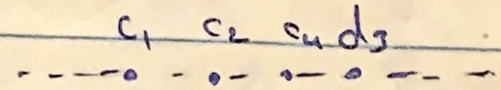
- a strategy for II is a function τ s.t. if $c_1, \dots, c_n$ a sequence of moves from I, then $\tau(c_1, \dots, c_n)$ is a legal move for II at stage n (i.e. if $c_n \in M$ $\tau(c_1, \dots, c_n) \in N$ + vice versa)

- a winning strategy for II is a τ s.t. no matter how I plays, II wins if they respond according to τ.
- winning strat for I defined sim'ly.

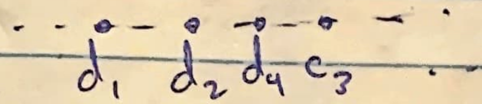
Ex's ① if $\mathcal{L} = \emptyset$ and $\mathcal{M}, \mathcal{N}$ are infinite $\mathcal{L}$-structures (i.e. infinite sets) then II has a winning strat: "make sure you end up w/ an injection"
(always possible since $\mathcal{M}, \mathcal{N}$ infinite)

OTOH: if $\mathcal{M}$ inf, $\mathcal{N}$ finite, I has winning strat: "play at least $|N|+1$ distinct elements of M"

② if $\mathcal{L} = \{<\}$ and $\mathcal{M}, \mathcal{N} \models DLO^*$ then II has a winning strat: "don't lose"



Pic:



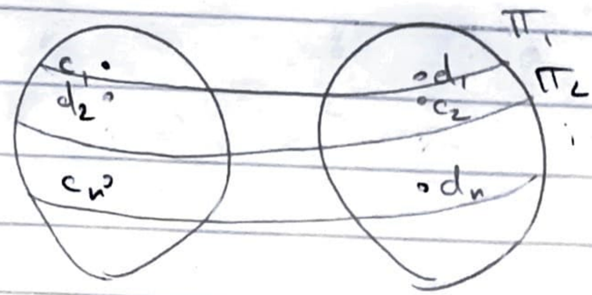
CTGH. if $\mathcal{M}, \mathcal{N}$ are LO's and $\mathcal{M}$ has min point but $\mathcal{N}$ doesn't, I has a winning strat
(what is it?)

Prop'n II has a winning strat in $G_\omega(\mathcal{M}, \mathcal{N})$ iff $\mathcal{M} \equiv_p \mathcal{N}$.

Pf: ($\Leftarrow$) - Sps we have family I of part. iso's from $\mathcal{M}$ to $\mathcal{N}$ w/ b-f prop.
- we describe strat for II.

Stage 1: if I plays $c_1 \in M$ find $\pi_1 \in I$ w/ $c_1 \in dom(\pi_1)$ and play $d_1 = \pi_1(c_1)$
(symmetric if $c_1 \in N$)

Stage n+1: sps we have $\pi_1 \subseteq \pi_2 \subseteq \dots \subseteq \pi_n$ s.t. π_k contains $\{\langle c_i, d_i \rangle : i \leq k\}$ $\forall k \leq n$



If I play $c_{n+1} \in M$ find $\pi_{n+1} \supseteq \pi_n$ in $\mathcal{I}$ w/ $c_{n+1} \in dom(\pi_{n+1})$ and play $d_{n+1} = \pi_{n+1}(c_{n+1})$ (sym. if $c_{n+1} \in N$)

Then $\pi = \bigcup \pi_n$ clearly a partial iso since it contains $f = \{\langle c_i, d_i \rangle : i \in \omega\}$
f is a partial iso too ✓
i.e. II wins ✓

($\Rightarrow$) – Sps II has winning strat τ.
we find $\mathcal{I}$ w/ b-f prop.
– say $\pi \in \mathcal{I}$ iff $\pi = \{\langle c_i, d_i \rangle : i \leq n\}$ for some partial run of the game up to stage n in which II plays according to τ. (i.e. $d_i = \tau(c_1, \dots, c_i)$ $\forall i \leq n$)

I	II	
c_1		
	$\tau(c_1) = d_1$	
c_2		π
	$\tau(c_1, c_2) = d_2$	
$\vdots$		
	$\tau(c_1, \dots, c_n) = d_n$	
c_n		