Ma 116a Homework #4

Due Tuesday, October 31st at 1:00pm

All numbered exercises are from Marker.

1) 2.5.9, 2.5.15 (a)

2) Read the definition of an "existentially complete model" of a theory T from Marker's book, Exercise 2.5.17. Let $\mathcal{L}$ be a countable language and T a theory in $\mathcal{L}$, which is $\forall\exists$-axiomatizable (see Exercise 2.5.15). Then for any countable model $\mathcal{M}$ of T, there is a countable model $\mathcal{N}$ of T such that $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{N}$ is existentially complete. (**Remark.** Exercise 2.5.17 (a) has a more general statement – note that in this exercise it should say that $\mathcal{N}$ is a model of T.)

3) Let $\mathcal{M} = \langle M, < \rangle$, where M is the set of real numbers whose decimal expansion has an even number of 7's and $<$ is the usual ordering of the reals. Let $\mathcal{N} = \langle \mathbb{Q} \times \mathbb{Q}, <_{\text{lex}} \rangle$, where $<_{\text{lex}}$ is the lexicographical ordering of the set of pairs of rationals. Show that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent.

4) Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are two structures for a language $\mathcal{L}$. Assume that there is a set of partial isomorphisms I between $\mathcal{M}$ and $\mathcal{N}$ that satisfies the back-and-forth property, and the following additional condition: for any $a_1, \ldots, a_n \in M$, there is $\pi \in I$ such that $\pi(a_i) = a_i, \forall i$. Show then that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$. Use this to prove that $\langle \mathbb{Q}, < \rangle$ is an elementary substructure of $\langle \mathbb{R}, < \rangle$.

5) In class we showed that the following is a set of axioms for $\text{Th}(\langle \mathbb{N}, S, 0 \rangle)$, where S is the successor function:

   - $\forall x(S(x) \neq 0)$
   - $\forall x \forall y(x \neq y \Rightarrow S(x) \neq S(y))$
   - $\forall y(y \neq 0 \Rightarrow \exists x(S(x) = y))$
   - $\forall x(S^n(x) \neq x)$, $n = 1, 2, \ldots$, where S^n means n applications of the function symbol S.

   Show that $\text{Th}(\langle \mathbb{N}, S, 0 \rangle)$ is not finitely axiomatizable, i.e., there is no finite set of axioms for this theory.