Ma 116a Homework #6
Due Tuesday, November 14th at 1:00pm

1) A *metric space* is a set M together with a function (called a *metric*) $d : M^2 \to \mathbb{R}$ such that $d(x, y) = d(y, x) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$, and $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z$. An *isometric embedding* of a metric space $\langle M_1, d_1 \rangle$ into a metric space $\langle M_2, d_2 \rangle$ is a one-to-one map $f : M_1 \to M_2$ such that $d_1(x, y) = d_2(f(x), f(y)), \forall x, y$. If f is also onto, then we say that f is an *isometry* of M_1 to M_2.

We can view metric spaces as structures in a first order language that has infinitely many binary relation symbols $R_q, q \in \mathbb{Q}^+$, indexed by the positive rationals. The metric space $\langle M, d \rangle$ is identified with the structure $\mathcal{M} = \langle M, \{R_q^{\mathcal{M}} : q \in \mathbb{Q}^+\} \rangle$, where $R_q^{\mathcal{M}}(x, y) \Leftrightarrow q < d(x, y), \forall x, y$. Under this identification, embedding of structures corresponds to isometric embedding and isomorphism to isometry.

Show that the class of finite metric spaces (when viewed as structures as above) with rational metric, i.e., with metric taking only rational values, satisfies all the properties required by Fraïssé's Theorem: it is countable (i.e., contains only countably many structures, up to isomorphism), and has HP (the hereditary property), JEP (the joint embedding property) and AP (the amalgamation property).

Note: The Fraïssé limit of this class is called the *rational Urysohn space* and it is the unique countable, ultrahomogenous and universal metric space with rational distances. Its completion is called the *Urysohn space* and it is the unique ultrahomogeneous, universal, complete separable metric space. It can be also viewed as the "random" complete separable metric space, and thus it is the analog in the category of such metric spaces of the random graph. It is the object of much current study in metric geometry.

2) Consider the class $\mathcal{E}$ of all finite equivalence relations, i.e, structures of the form $\langle X, E \rangle$, where X is a finite set and E is an equivalence relation. Show that $\mathcal{E}$ is countable (up to isomorphism) and satisfies, HP, JEP and AP. Identify its Fraïssé limit.