REMARKS ON THE SYMMETRIC POWERS OF CUSP FORMS ON GL(2)

DINAKAR RAMAKRISHNAN

To Steve Gelbart
On the occasion of his sixtieth birthday

Introduction

Let F be a number field, and π a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ of conductor $\mathcal{N}$. For every $m \geq 1$ one has its *symmetric m-th power L-function* $L(s, \pi; \mathrm{sym}^m)$, which is an Euler product over the places v of F, with the v-factors (for finite $v \nmid \mathcal{N}$ of norm q_v) being given by

$$L_v(s, \pi; \mathrm{sym}^m) \; = \; \prod_{j=0}^{m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1},$$

where the unordered pair $\{\alpha_v, \beta_v\}$ defines the diagonal conjugacy class in $\mathrm{GL}_2(\mathbb{C})$ attached to π_v. Even at a ramified (resp. archimedean) place v, one has by the local Langlands correspondence a 2-dimensional representation σ_v of the extended Weil group $W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$ (resp. of the Weil group W_{F_v}), and the v-factor of the symmetric m-th power L-function is associated to $\mathrm{sym}^m(\sigma_v)$. A special case of the *principle of functoriality* of Langlands asserts that there is, for each m, an (isobaric) automorphic representation $\mathrm{sym}^m(\pi)$ of $\mathrm{GL}(m+1, \mathbb{A})$ whose standard (degree $m+1$) L-function $L(s, \mathrm{sym}^m(\pi))$ agrees, at least at the primes not dividing $\mathcal{N}$, with $L(s, \pi; \mathrm{sym}^m)$. It is well known that such a result will have very strong consequences, such as the *Ramanujan conjecture* and the *Sato-Tate conjecture* for π. The *modularity*, also called *automorphy*, has long been known for $m = 2$ by the pioneering work of Gelbart and Jacquet ([GJ]); we will write $\mathrm{Ad}(\pi)$ for the selfdual representation $\mathrm{sym}^2(\pi) \otimes \omega^{-1}$, ω being the central character of π. A *major breakthrough, due to Kim and Shahidi* ([KS2, KS1, Kim])), has established the modularity of $\mathrm{sym}^m(\pi)$ for $m = 3, 4$, along with a useful cuspidality criterion (for $m \leq 4$). Furthermore, when $F = \mathbb{Q}$ and π is defined by a holomorphic newform f of weight 2, $\mathbb{Q}$-coefficients and level N, such that at some prime p, the component π_p is Steinberg, a recent *dramatic theorem* of Taylor, Harris, Clozel and Shepherd-Barron ([CHT], [HT2], [Tay3]), furnishes the *potential modularity* of $\mathrm{sym}^{2m}(\pi)$ (for every $m \geq 1$), i.e., its modularity over a number field K, thereby extracting the

Sato-Tate conjecture in this case by a clever finesse. It should however be noted that such a beautiful result is not (yet) available for π defined by newforms φ of higher weight, for instance for the ubiquitous cusp form $\Delta(z) = q\prod_{n\geq 1}(1 - q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$, where $z \in \mathcal{H}$ and $q = e^{2\pi i z}$, which is holomorphic of weight 12, level 1 and trivial character.

In this Note we consider the following more modest, but nevertheless basic, question:

Suppose $sym^m(\pi)$ is an automorphic representation of $GL_{m+1}(\mathbb{A}_F)$. When is it cuspidal?

If $sym^m(\pi_v)$ is, for some finite place v, in the discrete series, which happens for example when π_v is Steinberg, it is well known that the global representation $sym^m(\pi)$ will necessarily be cuspidal (once it is automorphic). On the other hand, one knows already for $m = 2$, as shown by Gelbart and Jacquet ([GJ]), that if π is dihedral, i.e., associated to an idele class character χ of a quadratic extension K of F, then $sym^2(\pi)$ is not cuspidal; in fact, this is *necessary and sufficient* condition. There is a non-trivial extension of such a criterion in the work of Kim and Shahidi ([KS1]), who show that for a non-dihedral π, $sym^3(\pi)$ is Eisensteinian iff π is *tetrahedral*, while $sym^4(\pi)$ is cuspidal iff π is not tetrahedral or octahedral. We will say that π is *solvable polyhedral* iff it is dihedral, tetrahedral or octahedral. Finally, if π is associated to an irreducible 2-dimensional Galois representation ρ which is icosahedral, i.e., with projective image isomorphic to the alternating group A_5, one knows that $sym^6(\rho)$ is reducible, suggesting that $sym^6(\pi)$ is not cuspidal. However, $sym^5(\rho)$ is, in the icosahedral case, necessarily a tensor product $sym^2(\rho') \otimes \rho$, where ρ' is the Galois conjugate representation of ρ (which is defined over $\mathbb{Q}[\sqrt{5}]$) (cf. [**?**], [Wan], for example). This allowed Wang to prove (in [Wan]) that $sym^5(\pi)$ is cuspidal by making use of the construction (cf [KS2]) of the functorial product $\Pi \boxtimes \pi'$ (in $GL(6)/F$), for Π (resp. π') a cusp form on $GL(3)/F$ (resp. $GL(2)/F$), and by developing a cuspidality criterion for this product.

In order to answer the question, we make the following definition: Call an irreducible cuspidal automorphic representation π of $GL(2, \mathbb{A}_F)$ *quasi-icosahedral* iff we have

   (i) $sym^m(\pi)$ is automorphic for every $m \leq 6$;
   (ii) $sym^m(\pi)$ is cuspidal for every $m \leq 4$; and
   (iii) $sym^6(\pi)$ is not cuspidal.

The key result which we prove (see part (b) of Theorem A below) is that, for every such quasi-icosahedral π of central character ω, there exists another cusp form π' of $GL(2)/F$ (of central character ω') such that the symmetric fifth power of such a quasi-icosahedral cusp form π is necessarily a character twist of the functorial product $Ad(\pi') \boxtimes \pi$, where $Ad(\pi') = sym^2(\pi') \otimes \omega'^{-1}$. If π were associated to an icosahedral Galois representation ρ, defined over

$\mathbb{Q}[\sqrt{5}]$, π' could be taken to correspond to the Galois conjugate representation $\rho^{[\theta]} : \sigma \to \rho(\theta\sigma\theta^{-1})$, where θ denotes the non-trivial automorphism of $\mathbb{Q}[\sqrt{5}]$. The beauty is that we can find π' by a purely automorphic argument.

All of this is consistent with the results of Wang , as well as with the philosophy of Langlands ([Lan4]), which predicts that any cuspidal π on $\mathrm{GL}(2)/F$ should be naturally associated to a reductive subgroup $H(\pi)$ of $\mathrm{GL}_2(\mathbb{C})$. In fact, one expects there to be a pro-reductive group $\mathcal{L}_F$ over $\mathbb{C}$ whose n-dimensional $\mathbb{C}$-representations σ classify (up to equivalence) the (isobaric) automorphic representations π of $\mathrm{GL}(n, \mathbb{A}_F)$, and $H(\pi)$ should be given by the image of σ.

Theorem A   *Let π a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$, which is not solvable polyhedral, of central character ω. Suppose $sym^m(\pi)$ is modular for all m. Then we have*

    (a) *$sym^5(\pi)$ is cuspidal.*
    (b) *$sym^6(\pi)$ is non-cuspidal iff we have*

$$\mathrm{sym}^5(\pi) \simeq \mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2,$$

       for a cuspidal automorphic representation π' of $GL_2(\mathbb{A}_F)$.
    (c) *If $sym^6(\pi)$ is cuspidal, then so is $sym^m(\pi)$ for all $m \geq 1$.*
    (d) *If $F = \mathbb{Q}$ and π is defined by a non-CM, holomorphic newform φ of weight $k \geq 2$, then $\mathrm{sym}^m(\pi)$ is cuspidal for all m.*

One can do a bit better than this in that for a given symmetric power, one does not need information on all the $\mathrm{sym}^m(\pi)$. See Theorem A$'$ in section 2 for a precise statement. The proofs are then given in sections 3 and 4.

In part (b), the cusp form π' is not uniquely determined, only up to a character twist. In a sequel we will show that, in fact, for a suitable choice of π', $\mathrm{sym}^5(\pi)$ is, in the quasi-icosahedral case, expressible as a character twist of $\mathrm{Ad}(\pi) \boxtimes \pi'$; π' will also turn out to be quasi-icosahedral. This is as predicted by looking at the Galois side, and it will help us normalize the choice of π', leading in addition to a precise rationality statement.

The results of this paper were essentially established some time ago, but the questions raised to me in the past two years by some colleagues have led me to believe in the possible usefulness of their being in print. While the inspiration for the results here came from Langlands (and the paper of Wang), and from a short conversation with Richard Taylor some time back, the proofs depend, at least partly, on the beautiful constructions [KS2, KS1, Kim] of Kim and Shahidi. Use is also made of the papers [Ram3, Ram7].

work with Jacquet, *A relation between automorphic forms on* GL(2) *and*
GL(3). His later works have also been influential. Furthermore, Steve has
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1. PRELIMINARIES

1.1. **The standard L-function of $\mathbf{GL}(n)$.** Let F be a number field with
adele ring $\mathbb{A}_F$. For each place v, denote by F_v the corresponding local com-
pletion of F, and for v finite, by $\mathfrak{O}_v$ the ring of integers of F_v with uniformizer
ϖ_v of norm q_v. For any algebraic group G over F, let $G(\mathbb{A}_F)$ denote the
restricted direct product $\prod'_v G(F_v)$, endowed with the usual locally compact
topology. For $m \geq 1$, let Z_m denote the center of $\mathrm{GL}(m)$. One knows that
the volume of $Z_m(\mathbb{A}_F)\mathrm{GL}_m(F)\backslash\mathrm{GL}(m, \mathbb{A}_F))$ is finite.

By a *unitary cuspidal* representation of $\mathrm{GL}_m(\mathbb{A}_F) = \mathrm{GL}_m(F_\infty) \times \mathrm{GL}_m(\mathbb{A}_{F,f})$,
we will always mean an irreducible, automorphic representation occurring
in the space of cusp forms in $L^2(Z_m(\mathbb{A}_F)G_m(F)\backslash G_m(\mathbb{A}_F), \omega)$ relative to a
character ω of $Z_m(\mathbb{A}_F)$, trivial on $Z_m(F)$. By a (general) *cuspidal represen-
tation* of $\mathrm{GL}_m(\mathbb{A}_F)$, we will mean an irreducible admissible representation
of $G_m(\mathbb{A}_F)$ for which there exists a real number, called the *weight of π* such
that $\pi \otimes |.|^{w/2}$ is a unitary cuspidal representation. Such a representation is
in particular a restricted tensor product $\pi = \otimes'_v \pi_v = \pi_\infty \otimes \pi_f$, where each
π_v is an (irreducible) admissible representation of $G(F_v)$, with π_v unramified
at almost all v.

For any irreducible, automorphic representation π of $GL(n, \mathbb{A}_F)$, let $L(s, \pi)$
$= L(s, \pi_\infty)L(s, \pi_f)$ denote the associated *standard $L-$function* ([Jac]) of π;
it has an Euler product expansion

$$(1.1.1) \qquad\qquad L(s, \pi) \;=\; \prod_v L(s, \pi_v),$$

convergent in a right-half plane. If v is an archimedean place, then one knows
(cf. [Lan3]) how to associate a semisimple $n-$dimensional $\mathbb{C}-$representation
$\sigma(\pi_v)$ of the Weil group W_{F_v}, and $L(\pi_v, s)$ identifies with $L(\sigma_v, s)$. On the
other hand, if v is a finite place where π_v is unramified, there is a correspond-
ing semisimple (Langlands) conjugacy class $A_v(\pi)$ (or $A(\pi_v)$) in $\mathrm{GL}(n, \mathbb{C})$
such that

$$(1.1.2) \qquad\qquad L(s, \pi_v) \;=\; \det(1 - A_v(\pi)T)^{-1}|_{T=q_v^{-s}}.$$

We may find a diagonal representative $\mathrm{diag}(\alpha_{1,v}(\pi), ..., \alpha_{n,v}(\pi))$, unique up
to permutation of the diagonal entries, for $A_v(\pi)$. Let $[\alpha_{1,v}(\pi), ..., \alpha_{n,v}(\pi)]$

denote the resulting unordered $n-$tuple. Since $W_{F,v}^{\mathrm{ab}} \simeq F_v^*$, $A_v(\pi)$ clearly defines an abelian $n-$dimensional representation $\sigma(\pi_v)$ of $W_{F,v}$. One has

Theorem 1.1.3 ([GJ, Jac])   *Let $n \geq 1$, and π a cuspidal representation of $GL(n, \mathbb{A}_F)$. When $n = 1$, assume that π is not of the form $|\cdot|^w$ for any $w \in \mathbb{C}$. Then $L(s, \pi)$ is entire.*

When $n = 1$, such a π is simply a unitary idele class character χ, and this result is due to Hecke. Also, when χ is trivial, $L(s, \pi_f)$ is the Dedekind zeta function $\zeta_F(s)$.

Given a pair of automorphic representations π, π' of $GL(n, \mathbb{A}_F)$, $GL(n', \mathbb{A}_F)$, respectively, one can associate an L-function $L(s, \pi \times \pi')$ which is meromorphic. We will postpone its definition till section 1.4.

For any L-function with an Euler product expansion (over F): $L(s) = \prod_v L_v(s)$, if S is any set of places of F, the associated *incomplete L-function* is defined as follows:

$$L^S(s) := \prod_{v \notin S} L_v(s).$$

1.2. Isobaric automorphic representations.

By the theory of Eisenstein series, one has a sum operation $\boxplus$ ([Lan2]) on a suitable set of automorphic representations of $GL(n)$ for all n. One has the following:

Theorem 1.2.1   ([JS])   *Given any $m-$tuple of cuspidal representations $\pi_1, ..., \pi_m$ of $GL(n_1, \mathbb{A}_F), ..., GL(n_m, \mathbb{A}_F)$ respectively, there exists an irreducible, automorphic representation $\pi_1 \boxplus ... \boxplus \pi_m$ of $GL(n, \mathbb{A}_F)$, $n = n_1 + ... + n_m$, which is unique and satisfies the following property: For any finite set S of places, we have, for every cuspidal automorphic representation π' of $GL(n', \mathbb{A}_F)$ (with n' arbitrary),*

$$L^S(s, (\boxplus_{j=1}^m \pi_j) \times \pi') = \prod_{j=1}^m L^S(s, \pi_j \times \pi').$$

The L-functions in the Theorem are the Rankin-Selberg L-functions attached to pairs of automorphic representations, which we briefly discuss in section 1.4 below.

Call such a (Langlands) sum $\pi \simeq \boxplus_{j=1}^m \pi_j$, with each π_j cuspidal, an *isobaric* representation. Denote by $\mathrm{ram}(\pi)$ the finite set of finite places where π is ramified, and let $\mathfrak{N}(\pi)$ be its conductor ([JPSS1]).

For every integer $n \geq 1$, set:

(1.2.2) $\mathcal{A}(n, F) = \{\pi : \text{isobaric representation of } GL(n, \mathbb{A}_F)\}/\simeq,$

and

$$\mathcal{A}_0(n, F) = \{\pi \in \mathcal{A}(n, F) | \pi \text{ cuspidal}\}.$$

Put $\mathcal{A}(F) = \cup_{n \geq 1} \mathcal{A}(n, F)$ and $\mathcal{A}_0(F) = \cup_{n \geq 1} \mathcal{A}_0(n, F)$.

Definition 1.2.3   *Given $\pi, \eta \in \mathcal{A}(F)$, if we can find an $\eta' \in \mathcal{A}(F)$ such that $\pi \simeq \eta \boxplus \eta'$, we will call η an isobaric summand of π and write*

$$[\pi : \eta] > 0.$$

Remark.   One can also define the analogues of $\mathcal{A}(n, F)$ for local fields F, where the "cuspidal" subset $\mathcal{A}_0(n, F)$ consists of essentially square-integrable representations of $\mathrm{GL}(n, F)$. See [Lan2] and [Ram1] for details.

1.3. Symmetric powers of GL(2).

1.3. **Symmetric powers of GL**(2). Since the L-group of $\mathrm{GL}(2)$ is $\mathrm{GL}(2, \mathbb{C}) \times W_F$, the principle of functoriality of Langlands ([Lan1]) predicts that for any algebraic representation

$$(1.3.1) \qquad\qquad\qquad r : \mathrm{GL}(2, \mathbb{C}) \rightarrow GL(N, \mathbb{C}),$$

and any number field F, there should be a map

$$(1.3.2) \qquad\qquad \mathcal{A}(2, F) \rightarrow \mathcal{A}(N, F),\ \pi \rightarrow r(\pi),$$

with compatible local maps, such that for all finite unramified places v (for π), we have the equality of Langlands classes

$$r(A(\pi_v)) = A(r(\pi)_v).$$

It suffices to establish this for irreducible representations r, which are all of the form $\mathrm{sym}^n(r_0) \otimes L^{\otimes k}$, with $n, k \in \mathbb{Z}, n \geq 0$; here r_0 denotes the standard representation of $\mathrm{GL}(2, \mathbb{C})$ with determinant L, and $\mathrm{sym}^n(r_0)$ denotes the symmetric n-th power representation of ρ.

It is enough to construct the $\mathrm{sym}^n(\pi)$'s for π cuspidal. When it exists, by which we mean it belongs to $\mathcal{A}(F)$, we will write (for $\pi \in \mathcal{A}(2, F)$)

$$\mathrm{sym}^n(\pi) = \mathrm{sym}^n(\rho)(\pi).$$

It may be useful to recall that if

$$L(s, \pi_v) = [(1 - \alpha_v q_v^{-s})(1 - \beta_v q_v^{-s})]^{-1}$$

at any unramified finite place v with norm q_v, with $A(\pi_v)$ being represented by the diagonal matrix with entries α_v, β_v, then for every $n \geq 1$,

$$(1.3.3) \qquad\quad L(s, \pi_v, \mathrm{sym}^n) = [\prod_{j=0}^{n}(1 - \alpha_v^j \beta_v^{n-j} q_v^{-s})]^{-1}.$$

It is well known that when $r = L$, $r(\pi) \in \mathcal{A}(1, F)$ is given by the central character $\omega = \omega_\pi$ of π. Consequently, if one can establish the lifting for $r = \mathrm{sym}^n(r_0)$, then one can also achieve it for $r = \mathrm{sym}^n(r_0) \otimes L^{\otimes k}$ by twisting by ω^k, i.e., by setting

$$\left(\mathrm{sym}^n(r_0) \otimes L^{\otimes k}\right)(\pi) = \mathrm{sym}^n(\pi) \otimes \omega^k.$$

So it suffices to establish the transfer $\pi \rightarrow r(\pi)$ for $\mathrm{sym}^n(r_0)$ for all n. Clearly, $\mathrm{sym}^1(\pi) = r_0(\pi) = \pi$.

Proposition 1.3.4   *Let π be a cuspidal automorphic representation of $GL(2, \mathbb{A}_F)$ which is associated to a two-dimensional, continuous $\mathbb{C}$-representation ρ of $\mathrm{Gal}(\overline{F}/F)$ so that $L(s, \rho) = L(s, \pi)$. Suppose $\mathrm{sym}^m(\pi)$ exists in $\mathcal{A}(F)$ for every $m \geq 1$. It is then cuspidal iff $\mathrm{sym}^m(\rho)$ is irreducible.*

Proof.   For any continuous finite-dimensional $\mathbb{C}$-representation σ of $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$, one knows (cf. [?]) the following fact about the Artin L-functions:

(a) $\qquad\qquad\qquad \mathrm{Hom}_{\Gamma_F}(\mathbb{C}, \sigma) = -\mathrm{ord}_{s=1} L^S(s, \sigma).$

Applying this to

$$\sigma := \mathrm{sym}^m(\rho) \otimes \mathrm{sym}^m(\overline{\rho}) \simeq \mathrm{End}(\rho),$$

we see by Schur's lemma that

(b) $\mathrm{sym}^m(\rho)$ *irreducible* $\iff -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^m(\rho) \otimes \mathrm{sym}^m(\overline{\rho})) = 1.$

On the other hand, by a result we will prove later in section 3.1 (see Lemma 3.1.1), $\mathrm{sym}^m(\pi)$ is, when it is automorphic, an isobaric sum of *unitary* cuspidal representations. This implies that

(c) $\mathrm{sym}^m(\pi)$ *cuspidal* $\iff -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^m(\pi) \times \mathrm{sym}^m(\overline{\pi})) = 1,$

where the L-function is the Rankin-Selberg L-function (see 1.3 below for its basic properties). Finally, since by hypothesis, ρ corresponds to π, the L-functions of (b) and (c) are the same. The assertion follows.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

One expects the same when ρ is an ℓ-adic Galois representation (attached to π), but this is unknown in general, except for small m (cf. [Ram7, Ram6]). The difficulty here is caused by the image of Galois not (usually) being finite.

As mentioned in the Introduction, a result of Gelbart and Jacquet ([GJ]) that $\mathrm{sym}^2(\pi)$ exists for any $\pi \in \mathcal{A}_0(2, F)$. It is cuspidal iff π is not *dihedral*, i.e., π is not automorphically induced by an idele class character of a quadratic field.

When π is dihedral, it is easy to see that $\mathrm{sym}^m(\pi)$ exists for all m, and that it is an isobaric sum of elements of $\mathcal{A}(1, F)$ and $\mathcal{A}_0(2, F)$. So we may, and we will, henceforth restrict our attention to non-dihedral forms π.

Here is a ground-breaking result due to Kim and Shahidi which we will need:

Theorem 1.3.5 (Kim-Shahidi [KS2], [KS1], Kim [Kim])   *Let $\pi \in \mathcal{A}_0(2, F)$ be non-dihedral. Then $\mathrm{sym}^n(\pi)$ exists in $\mathcal{A}(F)$ for all $n \leq 4$. Moreover, $\mathrm{sym}^3(\pi)$ (resp. $\mathrm{sym}^4(\pi)$) is cuspidal iff π is not tetrahedral (resp. octahedral).*

A non-dihedral π is *tetrahedral* iff $\mathrm{sym}^2(\pi)$ is monomial, while π is *octahedral* if it is not dihedral or tetrahedral but whose symmetric cube is not cuspidal upon base change to some quadratic extension K of F. We will say that π is *solvable polyhedral* if it is either dihedral, or tetrahedral, or octahedral.

1.4. **Rankin-Selberg L-functions.** Let π, π' be isobaric automorphic representations in $\mathcal{A}(n, F)$, $\mathcal{A}(n', F)$ respectively. Then there exists an associated Euler product $L(s, \pi \times \pi')$ ([JPSS2], [JS, JPSS2, Sha2, Sha1, MW]), which converges in some right half plane, even in $\{\Re(s) > 1\}$ if π, π' are unitary and cuspidal. It also admits a meromorphic continuation to the whole $s-$plane and satisfies the functional equation

(1.4.1) $$L(s, \pi \times \pi') \, = \, \varepsilon(s, \pi \times \pi') L(1 - s, \pi^\vee \times \pi'^\vee),$$

with

$$\varepsilon(s, \pi \times \pi') \, = \, W(\pi \times \pi') N(\pi \times \pi')^{\frac{1}{2} - s},$$

where $N(\pi \times \pi')$ is a positive integer not divisible by any rational prime not intersecting the ramification loci of $F/\mathbb{Q}$, π and π', while $W(\pi \times \pi')$ is a non-zero complex number, called the *root number of the pair* (π, π'). As in the Galois case, $W(\pi \times \pi') W(\pi^\vee \times \pi'^\vee) = 1$, so that $W(\pi \times \pi') = \pm 1$ when π, π' are self-dual.

When v is archimedean or a finite place unramified for π, π',

(1.4.2) $$L_v(s, \pi \times \pi') \, = \, L(s, \sigma(\pi_v) \otimes \sigma(\pi'_v)).$$

In the archimedean situation, $\pi_v \to \sigma(\pi_v)$ is the local Langlands correspondence ([La1]), with $\sigma(\pi_v)$ a representation of the Weil group W_{F_v}. When v is an unramified finite place, $\sigma(\pi_v)$ is defined in the obvious way as the sum of one dimensional representations defined by the Langlands class $A(\pi_v)$.

When $n = 1$, $L(s, \pi \times \pi') \, = \, L(s, \pi\pi')$, and when $n = 2$ and $F = \mathbb{Q}$, this function is the usual Rankin-Selberg $L-$function, extended to arbitrary global fields by Jacquet.

Theorem 1.4.3 [JS, JPSS2])   *Let $\pi \in \mathcal{A}_0(n, F)$, $\pi' \in \mathcal{A}_0(n', F)$, and S a finite set of places. Then $L^S(s, \pi \times \pi')$ is entire unless π is of the form $\pi'^\vee \otimes |.|^w$, in which case it is holomorphic outside $s = -w, 1 - w$, where it has simple poles. In particular, if π, π' are unitary cuspidal representations, $L^S(s, \pi \times \pi')$ is holomorphic in $\Re(s) > 1$, and moreover, there is a pole at $s = 1$ iff $\pi' \simeq \pi^\vee \simeq \bar{\pi}$.*

One also knows (cf.[Sha2], that for π, π' *unitary* cuspidal,

(1.4.4) $$L^S(1 + t, \pi \times \pi') \neq 0, \, \forall t \in \mathbb{R}.$$

Clearly, this continues to hold for isobaric sums π, π' of unitary cuspidal representations. (Note that there unitary isobaric representations which are not isobaric sums of unitary cuspidal representations, and the assertion will not hold for these representations.)

1.5. **The (conjectural) automorphic tensor product.** The *Principle of Functoriality* asserts that given isobaric automorphic representations π, π' $\mathrm{GL}_n(\mathbb{A}_F)$, $\mathrm{GL}_{n'}(\mathbb{A}_F)$ respectively, there should exist an isobaric automorphic

representation $\pi \boxtimes \pi'$, called the *automorphic tensor product*, or the *functorial product*, of $GL(nn', \mathbb{A}_F)$ such that

(1.5.1) $$L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi').$$

We will say that an automorphic $\pi \boxtimes \pi'$ is a *weak automorphic tensor product* of π, π' if the identity (1.5.1) of Euler products over F holds outside a finite set S of places, i.e, iff $L(s, \pi_v \boxtimes \pi'_v)$ equals $L(s, \pi_v \times \pi'_v)$ at every $v \notin S$.

The (conjectural) functorial product $\boxtimes$ is the automorphic analogue of the usual tensor product of Galois representations. For the importance of this product, see [Ram1], for example.

One can always define $\pi \boxtimes \pi'$ as an *admissible* representation of $GL_{nn'}(\mathbb{A}_F)$, but the subtlety lies in showing that this product is automorphic. Also, if one knows how to construct it for cuspidal π, π', then one can do it in general.

The automorphy of $\boxtimes$ is known in the following cases, which will be useful to us:

(**1.5.2**)

$(n, n') = (2, 2)$: ([Ram3])
$(n, n') = (2, 3)$: Kim-Shahidi ([KS2])

The reader is also referred to section 11 of [Ram5], which contains some refinements, explanations, and (minor) errata for [Ram3]. Furthermore, it may be worthwhile remarking that Kim and Shahidi effectively use their construction of the functorial product on $GL(2) \times GL(3)$ to prove the automorphy of *symmetric cube* transfer from $GL(2)$ to $GL(4)$, mentioned in section 1.3. A *cuspidality criterion* for the image under this transfer is proved in [RW], with an application to the cuspidal cohomology of congruence subgroups of $SL(6, \mathbb{Z})$.

2. STATEMENT OF THE MAIN RESULT

Here is a more precise, though a bit more cumbersome, version of Theorem A, which was stated in the Introduction.

Theorem A$'$ *Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ of central character ω. Assume, for the first three parts that π is not solvable polyhedral. Then we have the following:*

(a) *If $sym^5(\pi)$ is modular, then it is cuspidal.*
(b) *If $sym^5(\pi)$ and $sym^6(\pi)$ are both modular, then $sym^6(\pi)$ is non-cuspidal iff we have*

$$sym^5(\pi) \simeq Ad(\pi') \boxtimes \pi \otimes \omega^2,$$

   for a cuspidal automorphic representation π' of $GL_2(\mathbb{A}_F)$; in this case, $Ad(\pi')$ and $Ad(\pi)$ are not twist equivalent.
(c) *Let $m \geq 6$, and*
        assume that either

(i) $sym^j(\pi)$ *is modular for every* $j \leq 2m$,
   or

(ii) $\pi \boxtimes \tau$ *is modular for any cusp form* τ *on* $GL(r)/F$, *with* $r \leq \left[\frac{m}{2} + 1\right]$.

   Then $sym^m(\pi)$ *is cuspidal iff* $sym^6(\pi)$ *is cuspidal.*

(d) *If* $F = \mathbb{Q}$ *and* π *is defined by a non-CM, holomorphic newform* φ *of weight* $k \geq 2$, *then* $\mathrm{sym}^m(\pi)$ *is cuspidal whenever it is modular.*

3. Proof of Theorem A′, parts (a)–(c)

3.1. **Two lemmas.** In this and the following sections, S will always denote a finite set of places of F containing the archimedean and finite ramified (for π) places of F.

Lemma 3.1.1 *If* $sym^m(\pi)$ *is weakly modular, then it must be an isobaric sum of unitary cuspidal representations.*

Proof.   Assume $\mathrm{sym}^m(\pi)$ is weakly modular, i.e., for all places v outside a finite set S, $\mathrm{sym}^m(\pi_v)$ is the v-component of an isobaric automorphic representation Π. Suppose Π admits as an isobaric summand Π_0, which is cuspidal but not unitary. In other words, there is a non-zero real number t such that $\Pi_0 \otimes (|\cdot| \circ \det)$ is a unitary cuspidal representation. Then *every* local component $\Pi_{0,v}$ is necessarily non-unitary. As $\Pi_{0,v}$ must be a local isobaric summand of $\mathrm{sym}^m(\pi_v)$ for $v \notin S$, the latter must be non-tempered.

   On the other hand, since π is a cusp form on $GL(2)/F$, we know (cf. [Ram2]) that it contains infinitely many components π_v which are tempered. (In fact, more than $\frac{9}{10}$-th of the components are tempered.) This implies that for any finite set S of places of F, there exist places $v \notin S$ such that $\mathrm{sym}^m(\pi_v)$ is tempered. This gives the desired contradiction, yielding the Lemma.

$\square$

   We will use Lemma 3.1.1 repeatedly, often without specifically referring to it.

Lemma 3.1.2 *Suppose* $sym^r(\pi)$ *is modular for all* $r < m$. *Pick any positive integer* $i \leq m$. *Then* $sym^m(\pi)$ *is modular iff* $sym^i(\pi) \boxtimes sym^{m-i}(\pi)$ *is modular.*

Proof.   Since $\boxtimes$ is commutative, we may assume that $i \leq m/2$. By the Clebsch-Gordon identities, if r_0 denotes the standard 2-dimensional representation of $GL(2, \mathbb{C})$, we have

$$\mathrm{sym}^i(r_0) \times \mathrm{sym}^{m-i}(r_0) \simeq \oplus_{j=0}^i \mathrm{sym}^{m-2j}(r_0) \otimes \det^j.$$

It follows that

$$(3.1.3) \qquad L^S(s, \mathrm{sym}^i(\pi) \times \mathrm{sym}^{m-i}(\pi)) \ = \ \prod_{j=0}^{i} L^S(s, \mathrm{sym}^{m-2j}(\pi) \otimes \omega^j).$$

By hypothesis, $\mathrm{sym}^j(\pi)$ is modular for all $j < m$. If $\mathrm{sym}^m(\pi)$ is also modular, we may set

$$\mathrm{sym}^i(\pi) \boxtimes \mathrm{sym}^{m-i}(\pi) := \boxplus_{j=0}^{i} \mathrm{sym}^{m-2j}(\pi) \otimes \omega^j,$$

which defines the desired automorphic form on $\mathrm{GL}((i+1)(m-i+1))/F$. Conversely, if $\mathrm{sym}^i(\pi) \boxtimes \mathrm{sym}^{m-i}(\pi)$ is modular, then by (3.1.3), it must have a unique isobaric summand Π, with

$$\mathrm{sym}^i(\pi) \boxtimes \mathrm{sym}^{m-i}(\pi) := \Pi \boxplus \left(\boxplus_{j=1}^{i} \mathrm{sym}^{m-2j}(\pi) \otimes \omega^j \right).$$

It follows that at any unramified place v one has, for every integer $k \leq m$ and for every irreducible admissible representation η of $\mathrm{GL}_k(F_v)$, identities of the Rankin-Selberg local factors:

$$L(s, \Pi_v \times \eta) \ = \ L(s, \mathrm{sym}^m(\pi) \times \eta),$$

and

$$\varepsilon(s, \Pi_v \times \eta) \ = \ \varepsilon(s, \mathrm{sym}^m(\pi_v) \times \eta).$$

One gets the weak modularity of $\mathrm{sym}^m(\pi)$. In fact, these identities hold at every place v, as seen by using the local Langlands correspondence for $\mathrm{GL}(n)$ ([HT1], [Hen]). From the local converse theorem, one gets an isomorphism of Π_v with $\mathrm{sym}^m(\pi_v)$. Hence $\mathrm{sym}^m(\pi)$ is modular.

$\square$

3.2. **Proof of part (a) of Theorem A$'$.** By the work of Kim and Shahidi (see section 1), we know that for all $j \leq 4$, $\mathrm{sym}^j(\pi)$ is modular, even cuspidal since π is not solvable polyhedral. By hypothesis, $\mathrm{sym}^5(\pi)$ is modular. Applying Lemma 3.1.2 above with $i = 4$, we get the modularity of $\mathrm{sym}^4(\pi) \boxtimes \pi$. Suppose $\mathrm{sym}^5(\pi)$ is Eisensteinian. Then it must have an isobaric summand τ, say, which is cuspidal on $\mathrm{GL}(r)/F$ for some $r \leq 3$. By Lemma 3.1.1, τ must be unitary. We also know (see section 1) that $\pi \boxtimes \tau^\vee$ is automorphic on $\mathrm{GL}(2r)/F$. Using (3.1.3) we get the identity

$$L^S(s, \mathrm{sym}^4(\pi) \times (\pi \boxtimes \tau^\vee)) \ = \ L^S(s, \mathrm{sym}^5(\pi) \times \tau^\vee) L^S(s, \mathrm{sym}^3(\pi) \otimes \omega \times \tau^\vee).$$

As τ is a (unitary) cuspidal isobaric summand of $\mathrm{sym}^5(\pi)$, the first L-function on the right has a pole at $s = 1$. And by the Rankin-Selberg theory (see (1.4.4)), the second L-function on the right has no zero at $s = 1$. It follows that

$$-\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^4(\pi) \times (\pi \boxtimes \tau^\vee)) \ \geq \ 1.$$

Since $\mathrm{sym}^4(\pi)$ is a cusp form on $\mathrm{GL}(5)/F$, we are forced to have $2r \geq 5$, so $r = 3$. Comparing degrees, we must then have an isobaric sum decomposition

$$\pi^\vee \boxtimes \tau \ \simeq \ \mathrm{sym}^4(\pi) \boxplus \nu,$$

where ν is an idele class character of F. This implies that

$$-\mathrm{ord}_{s=1}L^S(s, \pi^\vee \boxtimes \tau \otimes \nu^{-1}) \geq 1,$$

which is impossible unless $r = 2$ and $\tau \simeq \pi \otimes \nu$. But we have $r = 3$, furnishing the desired contradiction. Hence $\mathrm{sym}^5(\pi)$ must be cuspidal.

$\square$

3.3. **Proof of part (b) of Theorem A$'$.** By hypothesis, $\mathrm{sym}^j(\pi)$ is modular for all $j \leq 6$, even cuspidal for $j \leq 5$ by part (a). By Lemma 1, $\mathrm{sym}^j(\pi) \boxtimes \pi$ is also modular for each $j \leq 5$.

First suppose we have an isomorphism

$$\mathrm{sym}^5(\pi) \simeq \mathrm{sym}^2(\pi') \boxtimes \pi \otimes \nu,$$

for a cusp form π' on $\mathrm{GL}(2)/F$ and an idele class character ν of F. This results in the identity:

$$(3.3.1) \qquad L^S(s, \mathrm{sym}^5(\pi) \boxtimes \pi) = L^S(s, (\mathrm{sym}^2(\pi') \boxtimes \pi) \times \pi \otimes \nu).$$

The L-function on the right is the same as

$$(3.3.2) \qquad L^S(s, \mathrm{sym}^2(\pi') \times \mathrm{sym}^2(\pi) \otimes \nu)L^S(s, \mathrm{sym}^2(\pi') \otimes \omega\nu).$$

As $\mathrm{sym}^2(\pi')^\vee \otimes (\omega\nu)^{-1}$ is equivalent to $\mathrm{sym}^2(\pi') \otimes \omega\nu^{-1}$, we see that by Lemma 3.1.2, $\Pi' := \mathrm{sym}^2(\pi') \boxtimes \mathrm{sym}^2(\pi')^\vee \otimes (\omega\nu)^{-1}$ makes sense as an automorphic form on $\mathrm{GL}(6)/F$. In addition, since $\mathrm{sym}^5(\pi) \boxtimes \pi$ is isomorphic to $\mathrm{sym}^6(\pi) \boxplus (\mathrm{sym}^4(\pi) \otimes \omega)$, we obtain by using (3.3.1) and (3.3.2):

$(3.3.3 - a)$
$$L^S(s, \mathrm{sym}^6(\pi) \times \mathrm{sym}^2(\pi')^\vee \otimes (\omega\nu)^{-1})L^S(s, \mathrm{sym}^4(\pi) \times \mathrm{sym}^2(\pi')^\vee \otimes \nu^{-1})$$

equals

$$(3.3.3 - b) \qquad L^S(s, \Pi' \times \mathrm{sym}^2(\pi'))L^S(s, \mathrm{sym}^2(\pi') \boxtimes \mathrm{sym}^2(\pi')^\vee).$$

The second L-function of (3.3.3-b) has a pole at $s = 1$. And since $\mathrm{sym}^4(\pi)$ is a cusp form on $\mathrm{GL}(5)/F$, the second L-function of (3.3.3-a) has no pole at $s = 1$, and the first L-function of (3.3.3-b) has no zero at $s = 1$. Consequently,

$$(*) \qquad -\mathrm{ord}_{s=1}L^S(s, \mathrm{sym}^6(\pi) \times \mathrm{sym}^2(\pi')^\vee \otimes (\omega\nu)^{-1}) \geq 1.$$

As $\mathrm{sym}^2(\pi')^\vee$ is automorphic on $\mathrm{GL}(3)/F$, $(*)$ cannot hold unless $\mathrm{sym}^6(\pi)$ is not cuspidal. We are done in this direction.

Now let us prove the *converse*, by supposing that $\mathrm{sym}^6(\pi)$ is Eisensteinian. In this case it must admit an isobaric summand τ which is cuspidal on $\mathrm{GL}(k)/F$ with $k \leq 3$. Since we have

$$\mathrm{sym}^6(\pi) \boxplus \mathrm{sym}^4(\pi) \simeq \mathrm{sym}^5(\pi) \boxtimes \pi,$$

τ must be an isobaric summand of $\mathrm{sym}^5(\pi) \boxtimes \pi$. It follows that

$$-\mathrm{ord}_{s=1}L^S(s, \mathrm{sym}^5(\pi) \times (\pi \boxtimes \tau^\vee)) \geq 1,$$

where $\pi \boxtimes \tau^\vee$ is modular since $k \leq 3$. Since $\mathrm{sym}^5(\pi)$ is a cusp form on $\mathrm{GL}(6)/F$, we are forced to have $k = 3$, and moreover,

$$(3.3.4) \qquad\qquad \mathrm{sym}^5(\pi) \simeq \pi^\vee \boxtimes \tau.$$

As $\mathrm{sym}^6(\pi)$ cannot have a $\mathrm{GL}(1)$ isobaric summand, no twist of τ can be an isobaric summand of $\mathrm{sym}^6(\pi)$ either, which has degree 7. On the other hand, since the dual of $\mathrm{sym}^6(\pi)$ is its twist by ω^{-6}, $\tau^\vee$ is an isobaric summand of $\mathrm{sym}^6(\pi) \otimes \omega^{-6}$. So we must have

$$(3.3.5) \qquad\qquad \tau^\vee \simeq \tau \otimes \omega^{-6},$$

showing τ is essentially selfdual. In fact, if we put

$$(3.3.6) \qquad\qquad \eta := \tau \otimes \omega^{-3},$$

it is immediate that η is even *selfdual*. It follows that

$$L^S(s, \eta, \mathrm{sym}^2) L^S(s, \eta, \Lambda^2) = L^S(s, \eta \times \eta^\vee),$$

showing that the left hand side has a pole at $s = 1$. Since η is a cusp form on $\mathrm{GL}(3)/F$, the second L-function cannot have a pole at $s = 1$ (see [JS]). Hence

$$(3.3.7) \qquad\qquad -\mathrm{ord}_{s=1} L^S(s, \eta, \mathrm{sym}^2) \geq 1.$$

By the backwards lifting results of Ginzburg, Rallis and Soudry (cf. [GRS], [?]), we then have a functorially associated cuspidal, necessarily generic, automorphic representation π_0' of $\mathrm{SL}(2, \mathbb{A}_F)$ $(= \mathrm{Sp}(2, \mathbb{A}_F))$ of trivial central character. We may extend it (cf. [?]) to an irreducible cusp form π' of $\mathrm{GL}(2)/F$ (with central character ω'), which is unique only up to twisting by a character, such that

$$(3.3.7) \qquad\qquad L^S(s, \mathrm{Ad}(\pi')) = L^S(s, \eta).$$

By the strong multiplicity one theorem, η is isomorphic to $\mathrm{Ad}(\pi')$, which is $\mathrm{sym}^2(\pi') \otimes \omega'^{-1}$.

Combining with (3.3.4) and (3.3.6), we get

$$\mathrm{sym}^5(\pi) \simeq \mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2,$$

as asserted in part (b) of Theorem A$'$.

Finally suppose $\mathrm{sym}(\pi)$ and $\mathrm{Ad}(\pi')$ are twist equivalent. Then $\mathrm{sym}^5(\pi)$ would need to be twist equivalent to $\mathrm{sym}^2(\pi) \boxtimes \pi$, which is Eisensteinian of the form $\mathrm{sym}^3(\pi) \boxplus \pi \otimes \omega$. This contradicts the cuspidality of $\mathrm{sym}^5(\pi)$, and we are done.

$\square$

3.4. **Proof of part (c) under Assumption (i).** There is nothing to prove if $m = 6$, so let $m \geq 7$, and assume by induction that the conclusion holds for all $n \leq m-1$. In particular, $\mathrm{sym}^n(\pi)$ is cuspidal for every $n < m$. Moreover, by hypothesis, $\mathrm{sym}^j(\pi)$ is modular for all $j \leq 2m$, and this implies, by Lemma 3.1.2, that $\mathrm{sym}^m(\pi) \boxtimes \mathrm{sym}^m(\pi)$ is modular.

Suppose $\mathrm{sym}^m(\pi)$ is not cuspidal. Then by [JS],

$$(3.4.1) \qquad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^m(\pi) \times \mathrm{sym}^m(\pi)^\vee) \geq 2.$$

We have by Clebsch-Gordon,

$$\mathrm{sym}^m(\pi) \boxtimes \mathrm{sym}^m(\pi)^\vee \simeq \boxplus_{j=0}^m \mathrm{sym}^{2j}(\pi) \otimes \omega^{-j},$$

and of course we have a similar formula for $\mathrm{sym}^{m-1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi)^\vee$, where the sum goes from $j = 0$ to $j = m - 1$. Consequently,
$$(3.4.2)$$
$$\mathrm{sym}^m(\pi) \boxtimes \mathrm{sym}^m(\pi)^\vee \simeq \left(\mathrm{sym}^{m-1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi)^\vee\right) \boxplus \left(\mathrm{sym}^{2m}(\pi) \otimes \omega^{-m}\right).$$

Since $\mathrm{sym}^{m-1}(\pi)$ is cuspidal, $L^S(s, \mathrm{sym}^{m-1}(\pi) \times \mathrm{sym}^{m-1}(\pi)^\vee)$ has a simple pole at $s = 1$ (cf. [JS]). Combining this with (3.4.1) and (3.4.2), we obtain

$$(3.4.3) \qquad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{2m}(\pi) \otimes \omega^{-m}) \geq 1.$$

Since $\mathrm{sym}^{2m}(\pi)$ is automorphic, it must admit ω^m as an isobaric summand.

On the other hand, we have (by Clebsch-Gordon)

$$(3.4.4) \qquad \mathrm{sym}^{m+1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi) \simeq \boxplus_{j=0}^{m-1} \mathrm{sym}^{2(m-j)}(\pi) \otimes \omega^j.$$

It follows that ω^m must be an isobaric summand of $\mathrm{sym}^{m+1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi)$, implying

$$(3.4.5) \qquad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{m+1}(\pi) \times \left(\mathrm{sym}^{m-1}(\pi) \otimes \omega^{-m}\right)) \geq 1.$$

Since $\mathrm{sym}^{m-1}(\pi)$ is cuspidal, this can only happen (cf. [JS]) if $\mathrm{sym}^{m-1}(\pi)^\vee \otimes \omega^m$ is an isobaric summand of $\mathrm{sym}^{m+1}(\pi)$. Therefore

$$\mathrm{sym}^{m+1}(\pi) \simeq \left(\mathrm{sym}^{m-1}(\pi)^\vee \otimes \omega^m\right) \boxplus \tau,$$

where τ is an (isobaric) automorphic form on $\mathrm{GL}(2)/F$.

Hence τ is an isobaric summand of $\mathrm{sym}^m(\pi) \boxtimes \pi$, which is isomorphic to $\mathrm{sym}^{m+1}(\pi) \boxplus \left(\mathrm{sym}^{m-1}(\pi) \otimes \omega\right)$. Recall that $\pi^\vee \boxtimes \tau$ is modular. Then there is an isobaric summand β of $\pi^\vee \boxtimes \tau$, which is cuspidal on $\mathrm{GL}(r)/F$ with $r \leq 4$, such that

$$-\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^m(\pi) \times \beta^\vee) \geq 1.$$

In other words, β is an isobaric summand of $\mathrm{sym}^m(\pi)$, and hence of $\mathrm{sym}^{m-1}(\pi) \boxtimes \pi$. Consequently,

$$(3.4.6) \qquad -\mathrm{ord}_{s=1} L^S(s, \left(\mathrm{sym}^{m-1}(\pi) \boxtimes \pi\right) \times \beta^\vee) \geq 1.$$

First suppose $r \leq 3$. Then we know that $\pi \boxtimes \beta^\vee$ is modular on $\mathrm{GL}(2r)$ (by [Ram3] for r=2, and [KS2] for $r = 3$). As $\mathrm{sym}^{m-1}(\pi)$ is by induction cuspidal, (3.4.6) forces the bound

$$(3.4.7) \qquad\qquad m \leq 2r \leq 6.$$

So we are done in this case.

Next suppose that $r = 4$, which means $\beta = \pi^\vee \boxtimes \tau$ is cuspidal. Since $\pi \boxtimes \pi^\vee \simeq \mathrm{sym}^2(\pi) \boxplus \omega$, it follows that $\pi \boxtimes \beta^\vee$ is modular, with

$$\pi \boxtimes \beta^\vee \simeq \left(\mathrm{sym}^2(\pi) \boxtimes \tau^\vee\right) \boxplus \left(\omega \otimes \tau^\vee\right),$$

where the first summand is on $\mathrm{GL}(6)/F$ and the second on $\mathrm{GL}(4)$. As a result, we have from (3.4.6),

$$(3.4.8) \qquad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{m-1}(\pi) \times \delta) \geq 1,$$

for an isobaric summand δ of $\pi \boxtimes \beta^\vee$, which is a cusp form on $\mathrm{GL}(n)$, for some $n \leq 6$. So, once again, the inequality (3.4.7) holds, and we are done.

$\square$

3.5. **Proof of part (c) under Assumption (ii).** The proof of part (c) in this case is a bit different because we are not assuming good properties of $\mathrm{sym}^j(\pi)$ for j all the way up to $2m$.

We may take $m > 6$ and assume by induction that $\mathrm{sym}^j(\pi)$ is cuspidal for all $j \leq m-1$. Suppose $\mathrm{sym}^m(\pi)$ is Eisensteinian. Then it must have an isobaric summand η, which is cuspidal on $\mathrm{GL}(r)/F$ with $r \leq \left[\frac{m+1}{2}\right]$. Then η must be an isobaric summand of $\mathrm{sym}^{m-1}(\pi) \boxtimes \pi$, because of the decomposition

$$\mathrm{sym}^{m-1}(\pi) \boxtimes \pi \simeq \mathrm{sym}^m(\pi) \boxplus \left(\mathrm{sym}^{m-2}(\pi) \otimes \omega\right).$$

By our hypothesis, $\pi \boxtimes \eta^\vee$ is modular on $\mathrm{GL}(2r)/F$. So we get

$$(3.5.1) \qquad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{m-1}(\pi) \times (\pi \boxtimes \eta^\vee)) \geq 1.$$

As $\mathrm{sym}^{m-1}(\pi)$ is cuspidal, we are forced to have

$$(3.5.2) \qquad m \leq 2r \leq m+1.$$

So the only possible (isobaric) decomposition of $\mathrm{sym}^m(\pi)$ we can have is

$$(3.5.3) \qquad \mathrm{sym}^m(\pi) \simeq \eta \boxplus \eta',$$

with

$$\eta \in \mathcal{A}_0([(m+1)/2], F) \quad \text{and} \quad \eta' \in \mathcal{A}_0(m+1 - [(m+1)/2], F).$$

In addition, by our hypothesis, $\eta \boxtimes \pi^\vee$ and $\eta' \boxtimes \pi^\vee$ are modular. We deduce that

$$(3.5.4) \qquad [\mathrm{sym}^{m-1}(\pi), \eta \boxtimes \pi^\vee] > 0, \quad \text{and} \quad [\mathrm{sym}^{m-1}(\pi), \eta' \boxtimes \pi^\vee] > 0.$$

First consider the case when m *is odd*. (This is similar to the argument above for $m = 5$.) Then $r = [(m+1)/2] = m + 1 - [(m+1)/2]$, and η, η' are both in $\mathcal{A}_0((m+1)/2, F)$. Since $\mathrm{sym}^{m-1}(\pi) \in \mathcal{A}_0(m, F)$, we must have

$$\eta \boxtimes \pi^\vee \simeq \mathrm{sym}^{m-1}(\pi) \boxplus \mu$$

and

$$\eta' \boxtimes \pi^\vee \simeq \mathrm{sym}^{m-1}(\pi) \boxplus \mu',$$

with μ, μ' in $\mathcal{A}(1, F)$. Then it follows that the Rankin-Selberg L-functions $L^S(s, \eta \times (\pi^\vee \otimes \mu^{-1}))$ and $L^S(s, \eta' \times (\pi^\vee \otimes {\mu'}^{-1}))$ both have poles at $s = 1$. This forces the following:

$$m = 3, \ \eta \simeq \pi \otimes \mu, \quad \text{and} \quad \eta' \simeq \pi \otimes \mu'.$$

So this cannot happen for $m \neq 3$.

Next consider the case when m *is even*. Then $\eta \in \mathcal{A}_0(m/2, F)$ and $\eta' \in \mathcal{A}_0(m/2 + 1, F)$. We get

$$\eta \boxtimes \pi^\vee \simeq \text{sym}^{m-1}(\pi)$$

and

$$\eta' \boxtimes \pi^\vee \simeq \text{sym}^{m-1}(\pi) \boxplus \tau,$$

with τ in $\mathcal{A}_0(2, F)$. Then η' must occur in $\pi \boxtimes \tau$, which is in $\mathcal{A}(4, F)$. So we must have

$$m/2 + 1 \leq 4.$$

In other words, m must be less than or equal to 6, which is not the case.

Thus we get a contradiction in either case. The only possibility is for $\text{sym}^m(\pi)$ to be cuspidal. Done proving part (c), and hence all of Theorem B.

$\square$

4. Proof of Theorem A′, part (d)

Finally, we want to restrict to $F = \mathbb{Q}$ and analyze the case of holomorphic newforms f of weight ≥ 2. One knows that the level N of f is the same as the conductor of the associated cuspidal automorphic representation π of $\text{GL}(2, \mathbb{A}_\mathbb{Q})$. Moreover, as f is not of CM type, π is not dihedral.

Fix a prime ℓ not dividing N and consider the *cyclotomic character*

$$(4.1) \qquad\qquad \chi_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_\ell^*,$$

defined by the Galois action on the projective system $\{\mu_{\ell^r} | r \geq 1\}$, where μ_{ℓ^r} denotes the group of ℓ^r-th roots of unity in $\overline{\mathbb{Q}}$. Then by a theorem of Deligne, one has at our disposal an irreducible, continuous representation

$$(4.2) \qquad\qquad \rho_\ell(\pi) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(2, \overline{\mathbb{Q}}_\ell),$$

unramified outside $N\ell$, such that for every prime p not dividing $N\ell$,

$$(4.3) \qquad\qquad \text{Tr}(\rho_\ell(\pi)(\text{Fr}_p)) = a_p,$$

where Fr_p denotes the Frobenius at p and a_p the p-th Hecke eigenvalue of f. Moreover,

$$(4.4) \qquad\qquad \det(\rho_\ell(\pi) = \omega \chi_\ell^{k-1}.$$

When f is of CM-type, there exists an imaginary quadratic field K, and an algebraic Hecke character Ψ of K such that

$$(4.5) \qquad \rho_\ell(\pi) \simeq \operatorname{Ind}_{Gal(\overline{\mathbb{Q}}/K)}^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}(\Psi_\ell),$$

where Ψ_ℓ is the ℓ-adic character associated to Ψ ([Ser]). Let θ denote the non-trivial automorphism of $\operatorname{Gal}(K/\mathbb{Q})$. Then it is an immediate exercise to check that for any $m \geq 1$, $\operatorname{sym}^m(\rho_\ell)$ is of the form $\oplus_j \beta_{j,\ell}$, where each $\beta_{j,\ell}$ is either one-dimensional defined by an idele class character of $\mathbb{Q}$ or a two-dimensional induced by $\Psi_\ell^a(\Psi_\ell^\theta)^{m-a}$ for some $a \geq 0$, with Ψ_ℓ^θ denoting the conjugate of Ψ_ℓ under θ. It is clear this is modular, but not cuspidal for any $m \geq 2$.

Let us assume henceforth that f is not of CM-type. Denote by G_ℓ the *Zariski closure of the image* of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ under $\rho_\ell(\pi)$; it is an ℓ-adic Lie group. Since f is of weight ≥ 2 and *not* of CM-type, a theorem of K. Ribet ([Rib]) asserts that for *large enough* ℓ,

$$(4.6) \qquad G_\ell = \operatorname{GL}(2, \overline{\mathbb{Q}}_\ell).$$

We will from now on consider only those ℓ large enough for this to hold. Since the symmetric power representations of the algebraic group GL(2) are irreducible, we get the following

Lemma 4.7 *For any non-CM newform f of weight $k \geq 2$ and for any $m \geq 1$ and large enough ℓ, the representation $\operatorname{sym}^m(\rho_\ell)$ is irreducible, and it remains so under restriction to $\operatorname{Gal}(\overline{\mathbb{Q}}/E)$ for any finite extension E of $\mathbb{Q}$.*

Since f is not of CM-type, $\operatorname{sym}^2(\pi)$ is cuspidal. In view of parts (a)–(c) (of Theorem A′), we need only prove the following to deduce part (d):

Proposition 4.8 *For any non-CM newform f of weight $k \geq 2$ and level N, with associated cuspidal automorphic representation π of $GL(2, \mathbb{A}_\mathbb{Q})$, assume that $\operatorname{sym}^m(\pi)$ is modular for all $m \geq 2$. Then the following hold:*

(i) *For any quadratic field K, the base change $\operatorname{sym}^3(\pi)_K$ to $GL(4)/K$ is cuspidal*

(ii) *$\operatorname{sym}^6(\pi)$ is cuspidal*

This Proposition suffices, because (i) implies that π is not solvable polyhedral, and (ii) implies what we want by part (c) of Theorem A′.

Let f be as in the Proposition. Suppose $m \geq 1$ is such that $\operatorname{sym}^j(\pi)$ is cuspidal for all $j < m$, but Eisensteinian for $j = m$. Then we have, as in the proof of the earlier parts of Theorem A′, a decomposition

$$(4.9) \qquad \operatorname{sym}^m(\pi) \simeq \eta \boxplus \eta',$$

with

$$\eta \in \mathcal{A}_0([(m+1)/2], \mathbb{Q}) \quad \text{and} \quad \eta' \in \mathcal{A}_0(m+1-[(m+1)/2], \mathbb{Q}),$$

with η, η' are essentially self-dual. Moreover, we have

Lemma 4.10 *The infinity types of η, η' are both algebraic and regular.*

Some explanation of the terminology is called for at this point. Recall that $W_{\mathbb{R}}$ is the unique non-split extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^*$, which is concretely described as $\mathbb{C}^* \cup j\mathbb{C}^*$, with $jzj^{-1} = \overline{z}$, for all $z \in \mathbb{C}^*$. Let Π be an irreducible automorphic representation of $\mathrm{GL}(n, \mathbb{A}_F)$. Since the restriction of $\sigma_\infty(\Pi)$ is semisimple and since $\mathbb{C}^*$ is abelian, we get a decomposition

$$\sigma_\infty(\Pi)|_{\mathbb{C}^*} \simeq \oplus_{i \in J} \chi_i,$$

where each χ_i is in $\mathrm{Hom}_{cont}(\mathbb{C}^*, \mathbb{C}^*)$. Π_∞ is said to be **regular** iff this decomposition is multiplicity-free, i.e., iff $\chi_i \neq \chi_r$ for $i \neq r$. It is **algebraic** ([Clo]) iff each $\chi_i| \cdot |^{(m-1)/2}$ is of the form $z \to z^{-a_i}\overline{z}^{-b_i}$, for some integers a_i, b_i. An algebraic Π is said to be **pure** if there is an integer w, called the **weight** of Π, such that $w = a_i + b_i$ for each $i \in J$.

It is well known that, since π is defined by a holomorphic newforms f of weight $k \geq 2$,

$$(4.11) \qquad\qquad \sigma_\infty(\pi) \otimes | \cdot |^{-1/2} \simeq \mathrm{Ind}(W_{\mathbb{R}}, \mathbb{C}^*; z_{1-k}),$$

where z_n denotes, for each integer n, the continuous homomorphism $\mathbb{C}^* \to \mathbb{C}^*$ given by $z \to z^n$. Note that π_∞ is regular (as $k > 1$) and algebraic of weight $k - 1$. From here on to the end of this chapter, we will simply write $I(-)$ for $\mathrm{Ind}(W_{\mathbb{R}}, \mathbb{C}^*; -)$. Set

$$\nu_{1-k} = z_{1-k}|_{\mathbb{R}^*}.$$

Then we have

$$(4.12) \qquad\qquad \omega_\infty = (\mathrm{sgn})\nu_{1-k},$$

where sgn denotes the sign character of $\mathbb{R}^*$. Indeed, $\omega_\infty = \mathrm{sgn}^{1-k}\nu_{1-k}$. But as f has trivial character, k is forced to be even, so $\mathrm{sgn}^{1-k} = \mathrm{sgn}$. (Here we have identified, as we may, ω_∞ with $\sigma_\infty(\omega)$.)

SubLemma 4.13 *For each $j \leq [m/2]$,*
(i)
$$\sigma_\infty(sym^{2j+1}(\pi)) \simeq I(z_{1-k}^{2j+1}) \oplus (I(z_{1-k}^{2j-1}) \otimes |.|^{1-k}) \oplus \ldots \oplus (I(z_{1-k}) \otimes |.|^{(1-k)j}),$$
and
(ii)
$$\sigma_\infty(sym^{2j}(\pi)) \simeq I(z_{1-k}^{2j}) \oplus (I(z_{1-k}^{2j-2}) \otimes |.|^{1-k}) \oplus \ldots \oplus (I(z_{1-k}^2) \otimes |.|^{(1-k)(j-1)}) \oplus \nu_{1-k}^j.$$

Proof of SubLemma.   Everything is fine for $j = 0$. So we may let $j > 0$ and assume by induction that the identities hold for all $r < j$. Applying (i) for $j - 1$ together with $(4.3)_{2j}$, (4.11) and (3.19), we see that

$$\sigma_\infty(sym^{2j}(\pi)) \oplus (\sigma_\infty(sym^{2j-2}(\pi)) \otimes |.|^{1-k})$$

is isomorphic to

$$(I(z_{1-k}^{2j-1}) \oplus (I(z_{1-k}^{2j-3}) \otimes |.|^{1-k}) \oplus \ldots \oplus (I(z_{1-k}) \otimes |.|^{(1-k)(j-1)}) \otimes I(z_{1-k}).$$

By Mackey theory, we have for all $a \geq b$,

$$I(z_{1-k}^a) \otimes I(z_{1-k}^b) \simeq I(z_{1-k}^{a+b}) \oplus I(z_{1-k}^a \bar{z}_{1-k}^b) \simeq I(z_{1-k}^{a+b}) \oplus (I(z_{1-k}^{a-b}) \otimes |.|^{1-k}).$$

Since $I(-) \otimes \mathrm{sgn} \simeq I(-)$, $I(-) \otimes |.|^{1-k}$ is isomorphic to $I(-) \otimes \nu_{1-k}$. Combining these and using the inductive assumption for $\sigma(sym^{2j-2}(\pi))$, we get (ii) for j. The proof of (ii) is similar and left to the reader.

$\square$

Now Lemma 4.10 follows easily from the SubLemma and the definition of regular algebraicity.

Proof of Proposition (contd.)   We need only examine $sym^m(\pi)$ for $m = 3$ and $m = 6$.

First suppose $m = 3$. Let K be any quadratic field. Then η_K and η'_K are both essentially self-dual forms on $GL(2)/K$ with algebraic, regular infinity types. Consequently, one knows that for $\beta \in \{\eta, \eta'\}$, there exists a semisimple representation

$$\rho_\ell(\beta) : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \rightarrow GL(2, \overline{\mathbb{Q}}_\ell)$$

such that for primes P in a set of Dirichlet density 1, we have

$$(4.14) \qquad L(s, \beta_P) = \det(1 - \mathrm{Fr}_P(NP)^{-s}|\rho_\ell(\beta))^{-1}.$$

If β is Eisensteinian, which in fact cannot happen, this is easy to establish. Ditto if it is dihedral. So we may take β to be cuspidal and non-dihedral. If K is totally real, the existence of $\rho_\ell(\beta)$ is a well known result, due independently to R. Taylor ([Tay1]) and to Blasius-Rogawski ([BR]); in fact a stronger assertion holds in that case. In this case, β corresponds to a Hilbert modular form, either one of weight $3k-2$ or to a twist of one of weight $3k-4$. If K is imaginary, the existence of $\rho_\ell(\beta)$ is a theorem of R. Taylor ([Tay2]), partly based on his joint work with M. Harris and D. Soudry. (Note that here, the central character of the unitary version of β is trivial.)

By part (a) of the Lemma, we then get the following at all primes P in a set of density 1:

$$(4.15) \qquad L(s, sym^3(\pi_K)_P) = \det(1 - \mathrm{Fr}_P(NP)^{-s}|\rho_\ell(\eta) \oplus \rho_\ell(\eta'))^{-1}.$$

But by construction,

$$(4.16) \qquad L(s, sym^3(\pi_K)_P) = \det(1 - \mathrm{Fr}_P(NP)^{-s}|sym^3(\rho_\ell(\pi)_K))^{-1}.$$

Thus we have, by the Tchebotarev density theorem,

$$sym^3(\rho_\ell(\pi)_K) \simeq \rho_\ell(\eta) \oplus \rho_\ell(\eta').$$

We get a contradiction as we know (cf. Lemma 4.7) that $sym^3(\rho_\ell(\pi)_K)$ is an irreducible representation.

Thus $sym^3(\pi_K)$ is cuspidal. This proves part (i) of the Proposition, and implies that π is not solvable polyhedral.

Next we turn to the question of cuspidality of $sym^6(\pi)$. Again, thanks to the hypothesis of modularity $sym^6(\pi)$, $sym^j(\pi)$ is cuspidal for all $j \leq 5$.

Suppose s$ym^6(\pi)$ is not cuspidal. Let η, η' be as in the the decomposition s$ym^m(\pi)$ given by (4.9). Since $m = 6$, $\eta \in \mathcal{A}_0(3, \mathbb{Q})$ and $\eta' \in \mathcal{A}_0(4, \mathbb{Q})$. Specializing Lemma 3.1.2 to $(i, m) = (5, 6)$, we get

$$(4.17) \qquad \mathrm{s}ym^5(\pi) \boxtimes \pi \simeq \eta \boxplus \eta' \boxplus (\mathrm{s}ym^4(\pi) \otimes |.|^{1-k}).$$

Lemma 4.18 Let $\beta \in \{\eta, \eta'\}$. Take $m = 3$ if $\beta = \eta$ and $m = 4$ if $\beta = \eta'$. Then for any prime ℓ away from the ramification locus of β, there exists a semisimple ℓ-adic representation

$$\rho_\ell(\beta) : \mathrm{G}al(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{G}L(m, \overline{\mathbb{Q}}_\ell)$$

such that for almost all primes p, we have

$$(4.19) \qquad L(s, \beta_p) = \mathrm{d}et(1 - \mathrm{F}r_p p^{-s}|\rho_\ell(\beta))^{-1}.$$

Proof of Lemma.   First Note that since the dual of s$ym^6(\pi)$ is s$ym^6(\pi) \otimes \omega^{-6}$, the twisted representation s$ym^6(\pi) \otimes \omega^{-3}$ is selfdual. So, we may, after replacing s$ym^6(\pi)$, η and η' by their respective twists by ω^3, assume that they are all selfdual. (Since η, η' are irreducible representations of unequal dimensions, they cannot be contragredients of each other, and so are forced to be selfdual themselves.) As we have seen, they are also regular and algebraic. Now the discussion in [Ram7] explains how to deduce the existence of the desired Galois representations attached to η, η' (see also [RS, Ram4, Lau, Wei]).

$\square$

Proof of Proposition 4.8 (contd.).   Applying Lemma 4.18 we get for almost all primes p,

$$L(s, \mathrm{s}ym^6(\pi)_p) = \mathrm{d}et(1 - \mathrm{F}r_p p^{-s}|\rho_\ell(\eta) \oplus \rho_\ell(\eta'))^{-1}.$$

By the Tchebotarev density theorem,

$$\mathrm{s}ym^6(\rho_\ell(\pi)) \simeq \rho_\ell(\eta) \oplus \rho_\ell(\eta').$$

Again we get a contradiction since by Lemma 4.7, s$ym^6(\rho_\ell(\pi))$ is an irreducible representation.

Thus s$ym^6(\pi)$ is cuspidal.

$\square$

We have now completely proved Theorem A$'$, which implies Theorem A of the Introduction.

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Dinakar Ramakrishnan
253-37 Caltech
Pasadena, CA 91125, USA.
dinakar@caltech.edu