BERNOULLI DISJOINTNESS (AFTER BERNSHTEYN)

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Fix an infinite (not necessarily countable) discrete group $G$. A $G$-flow is a nonempty compact Hausdorff space $X$ equipped with a continuous action of $G$. A very important $G$-flow is the Bernoulli shift $n^G$, where $n$ is finite.

A subflow of a $G$-flow $X$ is a non-empty closed $G$-invariant subset of $X$. Given two $G$-flows $X$ and $Y$, a joining of $X$ and $Y$ is a subflow of $X \times Y$ which projects onto $X$ and $Y$. We say that $X$ and $Y$ are disjoint, denoted $X \perp Y$, if the only joining of $X$ and $Y$ is the trivial joining $X \times Y$. This is equivalent to saying that if $Z$ is a $G$-flow which has $X$ and $Y$ as factors, then these factor maps factor through $X \times Y$.

A $G$-flow $X$ is minimal if every orbit of $X$ is dense. If a $G$-flow $X$ is disjoint from $n^G$, then it is easy to show that $X$ must be minimal. It was shown in [GTWZ] that this is the only restriction:

**Theorem 1** (Glasner-Tsankov-Weiss-Zucker). $X \perp n^G$ for any minimal $G$-flow $X$.

This property is called **Bernoulli disjointness** for obvious reasons.

1. **Proof of Bernoulli disjointness**

The original proof of Theorem 1 involved casework depending on various properties of $G$, and using many difficult results as a blackbox. Recently Anton Bernshteyn found a nicer proof of this result using the Lovász Local Lemma, which works uniformly for all groups $G$ (see [Ber]). His proof is as follows:

*Proof of Theorem 1.* Let $Z \subset X \times n^G$ be a joining. To show that $Z = X \times n^G$, it suffices to show that $Z$ is dense. So fix nonempty open sets $U \subset X$ and $V \subset n^G$. We need to show that $Z \cap (U \times V)$ is nonempty.

We claim that it suffices to find a subset $F \subset G$ satisfying the following two conditions:

1. $\bigcap_{f \in F} f \cdot U$ meets every orbit (in $X$).
2. $F \cdot V$ contains an orbit (in $n^G$).

To see this, suppose that the orbit $G \cdot y$ is contained in $F \cdot V$. Then there is some $x \in X$ with $(x, y) \in Z$. Now there is some $g \in G$ with $g \cdot x \in \bigcap_{f \in F} f \cdot U$. Since $g \cdot y \in F \cdot V$, there is some $f \in F$ with $g \cdot y \in f \cdot V$, so since $g \cdot x \in f \cdot U$ as well, we have $g \cdot (x, y) \in f \cdot (U \times V)$. Thus $f^{-1}g \cdot (x, y) \in U \times V$, and this is also in $Z$ since $Z$ is $G$-invariant.

We first find a family of subsets satisfying Condition 1. Fix any point $x_0 \in X$, and let $S = \{g \in G : x_0 \in g \cdot U\}$. Note that any finite subset $F$ of $S$ satisfies Condition 1, since the intersection is a nonempty open set (since it contains $x_0$) and thus meets every orbit by minimality of $X$. We claim that $S$ is infinite. Indeed, let $T \subset G$ be a finite subset such that $X = T \cdot U$ (this exists by minimality and compactness, since minimality implies $X = G \cdot U$). Then for every $g \in G$, we have $X = gT \cdot U$, and thus there is some $t \in T$ with $x_0 \in gt \cdot U$,
and thus \( gt \in S \), i.e. \( g \in t^{-1}S \). Thus \( G = T^{-1}S \), so since \( T \) is finite, \( S \) must be infinite (in fact left-syndetic).

So \( S \) has arbitrarily large finite subsets, and thus it suffices to show that a sufficiently large subset of \( S \) satisfies Condition 2. We will show the following stronger fact, which is interesting in its own right:

**Theorem 2** (Bernshteyn). Let \( V \) be a non-empty open subset of \( n^G \). Then for every sufficiently large finite subset \( F \subset G \), the set \( F \cdot V \) contains an orbit.

**Proof of Theorem 2.** Let \( F \subset G \) be a finite subset. Without loss of generality, we can shrink \( V \) so that \( V = V_{\phi} \), where \( V_{\phi} \) is the basic open neighbourhood defined by a finite partial function \( \phi : G \to 2 \), say with \( \text{dom} \phi = D \).

We claim that we can assume without loss of generality that \( F \) is \( D \)-separated, i.e. such that for any \( f, f' \in F \), we have \( fD \cap f'D = \emptyset \) (i.e. the \( D \)-balls in the Cayley graph are disjoint). To see that this, note that we can recursively construct a \( D \)-separated subset of \( F \) of size \( \geq |F|/2 \) as follows: pick any \( f_0 \in F \), then pick any \( f_1 \in F \setminus (D^{-1}Df_0) \), then pick any \( f_2 \in F \setminus (D^{-1}D\{f_0, f_1\}) \), and so on (each step removes at most \( |D| \) elements from \( F \)).

Now for \( F \cdot V_{\phi} \) to contain an orbit is equivalent to saying that the following intersection is nonempty:

\[
\bigcap_{g \in G} gF \cdot V_{\phi}.
\]

By compactness, it suffices to show that each finite intersection is nonempty.

Endow \( n^G \) with the product of the uniform probability measures. We recall the Lovász Local Lemma:

**Theorem 3** (Lovász Local Lemma). Let \( \mathcal{A} \) be a set of events in a probability space, each with probability \( \leq p \), such that for \( A \in \mathcal{A} \), there is a subset \( B \subset A \) with \( |A \setminus B| \leq d \) such that \( A \) is independent from \( B \). If

\[
4p(d + 1) < 1,
\]

then for any \( A_0, \ldots, A_k \in \mathcal{A} \), we have \( \mathbb{P}[\bar{A}_0 \cdots \bar{A}_k] > 0 \).

We verify the hypotheses of the Lovász Local Lemma for \( \mathcal{A} = \{ \neg(gF \cdot V_{\phi}) \}_{g \in G} \).

For a given \( g \in G \), since \( F \) is \( D \)-separated, the sets \( gf \cdot D \) are pairwise disjoint for distinct \( f \in F \), and thus

\[
\mathbb{P}[\neg(gF \cdot V_{\phi})] = \prod_{f \in F} \mathbb{P}[\neg(V_{g \cdot \phi})] = \left(1 - \frac{1}{n|D|}\right)^{|F|}.
\]

Now the event \( gF \cdot V_{\phi} \) is independent with the set \( \{ hF \cdot V_{\phi} : hF \text{ and } hF \text{ are disjoint} \} \). If \( gF \) and \( hF \) are not disjoint, then

\[
h \in gFDD^{-1}F^{-1},
\]

and thus the event \( gF \cdot V_{\phi} \) is independent with a set of cocardinality \( \leq |D|^2|F|^2 \). So for the Lovász Local Lemma to hold, we need the following inequality to hold:

\[
4 \cdot \left(1 - \frac{1}{n|D|}\right)^{|F|} \left(|D|^2|F|^2 + 1\right) < 1
\]

which clearly holds for \( F \) sufficiently large. \( \square \)

This concludes the proof of Bernoulli disjointness. \( \square \)
Appendix A. Proof of the Lovász Local Lemma

We restate the Lovász Local Lemma.

**Theorem 4** (Lovász Local Lemma). Let $\mathcal{A}$ be a set of events in a probability space, each with probability $\leq p$, such that for $A \in \mathcal{A}$, there is a subset $B \subset \mathcal{A}$ with $|A \setminus B| \leq d$ such that $A$ is independent from $B$. If

$$4p(d + 1) < 1,$$

then for any $A_0, \ldots, A_k \in \mathcal{A}$, we have $\mathbb{P}[\bar{A}_0 \cdots \bar{A}_k] > 0$.

This is the original proof (see Lemma, p.616 in [EL]):

**Proof.** We prove the following stronger claim:

**Proposition 1.** For any distinct $A_0, \ldots, A_k \in \mathcal{A}$, we have

1. $\mathbb{P}[\bar{A}_1 \cdots \bar{A}_k] > 0$ and
2. $\mathbb{P}[A_0 | \bar{A}_1 \cdots \bar{A}_k] \leq 2p$.

**Proof.** We proceed by strong induction on $k$.

Note that (1) clearly holds when $k = 0$, and if $k > 0$, then $\mathbb{P}[A_1 | \bar{A}_2 \cdots \bar{A}_k] \leq 2p$, so

$$\mathbb{P}[\bar{A}_1 | \bar{A}_2 \cdots \bar{A}_k] \geq 1 - 2p > 2p \geq 0$$

where we use that $4p < 1$, and thus $\mathbb{P}[\bar{A}_1 \cdots \bar{A}_k] > 0$.

For (2), assume wlog that $A_0$ is independent from $\{A_{q+1}, \ldots, A_k\}$, where $q \leq d$. Then we have

$$\mathbb{P}[A_0 | \bar{A}_1 \cdots \bar{A}_k] = \frac{\mathbb{P}[A_0 \bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k]}{\mathbb{P}[\bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k]}$$

The numerator is $\leq p$ as follows:

$$\mathbb{P}[A_0 \bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k] \leq \mathbb{P}[A_0 | \bar{A}_{q+1} \cdots \bar{A}_k] \leq \mathbb{P}[A_0] \leq p$$

The denominator is $> \frac{1}{2}$ as follows:

$$\mathbb{P}[\bar{A}_1 \cdots \bar{A}_q | \bar{A}_{q+1} \cdots \bar{A}_k] \geq 1 - \sum_{i=1}^{q} \mathbb{P}[A_i | \bar{A}_{q+1} \cdots \bar{A}_k] \geq 1 - \sum_{i=1}^{q} 2p \geq 1 - 2pd > \frac{1}{2}$$

where the last inequality uses that $4pd < 1$. So we are done.

**References**

