Lecture 10

In this and the next lecture, we will consider the extremal problem for some of the most obvious examples of bipartite graphs, cycles of even length. For cycles of length 4, we have already seen that $ex(n, C_4) \approx \frac{1}{2}n^{3/2}$.

The main theorem we will prove over the next couple of lectures is the upper bound $ex(n, C_{2k}) \leq cn^{1+1/k}$, due to Bondy and Simonovits. For $k = 2, 3$ and 5, that is, for C_4, C_6 and C_{10}, this is known to be sharp. A quick probabilistic argument, similar to that used earlier for complete bipartite graphs, gives the following general lower bound. We leave the details to the reader.

Theorem 1 *There exists a constant c such that*

$$ex(n, C_{2k}) \geq cn^{1+1/(2k-1)}.$$

There is also an explicit construction, due to Lazebnik, Ustimenko and Woldar, which does better, giving $ex(n, C_{2k}) \geq cn^{1+2/(3k-3)}$. However, this is beyond the reach of the course.

In order to prove the Bondy-Simonovits theorem, $ex(n, C_{2k}) \leq cn^{1+1/k}$, we will need some preliminary lemmas. Both concern cycles with an extra chord.

Lemma 1 *Let H be a cycle with an extra chord. Let (A, B) be a non-trivial partition of $V(H)$, that is, there is some edge crossing the partition. Then, unless H is bipartite between A and B, H contains paths of every length $\ell < |H|$ which begin in A and end in B.*

Proof Label the vertices of H as $0, 1, \ldots, t-1$, where $t = |H|$. Suppose that H does not contain cycles which start in A and end in B for every possible length $\ell < t$. We will focus on a particular class of path, saying that a path is good if it begins in A, ends in B and does not use the chord of H. Let s be the smallest integer such that there is no good path of length s. Then $s > 1$, since there is at least one edge between A and B. If this edge is the chord, it will automatically imply that there is some other edge across the partition. We also have that $s \leq t/2$. This is because, by symmetry, the existence of a good path of length j implies the existence of a good path of length $t - j$.

Let χ be the characteristic function of A. Then, for any j, $\chi(j + s) = \chi(j)$, where addition is taken modulo t. Let $d = \text{hcf}(s, t)$. Then there are p and q such that $ps+qt = d$ and, therefore, $\chi(j) = \chi(j+d)$, for all j. But then there is no good path of length d. Therefore, since s was the smallest number with this property, $d = s$ and s divides t. This also implies that for every i which is not a multiple of s, there will be good paths of length i.

We will now find paths of all remaining lengths is, where $1 \leq i \leq t/s - 1$, by using the chord. Suppose first that the chord joins two vertices at distance r, where $1 < r \leq s$, say 0 and r. We know from above that there are good paths of length $s + r - 1$. In particular, there is some j such that $\chi(j) \neq \chi(j+s+r-1)$. By shifting, we may assume that $-s < j \leq 0$. Therefore, since $j+s+r-1 \geq r$ and $\chi(j) \neq \chi(j + is + r - 1)$, the path $j, j + 1, \ldots, 0, r, r + 1, \ldots, j + is + r - 1$ is a path of length is beginning in A and ending in B. We need to verify that $j + is + r - 1 < t + j$, that is, that it doesn't loop all the way around, but this follows easily for $i \leq t/s - 1$.

We therefore assume that the chord is $0r$, where $s < r < t - s$. Let $-s < j < 0$ and consider the paths $j, j+1, \ldots, 0, r, r-1, \ldots, r-j-s+1$ and $s+j, s+j-1, \ldots, 0, r, r+1, \ldots, r-j-1$, each of length s. If either of them is a path starting in A and ending in B, we may extend it to produce a well-behaved path of length is until the number of unused vertices in the two arcs defined by the chord is less than s in both arcs. At this point, $is+1 \geq t-2(s-1)$ and, since s divides t, $is = t-s$, so we already have everything. Similarly, if either of the paths $0, r, r-1, \ldots, r-s+1$ or $0, r, r+1, \ldots, r+s-1$ begin in A and end in B, then H contains well-behaved paths of all lengths less than t.

We may therefore assume that, for $-s < j < 0$,

$$\chi(r-j+1) = \chi(r-j-s+1) = \chi(j) = \chi(s+j) = \chi(r-j-1).$$

The first and third equalities are by shifting. The second and fourth follow because the paths $j, j+1, \ldots, 0, r, r-1, \ldots, r-j-s+1$ and $s+j, s+j-1, \ldots, 0, r, r+1, \ldots, r-j-1$ must each have both endpoints in one of A or B. Similarly, we may assume that $\chi(r+s+1) = \chi(r+s-1)$. This implies that $\chi(i) = \chi(i+2)$ for every vertex i. Therefore, $s = 2$.

We may conclude therefore that t is even and that the vertices of the cycle alternate between A and B. It is easy now to see that if the chord is contained with one of A or B, then the graph contains paths of all length less than t which start in A and end in B. Therefore, the chord goes between A and B and H is bipartite, as required. $\square$

The second lemma we need is a condition for a graph to contain a cycle with an extra chord.

Lemma 2 *Any bipartite graph G with minimum degree $d \geq 3$ contains a cycle of length at least $2d$ with an extra chord.*

Proof Let P be the longest path in G, visiting vertices $x_1, \ldots, x_p$ in that order. x_1 has at least $d \geq 3$ neighbours in G. By the maximality of P, these must all lie in P. Suppose that they are $x_{i_1}, \ldots, x_{i_d}$ with $i_1 < \cdots < i_d$. Every two neighbours of x_1 must be at least 2 apart, since G is bipartite. Therefore, since $i_1 = 2$, we must have $i_d \geq 2d$. The required cycle with chord is formed by taking the path from x_1 to x_{i_d} and adding the edges $x_1 x_{i_2}$ and $x_1 x_{i_d}$. $\square$

We will also need two simple lemmas which we have proved in previous lectures.

Lemma 3 *Every graph G contains a subgraph whose minimum degree is at least half the average degree of G.*

Lemma 4 *Every graph G contains a bipartite subgraph with at least half the edges of G.*