

ABAB OR ABBA? THE ARITHMETICS OF PENALTY SHOOTOUTS IN SOCCER.

FEDERICO ECHENIQUE

FIFA and UEFA are experimenting with a new system for penalty shootouts. The new system is designed to attenuate the well known bias in favor of the first shooter in a shootout sequence (Apesteguia and Palacios-Huerta (2010)). The system currently in place, called ABAB, has shooters from each team alternate in shooting a penalty. If the two teams are A and B, then the sequence goes ABABABA... The new system is called ABBA; it has teams line up shooters according to the sequence ABBAABBAABB...

For example, last June 4th, Uruguay and Portugal tied in the U20 Soccer World Cup quarterfinals. A penalty shootout was to determine which team would advance to semifinals. Portugal drew the first place in the penalty shootout. There is a well-known “first shooter advantage:” other aspects being equal, the team to shoot first has a probability of between 60% and 70% of winning.¹ Portugal’s Rúben Diaz shot the first penalty and scored. The second shooter was Federico Valverde, from Uruguay. Valverde scored. Surprisingly, the third shooter was also wearing *la celeste*: José Luis Rodriguez shot and scored the third penalty. In the ABAB system, the third shooter would have been from Portugal. In the ABBA system, the third penalty was shot by

Division of the Humanities and Social Sciences, California Institute of Technology. I wish to thank Jose Apesteguia and Ignacio Palacios-Huerta for helpful discussions, and the commentators of the Fox Sports broadcast of the Uruguay-Portugal U20 World Cup semifinal for the motivation to write this note.

Since first writing this note, I have found that essentially the same calculations are already contained in Brams and Ismail (2017). They deserve credit for first making the points I am making here.

¹Apesteguia and Palacios-Huerta (2010); Ignacio Palacios-Huerta has been instrumental in convincing the soccer governing bodies to try the new system.

a Uruguayan while the fourth and fifth penalties were shot by Portuguese players.²

I present some simple calculations that support the move from ABAB to ABBA, and use these to calibrate the effect of first shooters. I track down the conditions that lie behind the first shooter advantage, and show that ABBA is less biased than ABAB under such conditions. I also perform a simple back of the envelope calculation to argue that if the first mover advantage ranges between 60 and 70% in the ABAB system, it should move to the 52-55% range under the ABBA system. Arguably this is a substantial reduction in first shooter advantage.

1. ANALYSIS

The two teams are called A and B. Team A shoots first. One team wins when 1) it has scored more goals, and 2) both teams have shot the same number of penalties. (In reality, both teams are guaranteed 5 penalties, but I abstract away from this fact so as to make my calculations simpler.) I assume that the probability of scoring depends on the net score in favor of each team: whether the team is up by one goal, down by one, or if the score is even. Conditional on the net score, the probability and success of a penalty is independent of past penalties. Teams and players are identical: one is not better than another at scoring or goalkeeping. Of course this assumption is never correct, but it is unavoidable in order to understand the pure bias built into the system.

The crucial property capturing the psychological advantage of shooting first is that *the probability that the first shooter scores and the second shooter misses is greater than the probability that the first shooter misses and the second scores*. Lets call this property ♠:

$$(\spadesuit) \quad \text{Prob}(A \text{ scores and } B \text{ misses}) > \text{Prob}(A \text{ misses and } B \text{ scores})$$

Think of a single round AB, where A shoots first and B second. Condition ♠ says that the probability that A wins in a single round is greater than the probability that B wins in a single round. A second shooter faces psychological

²In the final outcome, Uruguay qualified to the semis. Uruguay qualified thanks to Santiago Mele's fantastic goalkeeping, but the ABBA system may have contributed to attenuate the bias in favor of Portugal as a first shooter.

pressure regardless of whether the first shooter scored or missed. If the first shooter scored, the second shooter knows that if he misses, their team has lost the game and is out of the tournament. If the first shooter misses, then the second shooter will bask in the glory of having won the tournaments for his team mates and his country. Condition ♠ captures the effects of this psychological pressure.

Proposition 1. *The probability that A wins in system ABAB equals the ratio*

$$\frac{\text{Prob}(A \text{ wins in a single penalty round})}{\text{Prob}(A \text{ wins in a single penalty round}) + \text{Prob}(B \text{ wins in a single penalty round})}$$

The probability that A wins in system ABBA equals the ratio

$$\frac{\text{Prob}(B \text{ doesn't win in a single penalty round})}{\text{Prob}(A \text{ doesn't win in a single penalty round}) + \text{Prob}(B \text{ doesn't win in a single penalty round})}$$

Note that the probabilities in the denominators do not add up to one because in a single round it is possible that both teams score, or that both miss.

Say that there is a first shooter advantage if the probability v that A wins is strictly greater than 50%.

Proposition 2. *Regardless of whether the system is ABAB or ABBA, there is a first shooter advantage if and only if property ♠ holds. Moreover, when ♠ holds, ABAB confers a larger first shooter advantage than ABBA.*

We can use the formulas in Proposition 2 to estimate the effect of moving from ABAB to ABBA. For a quick calculation, I assume that the probability of scoring for the second mover is the same regardless of whether the first shooter scores or misses. If the first-shooter advantage under the ABAB system is 60%, and we suppose that a penalty shooter scores with 79% in normal circumstances (estimates taken from Apesteguia and Palacios-Huerta (2010)), then the first shooter advantage under the ABBA system would go down to 52%. If the advantage under the ABAB system is instead 70%, then the advantage would be reduced to 55% under ABBA. Of course, more accurate calculations are possible with better data on success probabilities.³

³It should be pointed out that the bias estimated by Apesteguia and Palacios-Huerta for the penalties beyond the first 5 is much smaller than 60%, but, as they point out, there's an obvious selection effect for games that go beyond the first 5, and it's not clear how to interpret the bias in these cases.

My estimates are consistent with experimental evidence reported in Palacios-Huerta (2014). Palacios-Huerta conducts experiments on the ABBA system using professional Spanish soccer players, and finds a first-shooter advantage of 54%.⁴

2. RELATED WORK

The first-shooter advantage has been documented by Apesteguia and Palacios-Huerta (2010), as discussed above. The paper by Vandebroek, McCann, and Vroom (2017) provides a detailed analysis of shootout sequences and the first-shooter advantage. In particular, they look at versions of the shootout problem that are non-stationary, and allow for a fixed number of initial rounds of penalty shots. These authors also calculate the effects of changing from ABAB to ABBA, and find significant reductions in first-shooter advantage. Anbarci, Sun, and Ünver (2015) adopt a mechanism design approach and characterize sequentially fair shootout mechanisms.

In my note I provide simple calculations that hopefully provide a basic understanding of how ABAB and ABBA differ. My calculations also enable a back-of-the-envelope prediction of the first shooter advantage in ABBA.

3. PROOFS.

Let $f(k)$ be the probability of scoring a penalty when the net score is k in favor of team A . Suppose that $f(k) > 0$ and $f(k) < 1$ for all k . Write $p = f(0)$, $q = f(1)$ and $q' = f(-1)$.

First consider the model ABAB of pure alternation of shooters. Let v be the value of the game where team A gets +1 if it wins and 0 if it loses. The value of the game will then equal the probability of team A winning. With probability $pq + (1 - p)(1 - q)$ the teams tie and the game continues; the continuation value to A being v . With probability $p(1 - q)$ the first shooter A wins and receives +1. With probability $(1 - p)q'$ the second shooter B wins and A receives a payoff of 0. Thus the relevant “Bellman equation” is:

$$v = pq \cdot v + p(1 - q) \cdot 1 + (1 - p)q' \cdot 0 + (1 - p)(1 - q') \cdot v.$$

⁴Palacios-Huerta actually favors the sequence ABBABAAB (see Palacios-Huerta (2012)), for which his experiments give an advantage of only 51%.

Therefore,

$$v = \frac{p(1-q)}{1-pq - (1-p)(1-q')} = \frac{p(1-q)}{p(1-q) + (1-p)q'}.$$

Note that $v \geq 1/2$ iff $p(1-q) \geq (1-p)q'$, which is condition ♠.

Note also that $p(1-q)$ is the probability of A winning in one round, and $(1-p)q'$ is the probability of B winning in one round.

In second place, consider the ABBA model. The difference is that now if A does not win in a single round, B becomes the first mover. By stationarity, v is the expected value of the game to B . So the value to A of the event that A does not win in the first round is $1-v$. Thus we have:

$$v = pq \cdot (1-v) + p(1-q) \cdot 1 + (1-p)q' \cdot 0 + (1-p)(1-q') \cdot (1-v),$$

or

$$[1 + pq + (1-p)(1-q')]v = pq + p(1-q) + (1-p)(1-q') = p + (1-p)(1-q').$$

Note that $1 + pq + (1-p)(1-q') = 1 - p(1-q) + p + (1-p)(1-q')$. So,

$$v = \frac{p + (1-p)(1-q')}{1 - p(1-q) + p + (1-p)(1-q')}.$$

Then $v \geq 1/2$ iff $p + (1-p)(1-q') \geq 1 - p(1-q) = pq + (1-p)q' + (1-p)(1-q')$. So, $v \geq 1/2$ iff $p \geq pq + (1-p)q'$ iff $p(1-q) \geq (1-p)q'$. Again this is condition ♠.

Observe that $p + (1-p)(1-q') = 1 - (1-p)q'$ is the probability of B not winning in one round while

$$1 - p(1-q) + p + (1-p)(1-q') = 1 - p(1-q) + p + (1-p) - q'(1-p) = [1 - p(1-q)] + [1 - (1-p)q']$$

is the sum of the probability that A does not win in one round and the probability that B does not win in one round.

These calculations prove the first proposition stated in the text.

To compare the value of shooting first under the two rules, consider the right-hand side of the equations for v in each of two systems. Let

$$h(v) = pq \cdot v + p(1-q) \cdot 1 + (1-p)(1-q') \cdot v$$

and

$$g(v) = pq \cdot (1 - v) + p(1 - q) \cdot 1 + (1 - p)(1 - q') \cdot (1 - v),$$

for $v \in [0, 1]$.

Note that $g(v) = h(1 - v)$ and $h'(v) = -g'(v) > 0$. So $g(v) = h(v)$ iff $v = 1/2$. Note that if $v > 1/2$ then $h(v) > g(v)$. Under condition \spadesuit , if $v_1 = h(v_1)$ and $v_2 = g(v_2)$, then $v_1, v_2 > 1/2$. Then

$$v_1 = h(v_1) > h(1/2) = g(1/2) > g(v_2) = v_2$$

since h is strictly increasing and g is strictly decreasing. Since $v_1 > v_2$, there is a greater first mover advantage under the ABAB rule than under ABBA. Thus the second proposition in the text is proven.

For the back-of-the-envelope calculations, we assume that $q = q'$. Then the value of ABAB is

$$v = \frac{p(1 - q)}{p(1 - q) + q(1 - p)} = \frac{1}{1 + l_p/l_q},$$

where $l_q = (1 - p)/p$. Then $l_p/l_q = l_v$. So if we assume that $v = .6$ and $p = .79$, a calculation gives $l_q = 1/2$. The implied q implies that the probability of the first shooter winning in the ABBA system is 52%.

REFERENCES

- ANBARCI, N., C.-J. SUN, AND M. U. ÜNVER (2015): “Designing Fair Tiebreak Mechanisms: The Case of FIFA Penalty Shootouts,” Available at SSRN: <https://ssrn.com/abstract=2558979>.
- APESTEGUIA, J., AND I. PALACIOS-HUERTA (2010): “Psychological pressure in competitive environments: evidence from a randomized natural experiment,” *American Economic Review*, 100(Dec), 2548–2564.
- BRAMS, S. J., AND M. S. ISMAIL (2017): “Making the Rules of Sports Fairer,” Forthcoming: SIAM review.
- PALACIOS-HUERTA, I. (2012): “Tournaments, Fairness and the Prouhet-Thue-Morse Sequence,” *Economic inquiry*, 50(3), 848–849.
- (2014): *Beautiful Game Theory: How Soccer Can Help Economics*. Princeton University Press.

VANDEBROEK, T. P., B. T. MCCANN, AND G. VROOM (2017): "Modeling the Effects of Psychological Pressure on First-Mover Advantage in Competitive Interactions The Case of Penalty Shoot-Outs," forthcoming, *Journal of Sports Economics*.