Ma 116a Homework #8

Due Thursday, November 30 at 1:00pm

This homework is an exploration of Exercise 4.5.2 in Marker and some related results.

1) Let $\mathcal{L} = \{s\}$ be a language with a single unary function symbol. Consider the $\mathcal{L}$-theory T^* consisting of the following axioms:

   i) $\forall x \forall y (x \neq y \Rightarrow s(x) \neq s(y))$

   ii) $\forall x \exists y (s(y) = x)$

   iii.n) $\forall x (s^n(x) \neq x)$

   where $s^n(x)$ is shorthand for $s(s(\ldots(s(x))\ldots))$ applied n-times, and we include iii.n in T for every $n \in \{1, 2, \ldots\}$.

   Show that T^* is κ-categorical for any uncountable cardinal κ, and hence complete by Vaught's test. Conclude that T^* axiomatizes $\mathrm{Th}(\mathbb{Z}, s)$, where $s(x) = x + 1$ is the successor function on $\mathbb{Z}$.

   Hint: see the class notes on completeness and categoricity for a very similar example.

2) (Exercise 4.5.2 in Marker) Let $T = \mathrm{Th}(\mathbb{Z}, s)$. For each fixed $n \in \mathbb{N}$, determine which types $p \in S_n(T)$ are isolated. That is, for each isolated $p \in S_n(T)$, describe a formula ϕ_p isolating p, and for each non-isolated p, justify that p is non-isolated by showing that for every formula $\phi \in p$, there is $q \in S_n(T)$ such that $q \neq p$ but $\phi \in q$.

   Note 1: $S_n(T)$ denotes the space of complete n-types over a theory T. (A collection of formulas p in the free variables $\bar{v} = (v_1, \ldots, v_n)$ is an *n-type over* T if $p \cup T$ is satisfiable; p is *complete* if for every $\phi(\bar{v})$ in the variables $v_1, \ldots, v_n$, either $\phi \in p$ or $\neg\phi \in p$.)

   Said another way: $p \in S_n(T)$ iff there is a model $\mathcal{M} \models T$ and $\bar{a} \in M^n$ such that $\bar{a}$ realizes p (i.e. $\mathrm{tp}^{\mathcal{M}}(\bar{a}/\emptyset) = p$).

   Note 2: By problem (1) we have $S_n(T) = S_n(T^*)$. Then, by the previous note, we have $p \in S_n(T)$ iff there is a model of T^* realizing p.

3) Let $T = \mathrm{Th}(\mathbb{Z}, s)$.

   Show that there is a countable model $\mathcal{M} \models T$ such that the only types that are realized in $\mathcal{M}$ are isolated. (Such a model is called *prime*.)

   Show that there is a countable model $\mathcal{N} \models T$ in which *every* type in $\bigcup_n S_n(T)$ is realized. (Such a model is called *saturated*.)