MODULARITY OF THE RANKIN-SELBERG L-SERIES, AND MULTIPLICITY ONE FOR SL(2)

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Contents

1. Introduction

Let f, g be primitive cusp forms, holomorphic or otherwise, on the upper half plane $\mathcal{H}$ of levels N, M repectively, with (unitarily normalized) L-functions

$$L(s,f) \;=\; \sum_{n\geq 1} \frac{a_n}{n^s} \;=\; \prod_{p \,\mathrm{prime}} [(1-\alpha_p p^{-s})(1-\beta_p p^{-s})]^{-1}$$

and

$$L(s,g) \;=\; \sum_{n\geq 1} \frac{b_n}{n^s} \;=\; \prod_{p \,\mathrm{prime}} [(1-\alpha'_p p^{-s})(1-\beta'_p p^{-s})]^{-1}.$$

When p does not divide N (resp. M), the inverse roots α_p, β_p (resp. α'_p, β'_p) are non-zero with sum a_p (resp. b_p). For every p prime to NM, set

$$L_p(s, f\times g) \;=\; [(1-\alpha_p\alpha'_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\beta_p\alpha'_p p^{-s})(1-\beta_p\beta'_p p^{-s})]^{-1}.$$

Let $L^*(s, f \times g)$ denote the (incomplete Euler) product of $L_p(s, f \times g)$ over all p not dividing NM. This is closely related to the convolution L-series $\sum_{n\geq 1} a_n b_n n^{-s}$, whose miraculous properties were first studied by Rankin and Selberg.

A fundamental question, first raised by Langlands, is to know whether this Rankin-Selberg product is modular, i.e., if there exists an automorphic form $f \boxtimes g$ on $\mathrm{GL}(4)/\mathbb{Q}$ whose standard L-function equals $L^*(s, f \times g)$ after removing the ramified and archimedean factors. The first main result of this paper is to answer it in the affirmative, in fact with the base field $\mathbb{Q}$ replaced by any number field F (see Theorem M, section 3). Our proof uses a mixture of converse theorems, base change and descent, and it also appeals to the local regularity properties of Eisenstein series and the scalar products of their truncations. Briefly, experts have long suspected that this result should follow from the converse theorem for GL(4) requiring only twists by GL(2) and GL(1), which has recently been published by Cogdell and Piatetski-Shapiro ([CoPS]). While this is morally true, *three difficulties* crop up when one tries to implement this, and new ideas are required to surmount them. The *first difficulty* arises because, given cuspidal automorphic representations π, π' on $\mathrm{GL}(2)/F$, one needs to have a definition of a candidate for $\pi \boxtimes \pi'$ as an *admissible* representation of $\mathrm{GL}(4, \mathbb{A}_F)$ (which is needed before we can test its analytic properties to show modularity). If one admits the local Langlands correspondence, then the local candidates $\pi_v \boxtimes \pi'_v$ can be defined as the admissible representations of $\mathrm{GL}(4, F_v)$ corresponding to $\sigma_v \otimes \sigma'_v$, where σ_v (resp. σ'_v) is the 2-dimensional representation of the Weil-Deligne group W'_{F_v} associated to π_v (resp. π'_v) by Kutzko ([Ku]). (Since the work on this paper was completed, a preprint of M. Harris and R.L. Taylor has appeared, establishing the local correspondence for $\mathrm{GL}(n)$. But we feel that it will be satisfying not to to have to appeal to it here; global arguments should always be able to circumvent fine local difficulties.) We get around this problem by appealing to the *base change* results of Arthur and Clozel ([AC]). To be precise, we first make use of the fact that every supercuspidal representation becomes a principal series representation after a finite normal solvable base change, so that a candidate for $\pi \boxtimes \pi'$ can be defined over suitable (infinite families of) global, solvable extensions K/F. Then, after proving modularity upstairs, we perform a simultaneous descent via an inductive argument in cyclic layers. At the end, we get as a byproduct the definition of $\pi_v \boxtimes \pi'_v$ at *any* v (and any F). Once one has the admissible Π on $\mathrm{GL}(4)/K$, one needs, for modularity via the converse theorem, good analytic information on $L(s, \Pi \times \eta)$, for any cuspidal η on $\mathrm{GL}(m)/K$, for $m \leq 2$. The $m = 1$ case is easy by the Rankin-Selberg theory as extended to general K by Jacquet ([J]). But the $m = 2$ case is subtle and leads to problems. One of them, which is the *second difficulty*, is caused by not knowing the equality, at **every** place, with each other and with $L(s, \Pi \times \eta)$, of the *three candidates* for the *triple product L-function* $L(s, \pi \times \pi' \times \eta)$. The first is defined formally as an Euler product of $\{L(s, \sigma_v \otimes \sigma'_v \otimes \tau_v)\}$, where τ_v is the 2-dimensional of W'_{F_v} associated to η_v; the second candidate is defined by the integral representation of Garrett ([G]), as generalized by Piatetski-Shapiro and Rallis ([PS-R2]); and the third is given by the machinery of Langlands-Shahidi ([La2], [Sh1]). Though the unramified local factors are known to be the same in all cases, something close to an equality is essential as no candidate has all the desired properties, and they have complementary strengths. (One simply cannot avoid dealing with the bad places!) By a careful

analysis and synthesis of known results due to Ikeda and others, and by using in addition some global arguments involving the works of Langlands and Tunnell on the Artin conjecture, we manage to prove at the end that the *first* candidate has *all* the desired properties under some local restrictions which can be achieved by solvable base change. So base change is used for yet another reason! (At the archimedean places, we make use of the work of Ikeda ([Ik2]) on the computation of the triple product L-factors for unramified representations.) We prove later the equality, at *each* place v, of $L(s, \pi_v \times \pi'_v \times \eta_v)$ (resp. $\varepsilon(s, \pi_v \times \pi'_v \times \eta_v)$) with the $\mathrm{GL}(4)\times\mathrm{GL}(2)$-factor $L(s, (\pi_v \boxtimes \pi'_v) \times \tau_v)$ (resp. $\varepsilon(s, (\pi_v \boxtimes \pi'_v) \times \eta_v)$). The *third* and final *difficulty* is the question of boundedness in vertical strips of the triple product L-function. This is absolutely crucial for applying the converse theorem, and this does not seem to be elementary as, for example, in the case of the standard L-function of $\mathrm{GL}(2)$. Our method is to use Arthur's truncation ([A1]) of the non-cuspidal Eisenstein series $E(g, s)$ on $\mathrm{GSp}(6)/F$ which occurs in the integral representation of $L(s, \pi \times \pi' \times \eta)$, and reduce the problem, via local regularity of eigenfunctions of the Laplacian, to the norms of the truncated Eisenstein series, making use of the fact that the intervening constant terms are in our case expressible in terms of abelian L-functions. The reader is referred to section 3.1 for a fuller discussion of the strategy of proof of the various parts. The details of proof occupy the sections 3.2- 3.7.

As a consequence of Theorem M, we settle a conjecture of Labesse and Langlands ([LL], [Lab1]) asserting that the space of cusp forms on $\mathrm{SL}(2)$ has multiplicity one (see Theorem 4.1.1). To see what this means concretely, consider f, g as above with trivial characters such that

$$a_p^2 \,=\, b_p^2$$

for almost all p. Then multiplicity one (in this context) implies that there exists a quadratic Dirichlet character χ such that (for almost all p)

$$a_p \,=\, b_p \chi(p).$$

(In fact, if N, M are in addition square-free, χ must be trivial.) When we started on this project in 1994, we were able to settle quickly the case when f and g are holomorphic, but then learnt that this case had been known to various people including D. Blasius and J.-P. Serre; here the idea is to make use of the associated ℓ-adic Galois representations (see [K], pp.90-91, and [Ra2], where there is also a mod ℓ analog and a density result). But this method *does not* work for Maass forms, and the starting point for this paper was our realization in Fall 1994 that both cases could be tackled simultaneously if one knew of the existence of $f \boxtimes g$. We first managed to prove multiplicity one for $\mathrm{SL}(2)$ over $\mathbb{Q}$ in Fall 96 by some special tricks and a weaker form of Theorem M. The proof given here works over arbitrary F, but is shorter, partly because some the earlier arguments over $\mathbb{Q}$ have been transplanted to the proof of Theorem M. If there is any creativity in this paper, it is perhaps foremost in the application of Theorem M to this problem, though it is not the most technically difficult part. It should also be remarked that it has been expected for some time now that the Labesse-Langlands conjecture should follow from the adjoint square lifting from $\mathrm{SL}(2)$ to $\mathrm{PGL}(3)$, more precisely from a careful comparison of the stable trace formula for $\mathrm{SL}(2)$ and the twisted trace formula for $\mathrm{PGL}(3)$ relative to $g \to {}^t g^{-1}$. This approach has been considered by Y. Flicker in a series of papers culminating in [F]. But we ignore the question of whether this program has been completed, as our approach is *totally* different and (hopefully)

has independent interest. We would also like to note in passing that, as shown by D. Blasius ([B]), multiplicity one fails for SL(n), for every $n > 2$.

A corollary of Theorem M is the deduction of the standard analytic properties of 4-fold convolutions $L(s, f_1 \times f_2 \times f_3 \times f_4)$ for quadruples (f_1, f_2, f_3, f_4) of primitive cusp forms. This is done by appealing, in addition, to the higher Rankin-Selberg theory for GL(4)×GL(4) due to Jacquet, Piatetski-Shapiro and Shalika ([JPSS1]), and Shahidi ([Sh 1,3,5]). We show, moreover, that $L(s, \mathrm{sym}^2(f_1) \times f_2 \times f_3)$ and $L(s, \mathrm{sym}^3(f_1) \times f_2)$ have meromorphic continuations and functional equations (see Theorem 4.2.1) and are non-zero in $\{\Re(s) \geq 1\}$ without any pole there except possibly at $s = 1$.

By using the existence of $\boxtimes$ on GL(2)×GL(2) in conjunction with a global argument we prove an identity, even at the ramified places v, equating the Rankin-Selberg L and ε-factors of $(\pi_{1,v} \boxtimes \pi_{2,v}, \pi_{3,v} \boxtimes \pi_{4,v})$ with those of $\sigma_{1,v} \otimes \sigma_{2,v} \otimes \sigma_{3,v} \otimes \sigma_{4,v}$, where $\sigma_{j,v}$ denotes, for each $j \leq 4$, the 2-dimensional representation of W'_{F_v} associated to $\pi_{j,v}$ (cf. [Ku]). This gives a little bit of information (see Proposition 4.3.3) on the local Langlands correspondence for GL(4). Moreover, we use the identities and the method of [PR] to prove that when π_j has trivial central character for every j, the *global root number* $W((\pi_1 \boxtimes \pi_2) \times (\pi_3 \boxtimes \pi_4))$ is 1 (see Theorem 4.3.4).

We then turn the tables and prove in section 4.4, that as a *consequence* of Theorem M, the three candidates for the triple product L-functions on GL(2) all agree at *all* places. (Recall that we needed to know that they were very nearly the same to get Theorem M in the first place.) This may be of independent interest.

The final application of our main result is the proof of the Tate conjecture for 4-fold products V of modular curves, asserting in particular that the order of pole at $s = 2$ of the L-function over any solvable (normal) number field K of the Galois module $W_\ell := H^4_{\mathrm{et}}(V_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ equals the rank of the group of K-rational codimension 2 Tate cycles on $V_{\overline{\mathbb{Q}}}$ (see Theorem 4.5.1). Moreover we show, in line with the works of Ribet ([Ri1]) and V.K. Murty ([Mu]) on the *Jacobians* of modular curves, that the latter number can also be computed with the *Tate cycles* replaced by the *algebraic cycles* modulo homological equivalence if the level of at least one of the curves is square-free. We refer to chapter 5 for a precise statement.

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2. **Notations and Preliminaries**

2.1. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For any subfield K of $\overline{\mathbb{Q}}$, let $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ denote the Galois group of $\overline{\mathbb{Q}}$ over K, together with the profinite topology. For any number field F with ring of integers $\mathfrak{O}_F$, let $\Sigma(F)$ (resp. $\Sigma_\infty(F)$, resp. $\Sigma_0(F)$) denote the places (resp. archimedean places, resp. finite places) of F. At each v, let F_v denote the completion of F relative to v, and if moreover v is finite, let $\mathfrak{O}_v$, $\mathfrak{P}_v$, $\mathbb{F}_v$, $|.|_v$ and $Nv = q_v$ respectively denote the ring of integers of F_v, the maximal ideal, the residue field, the normalized absolute value and the norm of v. When $F = \mathbb{Q}$, there is a unique archimedean place ∞ given by the canonical imbedding of $\mathbb{Q}$ in $\mathbb{R}$, and every finite place v corresponds to a rational prime number p, in which case we will write $\mathbb{Z}_p$ instead of $\mathcal{O}_v$. Denote by $\mathbb{A}_F$ the ring of adeles of F, which is the restricted direct product of $\{F_v\}$ relative to $\{\mathcal{O}_v\}$, equipped with its usual locally compact topology. Let I_F denote the group of ideles of F, and C_F the idele class group I_F/F^*.

We will fix a non-trivial unitary character ψ of $\mathbb{A}_F$ which is trivial on F, and let Ψ_v the v-component of Ψ. Various quantities, such as the ε-factors, will depend on this choice, which we will suppress in our notation.

In this paper we will systematically use the powerful language of automorphic representations, though at relevant places we will indicate briefly the essence of what we do in the classical language. Given any primitive cusp form f on the upper half plane of level N, character ω, there exists a (unique) cuspidal automorphic representation $\pi = \pi(f)$ of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ of conductor N and central character ω such that, at every prime p, the p-Euler factor of $L(s, f)$ agrees with the L-factor $L(s, \pi_p)$ of the p-component π_p. We refer the reader to the expository monograph [Ge] to understand how to go back and forth between the two approaches.

2.2. For every algebraic group G over F, let $G(\mathbb{A}_F)$ denote the restricted direct product $\prod'_v G(F_v)$, endowed with the usual locally compact topology. By a *cuspidal* representation of $G(\mathbb{A}_F) = G(F_\infty) \times G(\mathbb{A}_{F,0})$, we will always mean an irreducible, unitary, cuspidal automorphic representation. Such a representation is in particular a restricted tensor product $\pi = \otimes'_v \pi_v = \pi_\infty \otimes \pi_0$, where each π_v is an (irreducible) admissible representation of $G(F_v)$ for v finite, and an admissible $(\mathrm{Lie} G_v, K_v)-$module for v archimedean, with K_v denoting a compact modulo center subgroup of $G(F_v)$; π_0 (resp. π_∞) is the restricted tensor product of π_v over all finite (resp. archimedan) places v. By definition, π_v must be unramified at almost all v.

For any irreducible, automorphic representation π of $GL(n, \mathbb{A}_F)$, let $L(s, \pi) = L(s, \pi_\infty)L(s, \pi_0)$ denote the associated *standard* $L-$function ([J]) of π; it has an

Euler product expansion

$$L(s,\pi) \;=\; \prod_v L(s,\pi_v), \tag{2.2.1}$$

convergent in a right-half plane. If v is an archimedean place, then one knows (cf. [La1]) how to associate a semisimple $n-$dimensional $\mathbb{C}-$representation $\sigma(\pi_v)$ of the Weil group W_{F_v}, and $L(\pi_v,s)$ identifies with $L(\sigma_v,s)$. On the other hand, if v is a finite place where π_v is unramified, there is a corresponding semisimple (Langlands) conjugacy class $A_v(\pi)$ in $\mathrm{GL}(n,\mathbb{C})$ such that

$$L(s,\pi_v) \;=\; \det(1-A_vT)^{-1}|_{T=q_v^{-s}}. \tag{2.2.2}$$

We may find a diagonal representative $\mathrm{diag}(\alpha_{1,v}(\pi),...,\alpha_{n,v}(\pi))$ for $A_v(\pi)$, which is unique up to permutation of the diagonal entries. Let $[\alpha_{1,v}(\pi),...,\alpha_{n,v}(\pi)]$ denote the resulting unordered $n-$tuple. Since $W_{F,v}^{\mathrm{ab}} \simeq F_v^*$, $A_v(\pi)$ clearly defines an abelian $n-$dimesional representation $\sigma(\pi_v)$ of $W_{F,v}$. One has

Theorem 2.2.3 ([JS1]) *Let $n \geq 1$, and π a non-trivial cuspidal representation of $GL(n,\mathbb{A}_F)$. Then $L(s,\pi)$ is entire. Moreover, for any finite set S of places of F, the incomplete $L-$function $L^S(s,\pi) = \prod_{v\notin S} L(s,\pi_v)$ is holomorphic in $\Re(s)>0$.*

When $n=1$, such a π is simply a unitary idele class character and this result is due to Hecke. Also, when π is trivial, $L(s,\pi) = \zeta_F(s)$.

By the theory of Eisenstein series, one has a sum operation $\boxplus$ ([La3]), which results in the following

Theorem 2.2.4 ([JS2]) *Given any $m-$tuple of cuspidal representations $\pi_1,...,\pi_m$ of $GL(n_1,\mathbb{A}_F),...,\ GL(n_m,\mathbb{A}_F)$ respectively, there exists a unitary, irreducible, automorphic representation $\pi_1\boxplus...\boxplus\pi_m$ of $GLn,\mathbb{A}_F)$, $n = n_1+...+n_m$, which is unique up to equivalence, such that for any finite set S of places,*

$$L^S(s,\boxplus_{j=1}^m\pi_j) \;=\; \prod_{j=1}^m L^S(s,\pi_j).$$

Call such a (Langlands) sum $\pi \simeq \boxplus_{j=1}^m\pi_j$, with each π_j cuspidal, an *isobaric* representation. Denote by $\mathrm{ram}(\pi)$ the finite set of finite places where π is ramified, and let $\mathfrak{N}(\pi)$ be its conductor ([JPSS2]).

For every integer $n\geq 1$, set:

$$\mathcal{A}(n,F) \;=\; \{\pi : \text{isobaric representation of } \mathrm{GL}(n,\mathbb{A}_F)\}/\simeq, \tag{2.2.5}$$

and

$$\mathcal{A}_0(n,F) \;=\; \{\pi\in\mathcal{A}(n,F)|\,\pi \text{ cuspidal}\}.$$

Put $\mathcal{A}(F) = \cup_{n\geq 1}\mathcal{A}(n,F)$ and $\mathcal{A}_0(F) = \cup_{n\geq 1}\mathcal{A}_0(n,F)$.

Remark. One can also define the analogs of $\mathcal{A}(n,F)$ for local fields F, where the "cuspidal" subset $\mathcal{A}_0(n,F)$ consists of essentially square-integrable representations of $\mathrm{GL}(n,F)$. See [La3] and [Ra1] for details.

Let π, π' be isobaric automorphic representations in $\mathcal{A}(n,F)$, $\mathcal{A}(n',F)$ respectively. Then there exist an associated Euler product $L(s,\pi\times\pi')$ ([JS2,4], [JPSS], [COPS2]; [Sh1,3]), which converges in $\{\Re(s)>1\}$, and admits a meromorphic continuation to the whole $s-$plane with a functional equation. When v is archimedean or a finite place outside $\mathrm{ram}(\pi)$, one has

$$L_v(s,\pi\times\pi') \;=\; L(s,\sigma(\pi_v)\otimes\sigma(\pi'_v)). \tag{2.2.6}$$

When $n = 1$, $L(s, \pi \times \pi') = L(s, \pi\pi')$, and when $n = 2$ and $F = \mathbb{Q}$, this function is the Rankin-Selberg $L-$function, extended to arbitrary global fields by Jacquet ([J2]).

Theorem 2.2.7 (Jacquet-Shalika [JS2]) *Let $\pi \in \mathfrak{A}_0(n, F)$, $\pi' \in \mathfrak{A}_0(n', F)$, and S a finite set of places. Then $L^S(s, \pi \times \pi')$ is holomorphic in $\{\Re(s) > 0\}$ unless $\pi \simeq \pi'^{\vee}$, in which case it has a unique pole (of order 1) at $s = 1$.*

We will also need the general construction of automorphic L-functions attached to any $\pi \in \mathcal{A}(n, F)$. Let S be any finite set of places containing the archimedean and ramified places (for π). If r is any algebraic representation of $\mathrm{GL}(n, \mathbb{C})$, which is the connected component of the L-group of $\mathrm{GL}(n)$, we put

$$L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi, r),$$

where

$$L(s, \pi, r) = \det(I - (Nv)^{-s} r(A_v(\pi)))^{-1}.$$

Of particular interest is when r is the exterior square Λ^2, or the symmetric kth power Sym^k, for some $k \geq 1$, of the standard represenation. When r is simply the standard representation, $L^S(s, \pi, r)$ is evidently just $L^S(s, \pi)$.

One knows (cf. [JS1], [BuF], [Sh1]) that $L^S(s, \pi, \Lambda^2)$ converges absolutely in $\Re(s) > 1$ and admits a meromorphic continuation with a functional equation of the usual type relating s and $1 - s$. One also knows, by [BuG], [Sh1], the analogous properties of $L^S(s, \pi, \mathrm{sym}^2)$. For $n = 2$, one knows more by [GJ], namely that there is an (isobaric) automorphic representation $\mathrm{sym}^2(\pi)$ such that

$$L^S(s, \mathrm{sym}^2(\pi)) = L^S(s, \pi, \mathrm{sym}^2).$$

For $n = 2$, one also knows the meromorphic continuation and functional equation (cf. [Sh2]) of $L^S(s, \pi, \mathrm{sym}^k)$ for all $k \leq 5$.

2.3. Base change and automorphic induction.

We will now review the results of Arthur and Clozel [AC] in a form which is suitable for the applications found in this article. See also [Lab2].

In [AC], chapter 3 (sec. 3 - 6), one finds a construction, for any cyclic extension K/F with $[K : F]$ a prime ℓ, of maps

$$b_{K/F} : \mathcal{A}(n, F) \to \mathcal{A}(n, K),\ \pi \to \pi_K \quad (\textit{base change})$$

and

$$I_{K/F} : \mathcal{A}(n, K) \to \mathcal{A}(n\ell, F),\ \pi \to I(\pi) \quad (\textit{automorphic induction}),$$

such that at every place v of F which is finite and unramified for the representations and K/F (or archimedean) and a place w of K above v, we have (respectively)

$$\mathrm{res}^{F_v}_{K_w}(\sigma(\pi_v)) \simeq \sigma((\pi_K)_w), \tag{3.1}$$

and

$$\sigma(I(\pi)_v) \simeq \mathrm{ind}^{F_v}_{K_w}(\pi_w).$$

Given any local field E, if β an unramified representation of $\mathrm{GL}(r, E)$ (or if E is archimedean), we write $\sigma(\beta)$ to signify the associated r-dimensional representation of W'_E (or W_E).

There are also local analogs of these maps, customarily called "local base change" and "local automorphic induction". The former is constructed and discussed in great detail in chapter 1 of [AC]. The later is discussed briefly in [Cℓ1] and the relevant assertions are consequences of the results of [AC]. An alternate construction of the local automorphic induction for essentially tempered representations, which works also in characteristic p, is given in the paper [HH] of Henniart and Herb. The properties we will need are summarized as follows:

Proposition 2.3.1 *Let K/F be a cyclic extension of number fields or local fields of degree ℓ, a prime. Let θ be a generator of $Gal(K/F)$, and let χ denote the character of the idele class group (resp. multiplicative group) of F in the global (resp. local) case associated to K. Then*

1. *The image of $b_{K/F}$ consists precisely of those $\beta \in \mathcal{A}(n,K)$ such that $\beta \simeq \beta\circ\theta$.*
2. *The image of $I_{K/F}$ consists precisely of those $\pi \in \mathcal{A}(n\ell,F)$ such that $\pi \simeq \pi \otimes \chi$.*
3. *For every $\pi \in \mathcal{A}(m,F)$ and $\beta \in \mathcal{A}(n,K)$, we have the* **adjointness** *property:*
$$L(s,\pi \times I(\beta)) \;=\; L(s,\pi_K \times \beta),$$
and
$$\varepsilon(s,\pi \times I(\beta))\varepsilon(s,1_K)^{nm} \;=\; \varepsilon(s,\pi_K \times \beta)\prod_{j=0}^{\ell-1}\varepsilon(s,\chi^j)^{nm}.$$
4. *Suppose β is cuspidal (resp. supercuspidal) in $\mathcal{A}(n,K)$, for K global (resp. local). Then*
$$I(\beta)_K \simeq \boxplus_{j=0}^{\ell-1}\,\beta\circ\theta^j.$$
Moreover, $I(\beta)$ is cuspidal (resp. supercuspidal) iff β is **not** *isomorphic to $\beta\circ\theta$.*
5. *Suppose π is cuspidal (resp. supercuspidal) in $\mathcal{A}(n,F)$, for F global (resp. local). Then*
$$I(\pi_K) \simeq \boxplus_{j=0}^{\ell-1}\,\pi\otimes\chi^j.$$
Moreover, π_K is cuspidal (resp. supercuspidal) iff π is not isomorphic to $\pi\otimes\chi$.

For a proof of 1., 2. and the second half of 5., see [AC], [Cℓ1] and [HH]. For the remaining assertions, see [PR], pages 7,8, where this Proposition is stated as Prop. 3.1, and where $\mathcal{A}(n,F)$ is denoted Isob(n,F).

3. Construction of $\boxtimes : \mathcal{A}(\mathrm{GL}(2)) \times \mathcal{A}(\mathrm{GL}(2)) \to \mathcal{A}(\mathrm{GL}(4))$

The object of this chapter is to prove the following

Theorem M *Let π,π' be in $\mathcal{A}(2,F)$. Then we have:*

Existence*: There exists an isobaric automorphic representation $\pi \boxtimes \pi'$ of $GL(4,\mathbb{A}_F)$ satisfying (at every finite place v)*

(L_v) $$L(s,(\pi\boxtimes\pi')_v) \;=\; L(s,\pi_v\times\pi'_v),$$

and

(ε_v) $$\varepsilon(s,(\pi\boxtimes\pi')_v) \;=\; \varepsilon(s,\pi_v\times\pi'_v).$$

We also have

$$(L_\infty) \qquad L(s,(\pi\boxtimes\pi')_\infty) \;=\; L(s,\pi_\infty\times\pi'_\infty).$$

Cuspidality Criterion: *Suppose π,π' are both cuspidal.*

If neither of them is associated to a character of a quadratic extension, then $\pi\boxtimes\pi'$ is cuspidal iff we have

(C) π' is not equivalent to $\pi\otimes\chi$, for any idele class character χ of F.

If $\pi' = I_K^F(\mu)$, for a character μ of a quadratic extension K, then $\pi\boxtimes\pi'$ is cuspidal iff the base change π_K is cuspidal and not isomorphic to $\pi_K\otimes(\mu\circ\theta)\mu^{-1}$, where θ denotes the non-trivial automorphism of K/F.

We will call an isobaric representation $\pi\boxtimes\pi'$ satisfying (L_v) and (ε_v), for all v, a **strong lifting** to $\mathrm{GL}(4)/F$, or the **automorphic tensor product**, of the pair (π,π'). If it only satisfies (L_v) for almost all v, and not necessarily (ε_v) or (L_∞), we will call it a **weak lifting**.

We will briefly explain the arithmetic motivation for the cuspidality criterion above. Suppose π,π' correspond to irreducible, continuous 2-dimensional representations σ,σ' over $\mathbb{C}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then, since the operation $\boxtimes$ on $\mathcal{A}(2,F)\times\mathcal{A}(2,F)$ is supposed to correspond to the tensor product on the Galois side, $\pi\boxtimes\pi'$ should be cuspidal iff $\sigma\otimes\sigma'$ is irreducible. When σ,σ' are non-dihedral, reducibility happens iff one is a twist of the other by a character. That the criterion should be the same for all non-dihedral cuspidal pairs (π,π') is motivated by the hope (see [La3], [Cl2], and [Ra1]) that there is a group $\mathcal{L}_F$, whose connected component $\mathcal{L}_F^0$ is pro-reductive with $\mathcal{L}_F/\mathcal{L}_F^0 \simeq \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, and whose irreducible n-dimensional representations parametrize cuspidal automorphic representations of $\mathrm{GL}(n,\mathbb{A}_F)$.

3.1. **Relevant objects and the strategy.** We begin with a simple

Lemma 3.1.1. *Theorem M holds in the following three special cases:*

(I) At least one of $\{\pi,\pi'\}$ is not cuspidal.
(II) At least one of $\{\pi,\pi'\}$ is automorphically induced by a character μ of (the idele class group of) a quadratic extension K of F.
(III) π' is a twist of π, i.e., there exists a character χ of C_F such that

$$\pi' \;\simeq\; \pi\otimes\chi.$$

Proof. Suppose π' is not cuspidal. Then as we have seen in section 2, there exist idele class characters μ_1,μ_2 of F such that $\pi' = \mu_1\boxplus\mu_2$. In this case, we set

$$\pi\boxtimes\pi' \;=\; (\pi\boxtimes\mu_1)\boxplus(\pi\boxtimes\mu_2),$$

which is not cuspidal. In this case one knows ([JS2]) that the identities (L_v), (ε_v), and (L_∞) are all satisfied everywhere. Thus we are done in case (I).

Suppose π' is of the form $I_K^F(\mu)$, for a character μ of C_K, for some K. Then we set

$$\pi\boxtimes\pi' \;=\; I_K^F(\pi_K\otimes\mu),$$

where π_k denotes the base change of π to K. The identity (L_v) at the unramified places v is a direct consequence of the identities of Proposition 2.3.1. The fact that

(L_v) holds at every place, and that (L_∞) and (ε_v) also hold, will be shown as part of a general assertion in the next section (see Proposition 3.2.1).

It is left to consider (III). In this case, we set

$$\pi \boxtimes \pi' \,=\, (\mathrm{sym}^2(\pi) \otimes \chi) \boxplus \omega\chi,$$

where ω is the central character of π. The asserted identities are then consequences of the work of Gelbart and Jacquet ([GJ]) and the strong multiplicity one theorem ([JS2]).

□

We will say that (π, π') is of **general type** if we are not in either of these three special cases.

Given any pair (Π, η) of irreducible, generic, admissible representations of $\mathrm{GL}(n, \mathbb{A}_F)$, $\mathrm{GL}(m, \mathbb{A}_F)$ respectively, which are not necessarily automorphic, we may set

$$L(s, \Pi \times \eta) \,=\, \prod_v L(s, \Pi_v \times \eta_v) \tag{3.1.2}$$

and

$$\varepsilon(s, \Pi \times \eta) \,=\, \prod_v \varepsilon(s, \Pi_v \times \eta_v).$$

The following result is a crucial ingredient of our approach.

Theorem 3.1.3 (Cogdell - Piatetski-Shapiro ([CoPS])). *Let T be a fixed finite set of finite places of F. Let Π be an irreducible unitary, admissible, generic representation of $GL(4, \mathbb{A}_F)$ which satisfies the following:*
For every $\eta \in \mathcal{A}_0(n, F)$, $n \leq 2$, with η_v unramified at every v in T, we have:

(MC) $\quad L(s, \Pi \times \eta)$ and $L(s, \Pi^\vee \times \eta^\vee)$ converge absolutely in large $\Re(s)$,

and they admit meromorphic continuations to the whole s-plane.

(E) $\quad L(s, \Pi \times \eta)$ and $L(s, \Pi^\vee \times \eta^\vee)$ are entire.

There is a functional equation

$$L(1-s, \Pi^\vee \times \eta^\vee) \,=\, \varepsilon(s, \Pi \times \eta) L(s, \Pi \times \eta). \tag{FE}$$

(BV) $\quad L(s, \Pi \times \eta)$ is bounded in vertical strips.

Then Π is nearly automorphic, i.e., there exists an automorphic representation Π_1 of $GL(4, \mathbb{A}_F)$ such that $\Pi_v \simeq \Pi_{1,v}$ for almost all v.

Consequently, Theorem M can be attacked for (π, π') of general type if we can solve the following two problems (for a finite set T of finite places):

(P1) Define an irreducible, generic admissible representation Π of $\mathrm{GL}(4, \mathbb{A}_F)$ such that we have **at every place** v,

$$L(s, \Pi_v) \,=\, L(s, \pi_v \times \pi'_v) \tag{3.1.4}$$

and

$$\varepsilon(s, \Pi_v) \,=\, \varepsilon(s, \pi_v \times \pi'_v).$$

(P2) Given any $\eta \in \mathcal{A}_0(m, F)$, $m \leq 2$, with η_v unramified at every v in T, the hypotheses (MC), (E), (FE) and (BV) hold for $L(s, \Pi \times \eta)$ and $L(s, \Pi^\vee \times \eta^\vee)$.

(P1) cannot be solved as posed without knowing the local Langlands correspondence for GL(4), which in any case we would like to avoid using. We can, however, define a candidate Π_v at almost all places, in particular at the archimedean places and at finite places v where π_v and π'_v are not both supercuspidal. (Even the supercuspidal cases can be dealt with in odd residual characteristics.) See section 3.7. Given any π, we can find,using [Ku], a chain $F = K_0 \subset K_1 \subset \ldots \subset K_m = K$ with each K_j/K_{j-1} cyclic of prime degree, in fact an infinite family $X = X(\pi)$ of such chains, such that the base change π_K of π to K, which exists by [AC], has the property that at *any* place w of K, $\pi_{K,w}$ is not supercuspidal. Hence (P1) can be solved over K, and if we can also solve (P2) over a "sufficiently large" subset Y of X, we can construct $\pi \boxtimes \pi'$ over each K in Y, and then try to find a common descent to F satisfying the desired properties (listed in Theorem M). The part dealing with descent follows an approach taken for GL(2) in our earlier joint work [BR] with D. Blasius. (There is a mistake in [BR], but it does not affect the relevant section there discussing the descent criterion; in any case, we give a complete argument here for GL(n) - see Proposition 3.6.1) The best possible situation will be one in which almost every finite place v of F splits completely in some $K \in Y$. There are complications with finding a common descent if (π_K, π'_K) is not of general type for too many K. For this and other reasons, it will be best for us to use an inductive argument in cyclic layers of prime degree. We refer to sections 3.6 and 3.7 for details.

After replacing F by a suitable larger field, let us suppose that we have solved (P1), and that (π, π') is of general type with π_v not supercuspidal. Fix any T, η as above. If $m = 1$, then all the hypotheses of Theorem 3.1.3 are just known assertions of the Rankin-Selberg theory. So we may assume that $m = 2$. (P2) can be solved if we can find a "triple product L-function" $L(s, \pi \times \pi' \times \eta)$ which satisfies all these hypotheses and satisfies (at every v)

$$L(s, \pi_v \times \pi' \times \eta_v) \;=\; L(s, \Pi_v \times \eta_v). \tag{3.1.5}$$

There are in fact three candidates for such an L-function, but unfortunately, none of them possesses apriori *all* the desired properties. The first one is in some sense the most natural one. To define it, let σ_v (resp. σ'_v) denote, at each v, the 2-dimensional representation of W'_{F_v} associated to π_v (resp. π'_v) by the local Langlands correspondence for GL(2) ([Ku]). (When v is archimedean, W'_{F_v} signifies just usual the Weil group by W'_{F_v}.) Put

$$L(s, \pi \times \pi' \times \eta) \;=\; \prod_v L(s, \sigma_v \otimes \sigma'_v \otimes \tau_v), \tag{3.1.6}$$

where τ_v denotes, at each v, the representation of W'_{F_v} attached to η_v. We do not know apriori any of the desired properties for this Euler product.

One defines the epsilon factor $\varepsilon(\pi \times \pi' \times \eta)$ as the product over all v of $\varepsilon(s, \sigma_v \otimes \sigma'_v \otimes \tau_v)$; these local factors will also be denoted $\varepsilon(s, \pi_v \times \pi'_v \times \eta_v)$.

The second candidate, which we denote by $L_1(s, \pi \times \pi' \times \eta)$, is the one defined by Piatetski-Shapiro and Rallis in [PSR2], generalizing the construction of Garrett ([G]) over $\mathbb{Q}$ when all three representations correspond to holomorphic modular forms. (see also Ikeda ([Ik1]). This is defined by using an integral representation, and is known to satisfy (MC) and (FE). It also satisfies (E) by the arguments of Ikeda (see Theorem 2.7 of [Ik2], and also the discussion following Theorem 3.3.11

of this paper). But one does *not yet* have (BV), which is problematic, which we propose to rectify in this paper. One knows that at the finite unramified places ([PSR1]) and the infinite places ([Ik3]) that

$$L_1(s, \pi_v \times \pi'_v \times \eta_v) \,=\, L(s, \pi_v \times \pi'_v \times \eta_v). \tag{3.1.9}$$

See section 3.3 for a more complete discussion.

The third candidate, which we denote by $L_2(s, \pi \times \pi' \times \eta)$, was constructed in the work of Shahidi (cf. [Sh1], page 582), developing the ideas of Langlands in the monograph ([La2]). (See also section 0 of [PSR2] for an exposition.) To be precise, let G denote the group $\mathrm{Spin}(4,4)$. It has a parabolic $P = MU$ with Levi component M which is a quotient (over F) of $\mathrm{GL}(1)\times \mathrm{SL}(2)\times \mathrm{SL}(2)\times \mathrm{SL}(2)$ by $\{1, -1\}$, where -1 signifies $H_1H_3H_4(-1)$, with the H_j denoting the standard simple coroots. Take the representation $\beta := \omega \otimes \pi_1 \otimes \pi'_1 \otimes \tau_1$ on $M(\mathbb{A}_F)$, with π_1 (resp. π'_1, resp. τ_1) being *any* irreducible component of the restriction of π (resp. π', resp. τ) to $\mathrm{SL}(2, \mathbb{A}_F)$ and ω being the product of the central characters of π, π' and τ. Extend β in the usual way to $P(\mathbb{A}_F)$ by letting $U(\mathbb{A}_F)$ act trivially. Then the Langlands Eisenstein series on $G(\mathbb{A}_F)$ associated to the representation of $G(\mathbb{A}_F)$ induced by β defines, and gives analytic information on, $L_2(s, \pi \times \pi' \times \eta)$. One knows that this L-function satisfies (MC) and (FE) (cf. [Sh1]), and also (E) by the very recent work of H. Kim and F. Shahidi([K-Sh]). If v is archimedean or *unramified* or *tempered*, we have

$$L_2(s, \pi_v \times \pi'_v \times \eta_v) \,=\, L(s, \pi_v \times \pi_v \times \eta_v) \tag{3.1.8}$$

There is also a definition of epsilon factors $\varepsilon_2(s, \pi_v \times \pi'_v \times \eta_v)$. The main use of this approach for us comes from knowing that $L_2(s, \pi \times \pi' \times \tau)$ is non-zero on the line $s = 1$.

We take T to be a (finite) set containing all the places v where π_v or π'_v is in the discrete series, and solve (P2) for the relevant types of (π, π'). In such a case, one first notes, by an extension of the local results of [Ik1], [Ik2] by global arguments, that the local gamma factors (called ε'-factors by some) of first and the third candidates agree at *all* the places, and that their L-factors also agree at all the places v where at least one of the representations π_v, π'_v is not supercuspidal It is not clear that the L-factors agree at all the places v. But luckily, it turns out not to matter by a base changing trick. We refer to sections 3.3 and 3.4 for details.

Next comes the establishment of (BV). The idea here is to make use of Arthur's truncation $\Lambda^T E(f_s)$ of the Eisenstein series $E(f_s)$ on $\mathrm{GSp}(6)/F$ asociated to the character $\det^{s+2}$ of the ($\mathrm{GL}(2)$ occurring in the Levi of the) Siegel parabolic P. (Here T is a sufficiently regular parameter.) The integral representation for $L_1(s, \pi_1 \times \pi_2 \times \pi_3)$ is given by the integral, over $\mathrm{GL}(2)^3$, of $E(f_s)$ against a function φ in the space of $\pi_1 \otimes \pi_2 \otimes \pi_3$. By the standard arguments using th functional equation and boundedness in $\Re(s) >> 0$, it suffices to show that this integral is of bounded order in vertical strips Ξ. Since φ decreases rapidly at infinity, it suffices to check that for s in Ξ, $E(f_s)(g)$ is bounded in absolute value by a function of s of bounded order times $||g||^N$ for a uniform N as N goes to infinity. Now, using the fact that $E(f_s)$ is an eigenfunction, though not square-integrable globally, of the Laplacian Δ, we use standard Sobolev estimates and bound $|E(f_s)(g)|$ by its L^2-norm in a small neighborhood V of controlled size. The L^2-norm over V of $E(f_s) - \wedge^T E(f_s)$ can be estimated by use of the constant terms; these are essentially given by abelian

L-functions, which are known by Hecke to be of bounded order. It then suffices to estimate the L^2 norm of $\wedge^T E(f_s)$ over the whole locally symmetric manifold, which can be understood by the works of Langlands and Arthur.

Finally, to be able to honestly apply the converse theorem, we must also check (3.1.5). The details are all in sections 3.4 and 3.5.

3.2. **Weak to strong lifting, and the cuspidality criterion.** In this section we prove the following

Proposition 3.2.1 *Let π, π' be in $\mathcal{A}(2, F)$. Suppose we have constructed a weak lifting, i.e., an isobaric automorphic representation $\pi \boxtimes \pi'$ of $GL(4, \mathbb{A}_F)$ satisfying the identity (L_v) (of theorem M) at almost all v. Then we have (L_v) and (ε_v) at* **all** *the finite places v, and (L_∞) as well. In addition, the cuspidality criterion (of Theorem M) holds.*

Proof. Let S be the (finite) set of places outside which (L_v) holds. We may assume (see (I) of section 3.1) that π and π' are both cuspidal and non-dihedral. Note also that the central character of $\pi \boxtimes \pi'$ is simply $(\omega\omega')^2$, where as before, ω, ω' denote the respective central character of π, π'. This is so because the two idele class characters agree almost everywhere. (In fact, by a theorem of Hecke, it suffices to know that they agree at a set of primes of density $> 1/2$.)

First some notations. If $f(s), g(s)$ are two meromorphic functions of s such that their quotient is invertible, we will write $f(s) \equiv g(s)$. At any place v, given a character ν of F_v^*, we can write it as $\nu_0|.|^z$, for a unitary character ν_0 and a complex number z. The real part of z is uniquely defined; we will call it the *exponent of* ν, and denote it $e(\nu)$.

Choose a finite order character μ of C_F such that μ_v is sufficiently ramified at every finite place u in S so as to make the Euler u-factors of $\pi \boxtimes \pi'$, $\pi \times \pi'$, and their contragredients, equal 1. This is possible by using the results of [JPSS]. Then, comparing the global functional equations of both L-functions, we get

(3.2.2)
$$\prod_{w|\infty} L(s, (\pi \boxtimes \pi')_w) L(1-s, \pi_w^\vee \times {\pi'_w}^\vee) \equiv \prod_{w|\infty} L(1-s, (\pi \boxtimes \pi')_w^\vee) L(s, \pi_w \times \pi'_w).$$

For *any* place v, archimedean or otherwise, for any $n \geq 1$, and for any cuspidal automorphic representation Π of $\mathrm{GL}(n, \mathbb{A}_F)$ (such as $\pi \boxtimes \pi'$), it is known that $L(s, \Pi_v)$ is holomorphic in $\Re(s) > \frac{1}{2} - t$, for some $t = t(\Pi, v) > 0$ (see [BaR], Prop. 2.1, part B). Consequently, $L(s, (\pi \boxtimes \pi')_v)$ has no pole in common with $L(1-s, (\pi \boxtimes \pi')_v^\vee)$.

Next we claim that, since π, π' are (cuspidal) automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$, $L(s, \pi_v \times \pi'_v)$ is also holomorphic in $\Re(s) > \frac{1}{2} - t$, for some $t > 0$. If π_v and π'_v are both in the discrete series, then one has holomorphy even in $\Re(s) > 0$ (see [BaR], Lemma 2.3). If π_v is a principal series representation defined by (quasi)characters ν, ν', then by [GJ], we have $e(\nu) < 1/4$ and $e(\nu') < 1/4$. (Strictly speaking, when v is archimedean, one finds in [GJ] only the assertion that these exponents are $\leq 1/4$, but one can eliminate the possibility of exponent $1/4$ by using a simple version of the argument of [LRS1].) At the finite places, one can even replace $1/4$ by $1/5$ by using [Sh2]; see also [LRS2]. Over $\mathbb{Q}$, one can do still

better and reduce the bound to 5/28 ([BuDHI]). But we will not need these finer results. In any case, the claim follows easily.

Consequently, $L(s, \pi_v \times \pi'_v)$ has no pole in common with $L(1-s, \pi_v^\vee \times \pi_v'^\vee)$ either. Applying this, along with the earlier remark on the relative primality of $L(s, (\pi \boxtimes \pi')_v)$ and $L(1-s, (\pi \boxtimes \pi')_v^\vee)$, we get (L_∞) from (3.2.2).

Next pick any finite $u_0 \in S$ and choose a μ which is 1 at u_0, but is highly ramified at all other finite u in S. Again, by comparing the functional equations, we get an identity such as (3.2.2), with the factors at infinity replaces by the corresponding ones at u_0. Using the relative primality results above, now with $v = u_0$, we get (L_{u_0}). Since u_0 was arbotrary, we get (L_v) now at every place v.

For the identity of epsilon factors, we fix a u_0 in S and note that by [JPSS2], we can choose a global character μ which is 1 at u_0 and sufficiently ramified at any other place u in S such that the epsilon factor of $\pi_u \times \pi'_u \otimes \mu_u$ depends only on μ_u and the square of the product of the central characters ω_u, ω'_u, and the dependence is simple. Similarly, the epsilon factor of $(\pi \boxtimes \pi')_u$ has the same dependence on μ_u and the central character of $(\pi \boxtimes \pi')_u$, which we noticed above to be $(\omega_u \omega'_u)^2$. The analogous statements hold for the contragredients, and this results in the identity $(\forall u \in S - \{u_0\})$

$$\frac{\varepsilon(s, (\pi \boxtimes \pi')_u) L(s, (\pi \boxtimes \pi')_u)}{L(1-s, (\pi \boxtimes \pi')_u^\vee)} = \frac{\varepsilon(s, \pi_u \times \pi'_u) L(s, \pi_u \times \pi'_u)}{L(1-s, \pi_u^\vee \times \pi_u'^\vee)}.$$

Comparing the global functional equations again, we get (ε_{u_0}) as we already know that the L-factors agree. This finishes the proof of the first part of the Proposition.

It is left to prove the **cuspidality criterion**. First suppose $\pi \boxtimes \pi'$ is cuspidal. Recall that by (I) of section 3.1, $\pi \boxtimes \pi'$ is not cuspidal if π or π' is not. So π and π' must both be cuspidal. Suppose $\pi' \simeq \pi \otimes \chi$, for an idele class character χ of F. Then, by [JS2] and (L_v) for almost all v, $L^S(s, \pi \boxtimes \pi' \otimes (\chi\omega)^{-1})$ must have a pole at $s = 1$, with S denoting the set of ramified and archimedean places, and ω the central character of π. (We are using the fact that $\pi \otimes \omega^{-1}$ is the contragredient $\pi^\vee$ of π.) Then $\pi \boxtimes \pi'$ cannot be cuspidal, leading to a contradiction.

Conversely, suppose that π, π' are cuspidal and (C) holds, but that $\pi \boxtimes \pi'$ is not cuspidal. We will show that it leads to a contradiction.

Since $\pi \boxtimes \pi'$ is isobaric, we can decompose it uniquely as

$$\pi \boxtimes \pi' \,=\, \boxplus_{j=1}^r \eta_j,$$

where each η_j is a cuspidal automorphic representation of $\mathrm{GL}(n_j, \mathbb{A}_F)$, with $\sum_{j=1}^r n_j$ being 4. Suppose some n_j, say n_1, is 1. Then η_1 is an idele class character, and we have, for any large enough finite set S of places containing the archimedean ones,

$$L^S(s, \pi \times \pi' \otimes \eta_1^{-1}) \,=\, \zeta_F^S(s) \prod_{j \neq 1} L^S(s, \eta_j \otimes \eta_1^{-1}).$$

One knows ([JS2]) that, since each $L^S(s, \eta_j \otimes \eta_1^{-1})$ is cuspidal, it cannot vanish at $s = 1$. Thus the pole (at $s = 1$) of $\zeta_F^S(s)$ induces one of the function on the left, which is not allowed by (C). Thus n_j must be > 1 for each j. Then we are forced to have

$$n_1 \,=\, n_2 \,=\, 2.$$

Now the key idea is to compute the exterior square L-function of $\pi \boxtimes \pi'$ in two different ways. On the one hand, since it is of the form $\eta_1 \boxplus \eta_2$, with each η_j a cuspidal of $\mathrm{GL}(2)/F$, we get

$$L^S(s, \pi \boxtimes \pi', \Lambda^2) \;=\; L^S(s, \eta_1 \times \eta_2) L^S(s, \omega_1) L^S(s, \omega_2),$$

where ω_j is the central character of η_j. This can be seen easily at the unramified places v. Indeed, if $\tau_{j,v}$ is the 2-dimensional representation of W'_{F_v} associated to $\eta_{j,v}$, then the above identity is induced by the following:

$$\Lambda^2(\tau_{1,v} \otimes \tau_{2,v}) \;\simeq\; (\tau_1 \otimes \tau_2) \;\oplus\; \det(\tau_{1,v}) \;\oplus\; \det(\tau_{2,v}),$$

which is easy to verify.

Consequently, $L^S(s, \pi \boxtimes \pi', \Lambda^2)$ is divisible by (two) abelian L-functions, namely $L^S(s, \omega_1)$ and $L^S(s, \omega_2)$.

On the other hand, denoting by σ_v (resp. σ'_v) the 2-dimensional representation of W'_{F_v} associated to π_v (resp. π'_v), we also have the identity

$$\Lambda^2(\sigma_v \otimes \sigma'_v) \;\simeq\; \Lambda^2(\sigma_v) \otimes \mathrm{sym}^2(\sigma'_v) \;\oplus\; \mathrm{sym}^2(\sigma_v) \otimes \Lambda^2(\sigma'_v).$$

This implies the following equality of L-functions:

$$L^S(s, \pi \boxtimes \pi', \Lambda^2) \;=\; L^S(s, \mathrm{sym}^2(\pi) \otimes \omega) L^S(s, \mathrm{sym}^2(\pi') \otimes \omega'),$$

where $\mathrm{sym}^2(\pi)$ (resp. $\mathrm{sym}^2(\pi')$) is the automorphic representation of $\mathrm{GL}(3, \mathbb{A}_F)$ associated to π (resp. π') by Gelbart-Jacquet ([GJ]), and ω (resp. ω') is the central character of π (resp. π').

Suppose π is not automorphically induced by an idele class character of a quadratic extension. Then the main theorem of [GJ] says that $\mathrm{sym}^2(\pi)$ is cuspidal; so $L^S(s, \mathrm{sym}^2(\pi) \otimes \omega')$ cannot be divisible by an abelian L-function. Consequently, if neither π nor π' is automorphically induced, we get a contradiction, and so $\pi \boxtimes \pi'$ must be cuspidal.

It remains to consider the case when π' is of the form $I_K^F(\mu)$, for μ an idele class character of a quadratic extension K. Then by (II) of section 3.1, we know that $\pi \boxtimes \pi'$ must be isomorphic to $I_K^F(\beta)$, where $\beta = \pi_K \otimes \mu$. By hypothesis, π_K is cuspidal and not isomorphic to $\pi_K \otimes (\mu \circ \theta)\mu^{-1}$, where θ is the non-trivial automorphism of K/F. Then β is a cuspidal, not isomorphic to $\beta \circ \theta$. So by the properties of automorphic induction (Prop. 2.3.3), $\pi \boxtimes \pi'$ is cuspidal.

□

3.3. **Triple product L-functions: local factors and holomorphy.** Let π, π', π'' be unitary, irreducible, cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$ of central characters $\omega, \omega', \omega''$ respectively. Put $\omega = \omega\omega'\omega''$. At each place v, let $\sigma_v, \sigma'_v, \sigma''_v$ be the 2-dimensional representations of W'_{F_v} associated to π_v, π'_v, π''_v respectively, by the local Langlands correspondence for GL(2) ([Ku]).

Recall that $L(s, \pi \times \pi' \times \pi'')$ is the (correct) triple product L-function defined by the Euler product $\prod_v L(s, \sigma_v \otimes \sigma'_v \otimes \sigma''_v)$, while $L_1(s, \pi \times \pi' \times \pi'')$ is the one given as the gcd of the integral representation of Garrett and Piatetski-Shapiro and Rallis, and $L_2(s, \pi \times \pi' \times \pi'')$ is the one defined by the Langlands-Shahidi method.

The following basic fact, due to Ikeda in the archimedean case ([Ik3], Theorem 1.1.0), and Rallis and Piatetski-Shapiro in the non-archimedean case ([PS-R2]), will be very important to us.

Proposition 3.3.1 *Let v be any place of F where π_v, π'_v, π''_v are all unramified. Then we have*

$$L(s, \pi_v \times \pi'_v \times \pi''_v) \;=\; L_1(s, \pi_v \times \pi'_v \times \pi''_v).$$

By a slight refinement of known results we will also establish the following

Proposition 3.3.2 *Let v be any place of F, where at least one of the local representations π_v, π'_v, π''_v is in the principal series (possibly complementary). Then*

(a) $$L(s, \pi_v \times \pi'_v \times \pi''_v) \;=\; L_1(s, \pi_v \times \pi'_v \times \pi''_v),$$

and

(b) $$\varepsilon(s, \pi_v \times \pi'_v \times \pi''_v) \;=\; \varepsilon_1(s, \pi_v \times \pi'_v \times \pi''_v).$$

Remark. When v is finite, if two of the local representations are non-supercuspidal, then such an identity already follows from the works of Ikeda ([Ik2,3]). In the special, but important, case when all of π_v, π'_v, π''_v are Steinberg, this was established earlier by Gross and Kudla ([GK]). When v is archimedean, this is treated in [Ik3]. In this case, the L_1-factor is defined only up to an invertible holomorphic function, and so the content of this Proposition is that we can normalize it appropriately so as to have the stated identities. Note also that one knows by the work of Shahidi ([Sh5]) the identity $L(s, \pi_v \times \pi'_v \times \pi''_v) \;=\; L_2(s, \pi_v \times \pi'_v \times \pi''_v)$ for any archimedean v.

We have stated here only what we need, and a stronger assertion will be proved later in section 4.4 *after* establishing Theorem M.

Proof. By hypothesis, there exist quasi-characters μ, ν of F_v^* such that one of the local representations, say π_v, is the principal series defined by (μ, ν). The starting point for us is the following

Lemma 3.3.3 *The set of poles of $L_1(s, \pi_v \times \pi'_v \times \pi''_v)$ is contained in the set of poles of $L(s, \pi_v \times \pi'_v \times \pi''_v)$.*

Proof of Lemma 3.3.3. We first recall the following

Proposition 3.3.4 *For any quasi-character λ of F_v^* one has*

$$L(s, \lambda \otimes \pi'_v \otimes \pi''_v) \;=\; L(s, \lambda \otimes \sigma'_v \otimes \sigma''_v)$$

and

$$\varepsilon(s, \lambda \otimes \pi'_v \otimes \pi''_v) \;=\; \varepsilon(s, \lambda \otimes \sigma'_v \otimes \sigma''_v).$$

For a proof, see the author's paper with D. Prasad ([PR], Prop. 4.2), where the identity is established via a global argument.

On the other hand, it is immediate that

(3.3.5) $$L(s, \sigma_v \otimes \sigma'_v \otimes \sigma''_v) \;=\; L(s, \mu \otimes \sigma'_v \otimes \sigma''_v) L(s, \nu \otimes \sigma'_v \otimes \sigma''_v),$$

and

$$\varepsilon(s, \sigma_v \otimes \sigma'_v \otimes \sigma''_v) \;=\; \varepsilon(s, \mu \otimes \sigma'_v \otimes \sigma''_v) \varepsilon(s, \nu \otimes \sigma'_v \otimes \sigma''_v).$$

One also has the following result of Ikeda:

Proposition 3.3.6([Ik3], Thm. 1.8) *The quotient*

$$\frac{L_1(s, \pi_v \times \pi'_v \times \pi''_v)}{L(s, \mu \otimes \pi'_v \otimes \pi''_v) L(s, \nu \otimes \pi'_v \otimes \pi''_v)}$$

is entire.

Set

$$\gamma(s,\pi_v\times\pi_v'\times\pi_v'') \;=\; \varepsilon(s,\pi_v\times\pi_v'\times\pi_v'')\frac{L(1-s,\pi_v^\vee\times(\pi_v')^\vee\times(\pi_v'')^\vee)}{L(s,\pi_v\times\pi_v'\times\pi_v'')}.$$

Define $\gamma_j(s,\pi_v\times\pi_v'\times\pi_v'')$ analogously, for $j=1,2$, using the L_j and ε_j-factors instead. (The gamma factor is called the ε'-factor by some, eg., in [Ik1,2].)

In view of Propositions 3.3.4, 3.3.6 and identity (3.3.5), Lemma 3.3.3 will follow once we establish

Proposition 3.3.7 *We have:*

$$\gamma(s,\pi_v\times\pi_v'\times\pi_v'') \;=\; \gamma_1(s,\pi_v\times\pi_v'\times\pi_v'').$$

Using [Sh4], one can show such an identity with γ_1 replaced by γ_2.

Proof of Proposition 3.3.7. By Theorem 3 of [Ik2], we know that

$$\gamma_1(s,\pi_v\times\pi_v'\times\pi_v'') \;=\; \gamma_1(s,\mu\otimes\pi_v'\times\pi_v'')\gamma_1(s,\nu\otimes\pi_v'\times\pi_v''). \tag{3.3.8}$$

By applying the identities (3.3.4), we then see that

$$\gamma_1(s,\pi_v\times\pi_v'\times\pi_v'') \;=\; \gamma_1(s,\mu\otimes\sigma_v'\otimes\sigma_v'')\gamma_1(s,\nu\otimes\sigma_v'\otimes\sigma_v''). \tag{3.3.9}$$

In view of (3.3.5), the right hand side of (3.3.9) equals $\gamma(s,\pi_v\times\pi_v'\times\pi_v'')$ as claimed. □

As noted above, this also proves Lemma 3.3.3.

Proof of Proposition 3.3.2: Define the *non-temperednes index* $\lambda(\tau)$ of a unitary irreducible τ of $\mathrm{GL}(2,F_v)$ to be t, resp. 0, if τ is a complelementary series attached to $(\mu|.|^t,\mu|.|^{-t})$ with $t>0$ and μ unitary, resp. τ is tempered, and set (as in [Ik1]) $\lambda(\pi_v,\pi_v',\pi_v'')=\lambda(\pi_v)+\lambda(\pi_v')+\lambda(\pi_v'')$. Then it is easy to see that

(3.3.10). $\quad L(s,\sigma_v\otimes\sigma_v'\otimes\sigma_v'')$ is holomorphic in $\Re(s)>\lambda(\pi_v,\pi_v',\pi_v'')$.

(This is the Galois analog of Lemma 2.1 of [Ik1].) We know by Shahidi ([Sh2]) that $\lambda(\tau)$ is always $<\frac{1}{5}$ for $\tau\in\{\pi_v,\pi_v',\pi_v''\}$. Consequently, $\lambda(\pi_v,\pi_v',\pi_v'')$ is always $<1/2$ unless all the three representations are non-tempered. In the former case, i.e., if $\lambda(\pi_v,\pi_v',\pi_v'')<1/2$, then $L(s,\sigma_v\otimes\sigma_v'\otimes\sigma_v'')$ and $L(1-s,\sigma_v^\vee\otimes(\sigma_v')^\vee\otimes(\sigma_v'')^\vee)$ have no common poles. Ditto for the L_1-functions by Lemma 3.3.3, and the assertion follows in this case.

It remains to treat the case when $\lambda(\pi_v,\pi_v',\pi_v'')\geq 1/2$ (which should not happen!). Suppose π_v is in the principal series associated to the characters μ,ν. The unitarity implies that the set $\{\overline{\mu},\overline{\nu}\}$ equals $\{\mu^{-1},\nu^{-1}\}$, and so one of the following happens: (i) $\overline{\mu}=\mu^{-1}$; (ii) $\overline{\mu}=\nu^{-1}$. In case (i), μ and ν are unitary, so we are in case (ii). So π_v is the complementary series representation associated to the pair of characters $(\mu,\overline{\mu}^{-1})$. We may write $\mu=\chi|.|^t$, for some unitary (possibly ramified) character χ and real number t. Then $\nu=(\overline{\chi}^{-1})|.|^{-t}=\chi|.|^{-t}$, and so π_v is $\pi_{1,v}\otimes\chi$ with $\pi_{1,v}$ unramified. Similarly, π_v' (resp. π_v'') is $\pi_{1,v}'\otimes\chi'$ (resp. $\pi_{1,v}''\otimes\chi''$), with $\pi_{1,v}',\pi_{1,v}''$ unramified and χ',χ'' unitary. Then the integral representation for triple product L-functions gives the identity (with $\xi=\chi\chi'\chi''$)

$$L_1(s,\pi_v\times\pi_v'\times\pi_v'')=L_1(s,\pi_{1,v}\times\pi_{1,v}'\times(\pi_{1,v}''\otimes\xi)).$$

Consequently, if all three local representations are non-tempered, we may assume, by replacing (π_v,π_v',π_v'') by $(\pi_{1,v},\pi_{1,v}',\pi_{1,v}''\otimes\xi)$, that at most one of them is ramified. If ξ is ramified, then both the L- and the L_1-functions of (π_v,π_v',π_v'') at s do not share any pole with the corresponding contragredient functions at $1-s$, and the

assertion of Proposition 3.3.2 follows. When ξ is unramified, we can appeal to Proposition 3.3.1.

□

Theorem 3.3.11 *Let π, π', π'' be cuspidal automorphic representations of $GL(2, \mathbb{A}_F)$ satisfying the following:*

- *At least one of $\{\pi, \pi', \pi''\}$ is non-dihedral; and*
- *At every finite place v, at least one of $\{\pi_v, \pi'_v, \pi''_v\}$ is in the principal series.*

Then $L(s, \pi \times \pi' \times \pi'')$ is entire.

In view of Proposition 3.3.2, it suffices to show that $L_1(s, \pi \times \pi' \times \pi'')$ is entire, and this follows from the work of Ikeda. But a remark is in order, however. In [Ik1], one finds an assertion (see Theorem 2.7 there) that $L_1(s, \pi \times \pi' \times \pi'')$ is entire as long as one of the three representations is not dihedral, *without* any local conditions. This is as it should be, but a small correction needs to be made in his proof. To be precise, in the case when $\omega^2 = 1, \omega \neq 1$ with associated quadratic extension K/F (see pp. 231-234 of [Ik1]), he first shows that $L_1(s, \pi_K \times \pi'_K \times \pi''_K)$ is entire and then proceeds to assert that **any pole** of $L_1(s, \pi \times \pi' \times \pi'')$ on $\Re(s) = 1$ is a pole of $L_1(s, \pi_K \times \pi'_K \times \pi''_K)$. We do not see how to verify this claim; the problem is at the bad factors. The way he tries to derive this is to assert the following two identities:

$$(Ik1) \qquad L_1(s, \pi \times \pi' \times \pi'') \;=\; L_2(s, \pi \times \pi' \times \pi''),$$

and

$$(Ik2) \qquad L_2(s, \pi_K \times \pi'_K \times \pi''_K) \;=\; L_2(s, \pi \times \pi' \times \pi'')L_2(s, \pi \times \pi' \times \pi'' \otimes \omega),$$

and then appeal to Shahidi's result ([Sh6]):

$$(3.3.12) \qquad L_2(s, \pi \times \pi' \times \pi'' \otimes \omega) \;\neq\; 0 \quad \text{if} \quad \Re(s) = 1.$$

We are unable to verify either of the identities (Ik1), (Ik2), though they both hold locally *almost everywhere*.

This can be easily fixed as follows. First note that at any place v, (3.3.8) holds with L replaced by L_2. Since $3/5$ is the maximum possible value of $\lambda(\pi, \pi', \pi'')$ (by [Sh1]), the local factors (of either of three L-functions) is invertible on $\{\Re(s) = 1\}$. So (3.3.12) holds with L_2 replaced by the incomplete L-function L_2^S, taken over the places outside a finite set S. Choose S to be such that [Ik1], [Ik2] both hold with L_1, L_2 replaced by L_1^S, L_2^S repectively. (Over K, we remove the factors at places above S.) It follows that $L_1(s, \pi \times \pi' \times \pi'')$ has a pole iff $L^S(s, \pi \times \pi' \times \pi'')$ does, which happens iff $L^S(s, \pi_K \times \pi'_K \times \pi''_K)$ and (hence) $L(s, \pi_K \times \pi'_K \times \pi''_K)$ do. The rest of Ikeda's argument goes through verbatim.

3.4. **Boundedness in vertical strips.** A very important prerequisite to the application of converse theorems is the knowledge that the relevant L-functions are, in addition to being entire, bounded in vertical strips. This is what we prove here for a class of triple product L-functions. After the proof of the main theorem in Section 3.8, the same will hold for all such L-functions.

Theorem 3.4.1 *Let F be a totally complex number field, and let π_1, π_2, π_3 be unitary, cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Assume that at any*

finite place v, at least one $\pi_{j,v}$ is in the principal series. Then the entire function $L(s, \pi_1 \times \pi_2 \times \pi_3)$ is bounded in vertical strips of finite width.

Proof. In view of Propositions 3.3.1 and 3.3.2, we only need to prove the assertion of the theorem for $L_1(s, \pi_1 \times \pi_2 \times \pi_3)$, which has an integral representation involving a Siegel Eisenstein series $\mathcal{E}(s)$ on GSp_6/F. Our key idea is to appeal to the fact that $\mathcal{E}(s)$ is an eigenfunction of the Laplacian and then to make use of Arthur's truncation of $\mathcal{E}(s)$.

We now need to set up some notations and preliminaries. Let

$$G = GSp(6)/F = \{g \in \mathrm{GL}_6/F \mid {}^t g J g = \lambda J, \text{ for some } \lambda \in \mathbb{G}_m\},$$

where $J = \left(\begin{smallmatrix} 0 & -I_3 \\ I_3 & 0 \end{smallmatrix}\right)$, with I_n denoting, for any $n \geq 1$, the identity $n \times n$-matrix. The root system relative to the diagonal torus A in G is of rank 3 and is described by

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i, j \leq 3\},$$

where e_1, e_2, e_3 are stadard basis vectors of Lie $T^*_{\mathbb{C}}$. Pick the following set of simple roots:

$$\Delta = \{\alpha = e_1 - e_2,\ \beta = e_2 - e_3,\ \gamma = 2e_3\}.$$

Denote by $P_0 = AU_0$ the corresponding minimal parabolic subgroup of G, whose elements are matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right), A, B, D \in M_3$, with A (resp. D) upper (resp. lower) triangular. For each subset θ of Δ, let A_θ denote $(\cap_{\delta \in \theta} \ker(\delta))^0 \subset A$, M_θ the centralizer of A_θ in G, and $P_\theta = M_\theta U_\theta$ the corresponding **standard parabolic** subgroup of G, split over F, containing P_0. The **Weyl group** W is a semidirect product $(\mathbb{Z}/2)^3 \rtimes S_3$, whose elements are made up of permutations of $\{e_1, e_2, e_3\}$ (possibly) composed with flips (sign changes) $e_i \mapsto -e_i$, for $i \in \{1, 2, 3\}$.

Let $P = P_{\{\alpha,\beta\}} = MU$ be the (maximal) Siegel parabolic subgroup with modular function δp. We have the matrix representation

$$P = \left\{ p = \left(\begin{array}{c|c} \lambda A & 0 \\ \hline 0 & {}^t A^{-1} \end{array}\right) \left(\begin{array}{c|c} I & X \\ \hline 0 & I \end{array}\right) \;\middle|\; \begin{array}{c} \lambda \in \mathbb{G}_m, A \in \mathrm{GL}_3 \\ X = {}^t X \end{array} \right\},$$

with $\delta_p(p) = |\det A|^6$. Let $K = \prod_v K_v$ be the standard maximal compact mod center subgroup of $G(\mathbb{A}_F)$.

Let ω_i be the central character of π_i, for each $i \leq 3$. Put $\omega = \omega_1\omega_2\omega_3$ and consider the space $J(\omega, s)$ of right K-finite functions f_s on $G(\mathbb{A}_F)$ satisfying forall $p \in P(\mathbb{A}_F)$ and $g \in G(\mathbb{A}_f)$):

$$f_s(pg) = \omega(\lambda)|\lambda|^{3s+\frac{3}{2}}\omega(\det A)|\det A|^{2s+1} f_s(g). \tag{3.4.2}$$

We will henceforth take f_s to be a *decomposable, entire and K-finite section*, which is *good* in the sense of [Ik1], [PS-R]. The corresponding local space at v will be denoted $J_v(\omega, s)$. The relevant Eisenstein series is then given by the formula ($\forall\, g \in G(\mathbb{A}_F)$)

$$E(f_s)(g) = \sum_{\gamma \in P(F)\backslash G(F)} f_s(\gamma g). \tag{3.4.3}$$

(Note that $J(\omega, s)$ corresponds to $I(\omega, 2s-1)$ in Proposition 1.6 of [Ik1].)

Let $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ be a cusp form in the space of $\pi_1 \otimes \pi_2 \otimes \pi_3$ on $H(\mathbb{A}_F)$, where

$$H = \{(g_1, g_2, g_3) \in \mathrm{GL}_2^3 \mid \det g_1 = \det g_2 = \det g_3)\}.$$

Denote the corresponding Whittaker function by $W = W_2 \otimes W_2 \otimes W_3$. Then one knows (see [Ik1], [PS-R2]) that in $\mathrm{Re}(s) \gg 0$ one has the identity (with $f_s = \otimes_v f_{v,s}$):

$$\langle E(f_s), \varphi \rangle_H := \int_{C(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} E(h, f_s)\varphi(h)dh) = \prod_v \Psi(f_{v,s}; W_v), \tag{3.4.4}$$

where each $\Psi(f_{v,s}; W_v)$ is a local integral having $L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ as its g.c.d. By [PS-R2] one knows that at every unramified v, $\Psi(f_{v,s}; W_v)$ coincides with $L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ for a suitable choice of $f_{v,s}$. Hence we get

$$L_1(s, \pi_1 \times \pi_2 \times \pi_3) \prod_{v \in S} \frac{\Psi(f_{v,s}; W_v)}{L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})} \; = \; \langle E(f_s), \varphi \rangle_H \,,$$

where S is a finite set of places containing the ramified and archimedean places.

Recall that $L_1(s, \pi_1 \times \pi_2 \times \pi_3)$ has an Euler product in $\Re(s) > 1$ and a functional equation. It is clear that it is bounded for $\Re(s) >> 0$. So by the Phragman-Lindelöf theorem, Theorem 3.4.1 will be proved if we establish that this L-function is of bounded order in vertical strips of finite width.

Lemma 3.4.5. *For each v, the function $\frac{\Psi(f_{v,s}; W_v)}{L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})}$ is entire and of bounded order, for a suitable choice of $f_{v,s}$.*

Proof of Lemma. Put $g(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) = \frac{\Psi(f_{v,s}; W_v)}{L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})}$. The entireness of this function follows from the fact that $L_1(s, \pi_{1,v} \times \pi_{2,v}$ is the gcd of the local integrals $\Psi(f_{v,s}; W_v)$ as $(f_{v,s}, W_v)$ vary. The fact that $g(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ is of bounded order is obvious at any v-adic place as the local integral is a rational function of Nv^{-s}. So let v be archimedean. Since we have taken F to be totally complex, F_v is $\mathbb{C}$. There is a local functional equation

$$g(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) \; = \; \varepsilon_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) g(1-s, \overline{\pi}_{1,v} \times \overline{\pi}_{2,v} \times \overline{\pi}_{3,v}^{\vee}),$$

where the intervening ε-factor is, thanks to Proposition 3.3.2 (b), an exponential function. So it suffices to prove that, for a suitable choice of $f_{v,s}$, the local integral $\Psi(f_{v,s}; W_v)$ is of bounded order in $\Re(s) \geq 1/2$.

Let λ denote $\lambda(\pi_{1,v}) + \lambda(\pi_{2,v}) + \lambda(\pi_{3,v})$, where, for each j, $\lambda(\pi_{j,v})$ denotes, as in the proof of Proposition 3.3.2 on page 17, the *non-temperedness index* of $\pi_{j,v}$. It is known ([Ik3]) that the integral $\Psi(f_{v,s}; W_v)$ is absolutely convergent in $\Re(s) > \lambda$, defining a holomorphic function there. Noting that $|f_{v,s}|$ transforms under the left action of $P(F_v)$ by a character depending only on $\Re(s)$, we see that $\Psi(f_{v,s}; W_v)$ is bounded in any vertical strip $\{a \leq \Re(s) \leq b\}$ if $a > \lambda$.

Now suppose that at least one $\pi_{j,v}$ is tempered. By Gelbart-Jacquet ([GJ]), one has $\lambda(\pi_{i,v}) < 1/4$ for any i. (For $F = \mathbb{Q}$, one even has the archimedean estimate: $\lambda(\pi_{i,v}) < 5/28$ by [LRS1].) So $\lambda < 1/2$ in this case, and by the remarks above, $\Psi(f_{v,s}; W_v)$ is bounded in any vertical strip $\{1/2 \leq \Re(s) \leq b\}$ of finite width.

It remains to consider when *every* π_j is non-tempered. Arguing as in the proof of Proposition 3.3.2, we may assume that $\pi_{1,v}$ and $\pi_{2,v}$ are unramified and that $\pi_{3,v}$ is the twist of an unramified representation by a unitary character ξ which is ramified if non-trivial. If $\xi = 1$, i.e., when each $\pi_{j,v}$ is unramified, it is proved in [Ik3] that there exists a function $f_{v,s}$ such that $\Psi(f_{v,s}; W_v)$ equals $L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$. So we are done in this case by the standard properties of the Gamma function. Now suppose ξ is non-trivial. We can then use the ξ-twisted form of the unramified $f_{v,s}$ emplyed by Ikeda, in which case $\Psi(f_{v,s}; W_v)$ will be essentially given

by the integral expression preceding Lemma 2.3 in [Ik3], the change being that the integrand will get multiplied by $\xi(\sqrt{xyz}R^{-1}z)\tilde{\xi}(k)$, where $\tilde{\xi}$ is the character of the maximal compact K_v whose restriction to $K_v \cap P(F_v)$ is ξ. (Here the notations are as in [Ik3].) Now we appeal to the fact that the Whittaker function $W_{1,v}$ (resp. $W_{2,v}$, resp. $W_{3,v}$) satisfies a second order differential equation on $\mathrm{diag}(x,1)$ (resp. $\mathrm{diag}(y,1)$, resp. $\mathrm{diag}(z,1)$). Using integration by parts, we can then express an elementary function times $\Psi(f_{v,s}; W_v)$ in terms of a finite sum of integrals, each of which converges absolutely in $\Re(s) > \lambda - 1/2$ and is bounded in $\{a \leq \Re(s) \leq b\}$ if $a > \lambda - 1/2$. Since λ is $< 3/4$ by [GJ], we are done.

□

Since an entire function which is the ratio of two entire functions of bounded order is itself of bounded order, it suffices then to establish that the entire function $\langle E(f_s), \varphi \rangle_H$ is of bounded order. (Such a reduction was employed in [RS] for an analogous situation.)

Since φ vanishes rapidly at infinity, it will suffice to establish the following

Proposition 3.4.6 *Fix a vertical strip Ξ in $\mathbb{C}$ of finite width, and a Siegel set $\mathcal{S}$ in $G(\mathbb{A}_F)$. Then there exist $N \in \mathbb{Z}$ and a function $\Phi(s)$ of of bounded order, depending only on the width of Ξ and $\mathcal{S}$, such that*

$$|E(f_s)(g)| \leq |\Phi(s)|\,||g||^N,$$

for all g in $\mathcal{S}$.

By a function of bounded order we mean a quotient $\xi_1(s)/\xi_2(s)$ such that, for some $t_0 \geq 0$, $C > 0$, and $M \in \mathbb{Z}_+$, we have for all $s = x + iy \in \Xi$ with $y \geq t_0$, the bound $|\xi_j(s)| \leq Ce^{|y|^M}$, for $j = 1, 2$.

Proof of Proposition. Let us choose a sufficiently small compact, open subgroup K of $G(\mathbb{A}_{F,f})$ such that $E(f_s)$ is right invariant under K, and such that $Y := G(F)Z_\infty^+\backslash G(\mathbb{A}_F)/K$ is a smooth manifold. Fix an $\epsilon > 0$, and also a chart on Y defining the differentiable structure. We will work with the Siegel set modulo K, though we will suppress this in our notation.

For every g in $\mathcal{S}$, choose a neighborhood $V = V_g$ of g with non-empty interior, which is small enough to lie inside the unit ball of radius ϵ in local coordinates.

Recall that that $E(f_s)$ is an eigenfunction of the Laplacian Δ on all of Y, as it is defined by the Casimir operator, and it is moreover *locally* in L^p, for any $p \geq 1$. we will exploit this.

Lemma 3.4.7 *Let g, V be as above. Then there is a polynomial $P(s)$ independent of g such that*

$$||E(f_s)(g)|| \leq |P(s)|.||E(f,s)||_{2,V}.$$

Proof of Lemma Let $V' = V'_g$ be any neighborhood of g with non-empty interior such that its closure is contained in the interior of V. It suffices to prove the asserted bound for $||E(f,s)||_{\infty,V'}$. (We write $||.||_{p,X}$ for the L^p-norm over X.) As remarked above, $E(f_s)$ is an eigenfunction, say with eigenvalue λ_s, of the second order elliptic differential operator Δ. It can be seen that $E(f_s)$ is in the localized Sobolev space $\mathcal{H}_n^{\mathrm{loc}}(V)$ for any n. And since $\bar{V}' \subset V$, $E(f_s)$ equals a function in $\mathcal{H}_n(V)$ on V'.

Lemma 3.4.8 *There exists a polynomial $Q(X)$ in $\mathbb{C}[X]$, depending only on ϵ and λ_s such that*

$$||E(f,s)||_{\infty,V'} \leq |Q(\lambda_s)|.||E(f,s)||_{2,V}.$$

Since λ_s is a polynomial, in fact quadratic, in s, Lemma 3.4.7 follows from this. When λ_s is real, which happens in our case iff s lies on the unitary cross, the results of section 4 of [So], in particular the identity (4.2) and Lemma 4.1 , easily imply Lemma 3.4.8. The methods of [So] extend to the case of complex eigenvalues as well, but T. Wolff has pointed out to us how this Lemma can be deduced more directly from the local regularity properties of elliptic differential operators; see Lemma 3.4.9 below. (The difficult part of [So] involves precisely bounding the degree of $Q(X)$, but it is not important for our purposes.) For any r, denote by $||.||_{(2,r);X}$ norm associated to the rth L^2-derivative on X so that $||u||_{(2,r);X}$ equals $\sum_{|\nu|\leq r}||\partial^\nu u||_{2,X}$, with the ∂^ν denoting distribution derivatives.

Lemma 3.4.9 *Let Ω be a subset of $\mathbb{R}^N$ contained in the unit ball of radius ϵ, and Ω' a subset of Ω with non-empty interior such that $\bar{\Omega}' \subset \Omega$. Then we have the following:*

- *(1) For any integer $r > N/2$, there exists a constant $C_1 > 0$ depending only on ϵ and Δ such that, for all u in $\mathcal{H}_r(\Omega)$,*

$$||u||_{\infty,\Omega} \leq C_1||u||_{(2,r);\Omega}.$$

- *(2) For any integer $i \geq 2$, there is a constant $C_i > 0$ depending only on ϵ and Δ, such that for any $u \in \mathcal{H}_i(\Omega)$:*

$$||u||_{(2,i);\Omega'} \leq C_i\left(||u||_{2,\Omega'} + ||\Delta u||_{(2,i-2);\Omega}\right).$$

Proof of Lemma 3.4.9. If we write $\mathcal{H}^0(\Omega)$ for the subspace of $\mathcal{H}(\Omega)$ obtained by completing $C_c^\infty(\Omega)$, then the first (resp. second) assertion of this lemma is proved on pages 148-151 (resp. 262-267) of [GiT] (resp. [Fo]). Moreover, given any u in $\mathcal{H}_r(\Omega)$, for any r, and a subset Ω' of non-empty interior with its closure contained in the interior of Ω, we can find a smooth cut-off function ψ such that $v = \psi u$ is in $\mathcal{H}_r^0(\Omega)$ and $v = u$ on Ω' (see, for example, pages 275-276 of [Fo]), and we can do this uniformly for all r. Since we can bound $||\psi u||_{(2,r),\Omega}$, by a universal constant times $||u||_{(2,r),\Omega}$, we get the first part of the Lemma, and for the second part we only need to bound $||\Delta(\psi u)||_{(2,i-2),\Omega}$ by a constant times $||\Delta(u)||_{(2,i-2),\Omega} + ||u||_{(2,i-1),\Omega}$. We will suppress Ω henceforth. Recall that $||\Delta(\psi u)||_{(2,i-2),\Omega}$ is (by definition) $\sum_{|\nu|\leq i-2}||D^\nu\Delta(\psi u)||$, which is $\sum_{|\nu|\leq i-2}||D^\nu((\Delta\psi)u + (\nabla\psi)(\nabla u) + \psi(\Delta u))||_2$; this can be bounded by

$$\sum_{|\nu|\leq i-2}\sum_{\beta+\gamma=\nu}(||D^\beta(\Delta\psi)D^\gamma(u)||_2 + \dots).$$

It is easy to estimate derivatives of ψ, and moreover, all but one of the terms involves at most $i-1$ derivatives of u and is hence bounded by a constant times $||u||(2, i-1)$. The remaining term involves $i-2$ derivatives of Δu, and this is bounded by a constant times $||\Delta u||(2, i-2)$. The constants involved are controllable in terms of ϵ.

□

Proof of Lemma 3.4.8. Fix a g_0 and identify an open neighborhood of g_0 containing V with an open neighborhood of $\mathbb{R}^N$, N =dim(Y), contained in the

unit ball of radius ϵ. By shrinking V, we may asume that $E(f,s)$ is in $\mathcal{H}_i(V)$ for all i. By the first part of Lemma 3.4.9, we are left to estimate $||E(f_s)||_{(2,r),V}$ for an $r > N/2$, which we will take to be of the form $2m$. Choose neighborhoods V_j, $0 \leq j \leq m-3$, of g such that we have the nested inclusions

$$V' = V_{m-3} \subset \ldots V_1 \subset V_0 = V,$$

such that the closure of each V_i is in the interior of V_{i+1}. Apply the second part of Lemma 3.4.9 succesively with $u = E(f_s)$ and $(i, \Omega', \Omega) = (2(m-j), V_{m-j}, V_{m-j+1})$, with j ranging over $\{0, \ldots, m-3\}$, repeatedly making use of the fact that $\Delta E(f_s)$ is $\lambda_s E(f_s)$. The assertion of Lemma 3.4.8 follows at g_0. Now if g is any other point, we can translate the neighborhood V to g and see that the G_∞-invariance of Δ gives the corresponding result with the same constant.

□

Let $T \in \mathrm{Lie}\, A_{\mathbb{R}}$ be a sufficiently regular parameter, i.e., with $\langle \nu, T \rangle$ being sufficiently large for each ν in Δ. Denote by $\wedge^T E(f_s)$ the Arthur truncation of $E(f_s)$ (cf. [A1] or [MW1], p. 35). One knows that $\wedge^T E(f_s)$ decreases rapidly at infinity, and is hence is in $L^p(Y)$ for all p. Clearly, $||\psi||_{2,C_g}$ is bounded above by $||\psi||_{2,G}$ for any square-integrable function ψ on Y. (By abuse of notation, we are writing $||.||_{p,G}$ instead of $||.||_{p,Y}$.) So, in view of Lemma 3.4.8, Proposition 3.4.6 (and the Theorem) will be proved once we establish the following

Proposition 3.4.10 *Fix $\Xi, \mathcal{S}$ as above. Then there exist $N \in \mathbb{Z}$ and functions $\Phi_1(s), \Phi_2(s)$ of bounded order such that, for all g in $\mathcal{S}$,*

$$(i) \qquad || \wedge^T E(f_s)||_{2,G} \leq |\Phi_1(s)|$$

and, up to shrinking $V = V_g$ to a smaller neighborhood of g with non-empty interior,

$$(ii) \qquad ||E(f,s) - \wedge^T E(f,s)||_{\infty,V} \leq |\Phi_2(s)|\, ||g||^N.$$

Our first object will be to prove the bound (i) of this Proposition. For the sake of an inductive argument later, we will in fact need to analyze the L^2-norm of the truncation of a class of *relative* Eisenstein series $E^{P'}(h_s^w)$ (see (3.4.15) below) attached to any standard parabolic subgroup $P' = M'U'$ and a Weyl element w representing a double coset in $W_{M'} \backslash W / W_M$. We set

$$h_s^w = M_w(f_s),$$

where M_w is the intertwining operator defined by

$$M_w(f_s)(g) = \int_{(U_0 \cap w\bar{U}_0 w^{-1})(\mathbb{A}_F)} f_s(w^{-1}ug)\, du.$$

Here $\bar{U}_0$ denotes the opposite unipotent subgroup of the maximal unipotent subgroup U_0.

Remark 3.4.11. Since f_s is good, it factorizes as $\prod_v f_{v,s}$ and at any v where ω is unramified, $f_{v,s}$ is the standard function $\phi_{v,s}$ and $M_w(\phi_{v,s}) = c_v^w(s)\phi_{v,s}^w$, where $c^w(s) = \prod_v c_v^w(s)$ is a ratio of **abelian L-functions** – see [Ik1], [PS-R1]. Moreover, the possible poles are contained (cf. [Ik1]) in the set $\{\frac{3}{2}, 1, 0, -\frac{1}{2}\}$.

Recall that every element w in the Weyl group W is determined by what it does to $\{e_1, e_2, e_3\}$, and that each e_i gets sent to $\pm e_j$, for some j. We will adopt the

following notation: If w sends e_1 to e_3, e_2 to $-e_2$ and e_3 to $-e_1$, for example, we will denote w by $[3, \bar{2}, \bar{1}]$. Put

$$\Sigma_M = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\} \tag{3.4.12}$$

with $w_1 = \text{id.} = [1, 2, 3]$, $w_2 = [\bar{2}, \bar{3}, 1]$, $w_3 = [\bar{1}, \bar{2}, \bar{3}]$, $w_4 = [1, 2, \bar{3}]$, $w_5 = [1, 3, \bar{2}]$, $w_6 = [1, \bar{3}, \bar{2}]$, $w_7 = [2, \bar{1}, 3]$ and $w_8 = [2, \bar{3}, 1]$.

For each (standard) parabolic $P' = M'U' \neq G$, define a subset $\Sigma_{M,M'}$ of Σ_M by the following table:

(3.4.13)

$\Delta_{M'}$	$\Sigma_{M,M'}$
$\{\alpha, \beta\}$	$\{w_1, w_2, w_4, w_6\}$
$\{\alpha, \gamma\}$	$\{w_1, w_2, w_5\}$
$\{\beta\gamma\}$	$\{w_1, w_7\}$
$\{\alpha\}$	$\{w_1, w_2, w_3, w_4, w_5, w_6\}$
$\{\beta\}$	$\{w_1, w_3, w_4, w_6, w_7, w_8\}$
$\{\gamma\}$	$\{w_1, w_2, w_5, w_7\}$
ϕ	Σ_M

Lemma 3.4.14 *Let $P' = U'M'$ be a standard parabolic. Then $\Sigma_{M,M'}$ is a set of representatives for $W_{M'}\backslash W/W_M$, where $W_{M'}$ (resp. W_M) denotes the Weyl group of the roots relative to M' (resp. M). Moreover, for any $w \in \Sigma_{M,M'}$, $Q'_w := wPw^{-1} \cap M'$ is a parabolic subgroup of M', $P'_w := Q'_w \cdot U'$ (but not $wPw^{-1} \cap P'$) is a (standard) parabolic subgroup of G contained in P', and if we put $U'_w = wPw^{-1} \cap U'$, then $U' = U'_w \cdot V'_w$, where $V'_w = w\bar{U}_0 w^{-1} \cap U_0$.*

Proof of Lemma. Put

$$W^*_{M,M'} = \left\{ w \in W \;\middle|\; \begin{array}{c} w(\nu) > 0, \ \forall\, \nu \in \Delta_M; \\ w^{-1}(\nu) > 0, \ \forall\, \nu \in \Delta_{M'} \end{array} \right\}.$$

Then it is well known that $W^*_{M,M'}$ is a set of representatives in W for $W_{M'}\backslash W/W_M$. Explicitly, one has, when $\Delta_{M'}$ is empty,

$$W^*_{M,M'} = \left\{ \begin{array}{c} w_1, \bar{w}_2 := [3, \bar{2}, \bar{1}], \ \bar{w}_3 := [\bar{3}, \bar{2}, \bar{1}], \ w_4, w_5, \\ w_6, \bar{w}_7 := [2, \bar{3}, \bar{1}], \ w_8 \end{array} \right\}.$$

The only difference between this set and the Σ_M defined in the previous page is that w_2, w_3, w_7 are replaced by $\bar{w}_2, \bar{w}_3, \bar{w}_7$. But it is easy to see that for $j = 2, 3, 7$, $\bar{w}_j = w_j w'_j$, with $w'_j \in W_M$. Thus, when $\Delta_{M'}$ is empty, $\Sigma_{M,M'}$ is also a set of representatives for $W_{M'}\backslash W/W_M$ in W. The other cases of M' can be checked by comparing with the table on page 21 for $\Sigma_{M,M'}$. The remaining assertions of the Lemma follow by a straightforward computation.

The relative Eisenstein series of interest to us attached to (P', w) is given by

$$E^{P'}(h^w_s)(g) = \sum_{\gamma \in Q'_w(F)\backslash M'(F)} h^w_s(\gamma g). \tag{3.4.15}$$

Denote by $\wedge^{T,P'} E^{P'}(h^w_s)$ the truncation of $E^{P'}(h^w_s)$ as defined on page 97 of [A1]. Let $\langle \wedge^{T,P'} E^{P'}(h^w_s), \wedge^{T,P'} E^{P'}(h^w_s) \rangle_{P',T}$ denote the scalar product defined on page 45 of [A2], which specializes to usual scalar product when $P' = G$.

Proposition 3.4.16 *$\langle \wedge^{T,P'} E^{P'}(h_s^w), \wedge^{T,P'} E^{P'}(h_s^w)\rangle_{P',T}$ is of bounded order.*

Proof of Proposition 3.4.16. To mesh better with the notations of Langlands and Arthur we will shift the parameter s by $1/2$ so that the unitary axis is the line $\Re(s) = 0$ (instead of $\Re(s) = 1/2$), but by abuse of notation we will continue to denote the Eisenstein series by $E^{P'}(h_s^w)$.

For s, s' in $\mathbb{C}$, set

$$h(s,s') \;=\; \langle \wedge^{T,P'} E^{P'}(h_s^w), \wedge^{T,P'} E^{P'}(h_{-\overline{s'}}^w)\rangle_{P',T}, \tag{3.4.16}$$

which is a meromorphic function in (s,s') whose poles are in a union of vertical lines $s = c_i$ and horizontal lines $s' = c'_j$, depending on the polar set of $E^{P'}(h_s^w)$, which is in turn governed (cf. [La4]) by the constant term along P_0. The intersection of this polar set of $h(s,s')$ with $\{\overline{s} + s' = 0\}$ has only a *finite* number of points in Ξ. So, for large enough t_0, the function $s \to h(s, -\overline{s})$ is well defined and finite on $\{s = x + iy \in \Xi \quad | \quad |y| \geq t_0\}$.

By Langlands we can write $E^{P'}(h_s^w)$ in terms of residues of *cuspidal* Eisenstein series. In the notation of [A2], page 43, it is a finite sum of terms of the form

$$\mathrm{R}es_{\Lambda \to \Lambda_S}\left(F_B(\Lambda), \Lambda + \beta s\right), \tag{3.4.17}$$

where S is a suitable flag of affine subspaces $\mathfrak{t} = \Lambda_{\mathfrak{t}} + \mathrm{Lie}(A_{P'_w})^*$ in $\mathrm{Lie}(A_B)^*$, β in $\mathrm{Lie}(A_{P'_w})^*$, $B = M_B U_B$ a relevant parabolic subgroup of P'_w, and F_B a meromorphic function of Λ on $\mathrm{Lie}(A_B)^*$ with singularities lying in hyperplanes.

Note that h_s^w is asociated to the representation induced from P'_w by a one-dimensional representation, which is residual on M'_w associated to a representation induced by a character ξ_Λ^w of the minimal paraboloic P_0, which is exponential in Λ and of bounded order in each coordinate direction. The construction of Langlands ([La4]) is natural relative to induction in stages, and we see that $E^{P'}(h_s^w)$ must be the residue of the cuspidal Eisenstein series defined by ξ_Λ^w. Hence the only relevant B occurring in (3.4.17) is P_0. In this case the function $F_B(\Lambda)$ is right K-finite in the space of the representation induced by ξ_Λ^w. By the Iwasawa decomposition we see then that $F_B(\Lambda)$ is of exponential type.

By Lemma 3.1 of [A2], we see that $h(s,s')$ is a finite sum of double residues

$$\mathrm{R}es_{\Lambda \to \Lambda_S} \mathrm{R}es_{\Lambda' \to \Lambda_{S'}} \omega^{T,P'}(\Lambda + \beta s, -\Lambda' + \beta s', F_B(\Lambda), F_B(\Lambda')), \tag{3.4.18}$$

where (cf. page 46 of [A2])

$$\omega^{T,P'}(\lambda, \lambda', \phi, \phi') \;=\; \sum_{P_1 \subset P'} \sum_{t,t' \in W^{P'}(B,P_1)} \frac{\langle M_{t,\lambda}(\phi), M_{t',-\overline{\lambda'}}(\phi')\rangle e^{(t\lambda - t'\lambda')(T)}}{a(P_1,P') \prod_{\nu \in \Delta_{P_1}^{P'}} (t\lambda - t'\lambda')(\nu^\vee)}.$$

Here $a(P_1,P')$ is a non-zero constant, and $W^{P'}(B,P_1)$ is the set of isomorphisms, possibly empty, of $\mathrm{Lie}A_B$ onto $\mathrm{Lie}A_{P_1}$ leaving $\mathrm{Lie}A_{P'}$ pointwise fixed which arise from the restricted Weyl group $W(G,A_0)$. (In our particular case, P_1 has to be $B = P_0$.) We refer to [A2] for a definition of $\Delta_{P_1}^{P'}$.

The c-functions associated to the intertwining operators $M_{t,\lambda}$ (of (3.4.18)) acting on functions $\phi(\lambda)$ coming from induced representations from the Borel are, just like the c_s^w associated to M_w discussed in Remark 3.4.11, ratios of abelian L-functions ([La7]), hence bounded of order 1. Combining this with that fact that $F_B(\Lambda)$ is of pure exponential type, we see that the expression $\omega^{T,P'}(\Lambda + \beta s, -\Lambda' +$

$\beta s, F_B(\Lambda), F_B(\Lambda'))$ is of bounded order in Λ and Λ'. Its singularities are in hyperplanes, and evaluating the residues for $\Lambda \to \Lambda_S$ and $\Lambda' \to \Lambda_{S'}$, we see that the double residue of (3.4.18) must be of bounded order in (s, s') along any direction. This then implies the result we want about $h(s, -\overline{s})$ in $\{s = x+iy \in \Xi \quad | \quad |y| \geq t_0\}$. □

Proof of Proposition 3.4.10. Part (i) of this Proposition clearly follows by applying Proposition 3.4.16 in the special case $P' = G$. It remains to prove (ii). First we need to derive a manageable expression for $E(f_s) - \wedge^T E(f_s)$. This takes some work and we start with some preliminaries.

For each standard parabolic $P' = M'U'$, $M' = \mathrm{Cent}_G(A')$, put

$$M_0' = \left\{ m \in M'(\mathbb{A}_F) \;\middle|\; \begin{array}{c} |\chi(m)| = 1, \\ \forall\, \chi \in \mathrm{Hom}(M', \mathbb{G}_m) \end{array} \right\}.$$

Then there exists a map

$$\log_{M'} : M'(\mathbb{A}_F) \longrightarrow \mathrm{Lie}(A')_{\mathbb{C}}$$

defined so as to satisfy $\langle \chi, \log_{M'}(m) \rangle = \log |\chi(m)|$, for all $m \in M'(\mathbb{A}_F)$ and $\chi \in \mathrm{Hom}(M', \mathbb{G}_m)$. Then $M_0'(\mathbb{A}_F)$ is precisely $\ker(\log_{M'})$. On the other hand, the Iwasawa decomposition $G(\mathbb{A}_F) = U'(\mathbb{A}_F)M'(\mathbb{A}_F)K$ leads to a map

$$m_{P'} : G(\mathbb{A}_F) \longrightarrow M_0'(\mathbb{A}_F) \backslash M'(\mathbb{A}_F),$$

given by sending g to the coset $M_0'(\mathbb{A}_F)m$, if $g = umk$, with $u \in U'(\mathbb{A}_F)$, $m \in M'(\mathbb{A}_F)$ and $k \in K$. Thus $\log_{M'} \circ m_{P'}$ is a map from $G(\mathbb{A}_F)$ into $\mathrm{Lie}(A')_{\mathbb{C}}$.

Note that A_Δ is the center Z of G. We have a decomposition $\mathrm{Lie}\, A'_{\mathbb{R}} = \mathrm{Lie}\, Z_{\mathbb{R}} \oplus \sigma^G_{M'}$, where $\sigma^G_{M'}$ is the subspace generated by the set $\Delta_{M'}$ of simple roots relative to M'. Denote by $\hat{\tau}_{P'}$ the characteristic function of the sum of $\mathrm{Lie}\, Z_{\mathbb{R}}$ and the interior of the cone of $\sigma^G_{M'}$ generated by $\Delta_{M'}$. Then one has (see p. 35 of [MW1]) $(\forall\, G \in G(\mathbb{A}_F))$:

(3.4.17)

$$\begin{aligned} &(E(f_s) - \wedge^T E(f_s))(g) \\ &= \sum_{P_0 \subseteq P' \underset{+}{\subset} G} (-1)^{(\dim A' - \dim A_0)} \sum_{\gamma \in P'(F) \backslash G(F)} \hat{\tau}_{P'}(\log_{M'}(m_{P'}(\gamma g) - T') E(f_s)_{P'}(\gamma g), \end{aligned}$$

where T' is the projection of T in $\mathrm{Lie}\, A'_{\mathbb{R}}$, and $E(f_s)_{P'}$ is the **constant term** of $E(f_s)$ along P', namely:

$$E(f_s)_{P'}(g) = \int_{U'(F) \backslash U'(\mathbb{A}_F)} E(f_s)(ug)\, du. \tag{3.4.20}$$

Proposition 3.4.21 *(a) For each P', the constant term of $E(f_s)$ along P' is given by*

$$E(f_s)_{P'}(g) = \sum_{w \in \Sigma_{M, M'}} E^{P'}(h_s^w)(g)$$

for all g in $G(\mathbb{A}_F)$.

(b) For T large enough (and regular), we have

$$E(f_s) - \wedge^T E(f_s)$$
$$= \sum_{P_0 \subseteq P' \subset_+ G} (-1)^{(\dim A' - \dim A_0)} \sum_{w \in \Sigma_{M,M'}} \hat{\tau}_{P'}(\log_{M'}(m_{P'}(g) - T') E^{P'}(M_w(f_s))(g).$$

Remark: For $P' = P_0$, the expression given in part (a) is well known by the work of Rallis and Piatetski-Shapiro ([PS-R1]) – see also [Ik1]. For **cuspidal** Eisenstein series (which is **not** what we have here), there is a general formula for any reductive group G – see Section II.1.7 of [MW1], for example.

Proof of Proposition 3.4.21 (a) Combining (3.4.3) and Lemma 3.4.14, and remembering that U'_w denotes $wPw^{-1} \cap P'$, we get

$$E(f_s)_{P'}(g) = \sum_{w \in \Sigma_{M,M'}} \sum_{\gamma \in Q'_w(F) \backslash M'(F)} \int_{U'_w(F) \backslash U'(\mathbb{A}_F)} f_s(w^{-1} u \gamma g)\, du.$$

Now we use the decomposition $U' = U'_w \cdot V'_w$ of Lemma 3.4.14, and recall the $P(F)$-invariance of f_s together with the fact that the intertwining operator M_w is defined (see (3.4.12) by integration over $V'_w(\mathbb{A}_F)$. We see that, up to normalizing the measures so that volume of the compact group $(wUw^{-1} \cap U')(F) \backslash (wUw^{-1} \cap U')(\mathbb{A}_F)$ is 1,

$$E(f_s)_{P'}(g) = \sum_{w \in \Sigma_{M,M'}} \sum_{\gamma \in Q'_w(F) \backslash M'(F)} M_w(f_s)(\gamma g)\, du.$$

This proves (a).

(b) By definition, $Q'_w \backslash M' = P'_w \backslash P'$. Since $m_{P'}$ is left invariant under elements of $P'(F)$, the assertion is a consequence, in view of the identity of (a), of the following

Lemma 3.4.22 *For g lying in a fixed Siegel domain, we have for T sufficiently large (and regular):*

$$\hat{\tau}_{P'}(\log_{M'}(\gamma g)) - T') E_w^{P'}(f_s(\gamma g)) = 0 \quad \text{if} \quad \gamma \in G(F) - P'(F).$$

Proof of Lemma. We may restrict ourselves to a Siegel domain of the form $\mathfrak{S}_c = \Omega A'_c K$, where Ω is a compact subset of $P'(\mathbb{A}_F)$, c a positive real number, and A'_c the set of elements t in $A'(\mathbb{A}_F)$ such that $|\nu(t)| > c$ for each ν in $\Delta_{M'}$. As usual, one chooses c to be small enough and Ω large enough so that $G(\mathbb{A}_F) = P'(F)\mathfrak{S}_c$.

Let $\gamma \in G(F) - P'(F)$. Then by the Bruhat decomposition, $\gamma = \bar{u}w$, for some Weyl element w, $\gamma \in P'(F)$ and $\bar{u} \in \bar{U}'_w(F) := wU'(F)w^{-1}$.

By the Iwasawa decomposition relative to P', we can write any g in $G(\mathbb{A}_F)$ as a product $uxtk$, with $u \in U'(\mathbb{A}_F)$, $x \in M'_0(\mathbb{A}_F)$, $t \in A'(\mathbb{A}_F)$ and $k \in K$. For any root ν (in $\Phi_{M'}$) let us write $H_\nu(g)$ for $|\nu(t)|$, which is well defined. By the Iwasawa decomposition relative to P_0, we can also write $g = nt_0k_1$ with $n \in U_0(\mathbb{A}_F)$, $t_0 \in A_0(\mathbb{A})$ and $k \in K$. Then $H_\nu(g)$ identifies with ν evaluated at the projection of t_0 in $A'(\mathbb{A}_F)$. For $w \in W$, we will write $\nu^w(t) = \nu(t^w) := \nu(wtw^{-1})$.

Claim 3.4.23. Let $\bar{u} \in \bar{U}'_w(F)$. Then $\exists\, \nu \in \Phi^+_{M'}$ such that $\nu^w = \nu^{-1}$ on A' and $H_\nu(\bar{u}) \leq 1$.

Suppose the claim holds. Put $\gamma = y\bar{u}w \in G(F)$. Then for any $g = uzk \in G(\mathbb{A}_F)$, with $z \in U'(\mathbb{A}_F)M'(\mathbb{A}_F)$, $u \in U'(\mathbb{A}_F)$, $k \in K$, we have

$$H_\nu(\gamma g) = H_\nu(\bar{u}wuzk) = H\nu(\bar{u}u^w z^w) = H_\nu(z^w)H_\nu((z^w)^{-1}(\bar{u}u^w)z^w).$$

Note that $H_\nu(z^w) = H_{\nu^w}(z) = H_\nu(g)^{-1} < c^{-1}$. On the other hand, since the conjugate of $\bar{u}u^w$ by z^w lies in $\bar{U}'_w(F)$, we have by the claim, $H_\nu((z^w)^{-1}(\bar{u}u^w)z^w) \leq 1$. So $H_\nu(\gamma g) < c^{-1}$. Since ν is a positive sum of roots in $\Delta_{M'}$, we can take T to be large enough so that $\nu(T') > c^{-1}$. Then we must have $\hat{\tau}_{P'}(\log_{M'}(\gamma g)) - T') = 0$.

It remains to prove the claim. We can define local "heights" $H_{\nu,v}$ in the obvious way to get $H_\nu(g) = \prod_v H_{\nu,v}(g_v)$. So it suffices to find a ν with $\nu^w = \nu^{-1}$ such that $H_{\nu,v}(\bar{u}_v) \leq 1$ everywhere. It is instructive to begin by looking at the situation for the basic case $G' = SL_2$, and the argument for G is simply a jazzed up version. Recall that at each v, the standard maximal compact subgroup if $G'(F_v)$ is $G'(O_v)$ of F is p-adic, $SO(2)$ if $F_v = \mathbb{R}$, and $SU(2)$ if $F_v = \mathbb{C}$. Let $\bar{u} = \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$, with $x \in F^*$. By the Iwasawa decomposition, we can find $a \in F_v^*$ and $y \in F_v$ such that $k := \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a^{-1}(1+xy) & a^{-1}y \\ ax & a \end{smallmatrix}\right)$ lies in K_v. In the non-archimedean case, if $|x| \leq 1$, we take $a = 1$ (and $y = 0$) and if $|x| > 1$, we are forced to have $|a| \leq |x|^{-1}$. Suppose $F_v = \mathbb{R}$. Then $k^t k$ must be the identity matrix, and this forces $|a| = \frac{1}{\sqrt{x^2+1}}$ (and $y = -x$). The complex case is similar and we get $|a| = \frac{1}{\sqrt{|x|^2+1}}$. Now consider the global case and write a for the idele (a_v) with a_v being (for each v) the element chosen above (and denoted just by a for brevity of notation). Then we get

$$|a| \leq \prod_{v|\infty} \frac{1}{\sqrt{1+|x|_v^2}} \prod_{v \text{ finite}} |x|_v^{-1} \leq \prod_{v|\infty} \frac{|x|_v}{\sqrt{1+|x|_v^2}} \leq 1,$$

by Artin's product formula, which gives $\prod_{\text{all } v} |x|_v^{-1} = 1$ (since $x \in F^*$). Take ν to be the root $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mapsto a/b$. Clearly, $\bar{U} = w\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)w^{-1}$ and $\nu^w = \nu^{-1}$, with $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Moreover, since $|a| \leq 1$, we have $H_\nu(\bar{u}) \leq 1$.

In our case ($G = GSp(6)$), we will now give the argument for the most basic case $P' = P_0$ and $\bar{U}_w = \bar{U}_0$. The other cases can be easily extracted from this. We write

$$\bar{u} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ t_1 & 1 & 0 & & 0 & \\ t_3 & t_2 & 1 & & & \\ \hline u_1 & u_2 & u_3 & 1 & -t_1 & t_1t_2 - t_3 \\ v_1 & v_2 & v_3 & 0 & 1 & -t_2 \\ w_1 & w_2 & w_3 & 0 & 0 & 1 \end{array}\right)$$

with u_i, v_i, w_i, t_i in F. Let v be a place. By the Iwasawa decomposition (over F_v), we can find $t = \mathrm{diag}(a, b, c, a^{-1}, b^{-1}, c^{-1}) \in A_0(F_v)$ and $n \in U_0(F_v)$ such that $k := t^{-1}n^{-1}u$ lies in K'_v, the standard maximal compact subgroup of $Sp(6, F_v)$. If we write

$$n^{-1} = \left(\begin{array}{ccc|ccc} 1 & s_1 & s_3 & z_1 & z_2 & z_3 \\ 0 & 1 & s_2 & y_1 & y_2 & y_3 \\ 0 & 0 & 1 & x_1 & x_2 & x_3 \\ \hline & & & 1 & 0 & 0 \\ & 0 & & -s_1 & 1 & 0 \\ & & & -s'_1 & -s_2 & 1 \end{array}\right), \text{ with } s'_3 = s_3 - s_1s_2,$$

then we have:

(3.4.24) $$k = \left(\begin{array}{c|c} * & * \\ \hline \mathrm{X} & \mathrm{Y} \end{array}\right)$$

where

$$X = \begin{pmatrix} au_1 & au_2 & au_3 \\ b(-s_1u_1+v_1) & b(-s_1u_2+v_2) & b(-s_1u_3+v_3) \\ c(-s_3'u_1-s_2v_1+w_1) & c(-s_3'u_2-s_2v_2+w_2) & c(-s_3'u_3-s_2v_3+w_3) \end{pmatrix}$$

and

$$Y = \begin{pmatrix} a & -at_1 & -at_3' \\ -bs_1 & b(s_1t_1+1) & b(s_1t_3'-t_2) \\ -s_3'c & c(s_3't_1-s_2) & c(s_3't_3'+s_2t_2+1) \end{pmatrix}.$$

In the non-archimedean case, we are then forced to have: $|a|, |b|, |c| \leq 1$. Moreover, suppose some element, call it d, of the set $\{u_1, u_2, u_3, t_1, t_3'\}$ is non-zero. Then we can take $a = 1$ if d is integral at v and we are forced to have $|a| \leq |d|^{-1}$ if d is non-integral at v. In the archimedean case, just as in the SL(2) case, we have $a = \frac{1}{\sqrt{1+|x|_v^2}}$. Again the global a has absolute value bounded above by 1. In this case, we are done by taking $\nu = 2e_1$ and w to be the longest root. Suppose every element of this set is zero. then t_3 is also zero, and we get

$$\bar{u} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & t_2 & 1 & & & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ v_1 & v_2 & v_3 & 0 & 1 & -t_2 \\ w_1 & w_2 & w_3 & 0 & 0 & 1 \end{array}\right) k_1,$$

with k_1 in K_v'. So we may replace $\bar{u}$ by $\bar{u}k_1^{-1}$. Then we get a condition on b analogous to the one on a above if some element of the set $\{v_1, v_2, v_3, s_1, t_2\}$ is non-zero. In this case we take ν to be $2e_2$. If every element of this set is also zero, then since $\bar{u}$ is non-zero, some w_j must be non-zero, which leads to a condition on $|c|$. We are done by taking ν to be $2e_3$ in this case. This finishes the proof of the Claim, Lemma 3.4.22 and Proposition 3.4.21.

Proof of Proposition 3.4.10 (contd.) In view of part (b) of Proposition 3.4.21, it suffices to show that, for all $P' = M'U'$ and w in $W^*_{M,M'}$, we have the bound

(3.4.25) $$||E^{P'}(h_s^w)||_{2,V_g'} \leq |\Phi(s)|\,||g||^N,$$

for some function $\Phi(s)$ of bounded order and integer N.

Since Y has finite volume, say A, which is bigger than or equal to $\mathrm{vol}(U_g)$, we get the bound

(3.4.26) $$||E^{P'}(h_s^w)||_{2,V_g'} \leq A\,||E^{P'}(h_s^w)||_{\infty,V_g'}.$$

Let us call (P', w) **convenient** if $P_w' = P'$. This happens in two cases: (i) when $P' = P_0$ and w arbitrary, and (ii) when $P' = P$ and $w = w_1$ (identity). Suppose we are in either of these cases. Then $E^{P'}(h_s^w) = M_w(f_s)$. Recall that the effect of the intertwining operator M_w on f_s is given by a ratio of abelian L-functions (see Remark 3.4.11), which are functions of bounded order by Hecke. Moreover, the definition of f_s shows that its value at g is bounded by a constant (independent of s) times $||g||^r$, for some integer r depending on the width of Ξ. Making use of

(3.4.26), and possibly shrinking V_g slightly, we can find an N for which (3.4.25) holds in such a convenient case.

Now suppose (P', w) is *inconvenient*. Then $E^{P'}(M_w(f_s))$ is left-invariant under $U'(\mathbb{A}_F)M'(F)$. Let A' denote the center of M' and put $K' = K \cap M(\mathbb{A}_{F,f})$. Since P' commutes with A', we get the requisite behavior of $E^{P'}(M_w(f_s))$ along the root directions along A'. It remains to study the growth along the root directions which are trivial on A'. By the decomposition $G(\mathbb{A}_F) = P'(\mathbb{A}_F)K_\infty K$, the problem reduces, in view of (3.4.26), to bounding the growth, for each $k \in K_\infty/(K_\infty \cap M'_\infty)$, of the function on $Y' := M'(F)(Z'_\infty)^+\backslash M'(\mathbb{A}_F)/K'$ given by $g \to E^{P'}(h_s^w)(gk)$. Let W_g denote the corresponding neighborhood (of the image of g), which we may shrink to be in the open ball of radius ϵ in the local coordinates, and let W'_g be the subset defined by U'_g such that $\overline{W'_g} \subset W_g$. Note that the relative Eisenstein series is an eigenfunction of the Laplacian Δ' defined by the Casimir operator on $M'(F_\infty)$ with eigenvalue λ'_s, which is again a quadratic function of s. Applying the earlier argument (see Lemmas 3.4.8, 3.4.9), and making use of the K_∞-finitenes of f_s, we get for a polynomial $P'(s)$ independent of g and k:

$$||E^{P'}(h_s^w)||_{\infty,W'_g} \leq |P'(s)|.||E^{P'}(h_s^w)||_{2,W'_g}. \tag{3.4.27}$$

We can truncate $E^{P'}(h_s^w)$, and in view of Proposition 3.4.16, it is enough to estimate $||E^{P'}(h_s^w) - \wedge^{T,P'}E^{P'}(h_s^w)||_{2,W'_g}$. In view of the obvious analog of Proposition 3.4.21, it will suffice to bound the function $||E^{P''}(H_s^u)||_{\infty,W'_g}$ for any parabolic P'' contained in P', Weyl element u in $W_{M'_w}\backslash W_{M'}/W_{M''}$ and function $H_s^u = M_u(h_s^w)$. In the convenient case the proof goes as above. In the inconvenient cases, the parabolic taking the place of P'_w is necessarily the minimal parabolic P_0. So if we truncate one more time, all the resulting cases will be convenient. Done.

□

This finishes the proof of Propositions 3.4.10, 3.4.6, and hence Theorem 3.4.1.

Remark 3.4.28. There is clearly an inductive argument buried here which should help understand the nature of integral representations of L-functions for larger groups.

3.5. **Modularity in the good case.** Let F be a totally complex number field. We will call a pair (π, π') of cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$ **good** if at *each* finite place v, either π_v or π'_v is not supercuspidal.

Theorem 3.5.1 *For every good pair (π, π'), there exists an isobaric automorphic representation $\pi \boxtimes \pi'$ of $GL(4, \mathbb{A}_F)$ such that the identities $(L_v), (\varepsilon_v), (L_\infty)$ asserted in Theorem M hold, where v is an arbitrary finite place.*

Proof. In view of Lemma 3.1.1 we may, and we will, assume that neither π nor π' is dihedral, i.e., automorphically induced from a grossencharacter of a quadratic extension, and that π' is a not a character twist of π.

Our first object is to define, for each place v, an irreducible admissible representation Π_v of $\mathrm{GL}(4, F_v)$ canonically associated to the pair (π_v, π'_v). As before, let σ_v (resp. σ'_v) be the 2-dimensional representation of W'_{F_v} attached to π_v (resp. π'_v) by [Ku]. First let v be archimedean. Then the local Langlands correspondence exists,

and we may (and will) take Π_v to be the representation associated to $\sigma_v \otimes \sigma'_v$ by the method of [La 1].

Next consider the non-archimedean case. Here we will, for every $n \geq 1$ with d a divisor of n and a supercuspidal representation β of $\mathrm{GL}(d, F_v)$, denote the associated (generalizd) special representation of $\mathrm{GL}(n, F_v)$ by $\mathrm{St}_n(\beta)$. By hypothesis, one of the local representations, say π_v, is not supercuspidal. If π_v is a *principal series representation* attached to quasi-characters μ, ν of F_v^*, then let us set

$$\Pi_v := (\mu \otimes \pi'_v) \boxplus (\nu \otimes \pi'_v).$$

It remains to consider when π_v *is the special representation* $\mathrm{St}_2(\nu)$ defined by a quasi-character ν. If π'_v *is supercuspidal*, then let us set

$$\Pi_v := \mathrm{St}_4(\pi'_v \otimes \nu).$$

If π'_v *is a principal series representation* defined by quasi-characters μ', ν', then we set

$$\Pi_v := (\mathrm{St}_2(\nu) \otimes \mu') \boxplus (\mathrm{St}_2(\nu) \otimes \nu').$$

Finally, if π'_v is also a special representation $\mathrm{St}(\nu')$, for a quasi-character ν', then we set

$$\Pi_v := \mathrm{St}_3(\nu\nu') \boxplus \nu\nu'.$$

If we put $\Pi := \otimes_v \Pi_v$, then Π is an irreducible, admisible, generic representation of $\mathrm{GL}(4, \mathbb{A}_F)$. We want to apply the converse theorem for GL(4) (cf. [CoPS], which is recalled in Theorem 3.1.3). Let T be a finite set of finite places containing those v where both π_v and π'_v are in the discrete series, and let η be an arbitrary cuspidal automorphic representation of $\mathrm{GL}(m, \mathbb{A}_F)$ with $m = 1$ or 2, such that η_v is unramified at every v in T. We are interested in the formally defined Rankin-Selberg L-function $L(s, \Pi \times \eta)$, and more precisely in the local integrals which show up in the proof of the converse theorem in [CoPS]. By [JPSS] we know that, at each finite place v, the L-factor $L(s, \Pi_v \times \eta_v)$ is the gcd of the corresponding local integrals $\Psi(s, W_v, W'_v, \Phi_v)$; when v is archimedean, we know (cf. [JS4]) that the poles of $\Psi(s, W_v, W'_v, \Phi_v)$ are contained in the poles of the L-factor. Moreover, we have a local functional equation involving the correct L- and ε-factors. See also [MW2]. (When v is archimedean, it is not known if the L-factor is realized in terms of the local integrals, unless one extends the function spaces [JS4], but this will not play any role for us.) Using this in conjunction with Theorems 3.3.11 and 3.4.1, we see that we can apply Theorem 3.1.3 and conclude the near-automorphy of Π once we note the following

Lemma 3.5.2 *At* **every** *place v of F, we have the identities*

$$L(s, \Pi_v \times \eta_v) = L(s, \pi_v \times \pi'_v \times \eta_v)$$

and

$$\varepsilon(s, \Pi_v \times \eta_v) = \varepsilon(s, \pi_v \times \pi'_v \times \eta_v).$$

Proof. We have For $m = 1$, this is simply a rewording of Proposition 3.3.4. So assume that $m = 2$, and denote, for each v, the 2-dimensional representation of W'_{F_v} associated to η_v by τ_v. Suppose v is non-archimedean. If π_v is a principal series representation defined by quasi-characters μ, ν, then σ_v is simply $\mu \oplus \nu$, and $L(\pi_v \times \pi'_v \times \eta_v)$ (resp. $\varepsilon(\pi_v \times \pi'_v \times \eta_v)$) coincides with $L(s, \mu \otimes \sigma'_v \otimes \tau_v) L(s, \nu \otimes \sigma'_v \otimes \tau_v)$ (resp. $\varepsilon(s, \mu \otimes \sigma'_v \otimes \tau_v)\varepsilon(s, \nu \otimes \sigma'_v \otimes \tau_v)$). Again the assertion follows from Prop. 3.3.4, thanks to the definition of Π_v. It works similarly when π'_v is in the principal

series. So we may assume that both π_v and π'_v are in the discrete series. Then by the choice of (T,η), η_v is an unramified principal series. If π_v and π'_v are both supercuspidal, the assertion follows once again by Prop. 3.3.4. It remains to consider when (i) π_v is special and π'_v is supercuspidal, and (ii) π and π' are both special. The Lemma followsm in both cases, due to our choice of Π_v, from prop. 81 and Theorem 8.2 of [JPSS]. Finally let v be archimedean. But here, one knows by [La1] that $L(s,\Pi_v\times\eta_v)$ (resp. $\varepsilon(s,\Pi_v\times\eta_v)$) equals $L(s,\sigma_v\otimes\sigma'_v\otimes\tau_v)$ (resp. $\varepsilon(s,\sigma_v\otimes\sigma'_v\otimes\tau_v)$). Done.

So now we know that Π is nearly automorphic, i.e., that there exists an automorphic representation Π_1 of $\mathrm{GL}(4,\mathbb{A}_F)$ such that Π_v is isomorphic to $\Pi_{1,v}$ for almost all v.

Lemma 3.5.3. *There is an isobaric automorphic representation Π_2 of $GL(4,\mathbb{A}_F)$ such that $\Pi_v\simeq\Pi_{2,v}$ for almost all v.*

Proof. Indeed, this is obvious if Π_1 is cuspidal. So assume not. Then Π_1 is a Jordan-Hölder component, thanks to Prop.2 of [La6], of a representation induced from a cuspidal automorphic representation β of (the Levi component) of a standard parabolic subgroup P of GL(4). There is by definition an isobaric constituent Π_2 of this same induced representation, whose local components are at the unramified places the same as those of Π_1 (and those of Π). Moreover, by the reverse implication of prop.2 of [La6], Π_2 is also automorphic.

□

Now Theorem 3.5.1 follows by applying Proposition 3.2.1 by taking $\pi\boxtimes\pi'$ to be Π_2.

Remark 3.5.4. It is natural to wonder, since π is generic, if it is exactly the same as Π_2. We will show later (in section 4.3) that this is indeed the case. We will then have a uniqueness statement for $\pi\boxtimes\pi'$.

3.6. **A descent criterion.** The object of this section is to prove the following simple extension of the Proposition in section 4.2 of [BR]. (There is a mistake in [BR], but this does not affect sections 1-4. In any case, the proof given below is completely self-contained.)

Proposition 3.6.1 *Fix $n,p\in\mathbb{N}$ with p prime. Let F a number field, $\{K_j\mid j\in\mathbb{N}\}$ a family of cyclic extensions of F with $[K_j:F]=p$, and for each $j\in\mathbb{N}$, π_j a cuspidal automorphic representation of $GL(n,\mathbb{A}_{K_j})$. Suppose that, for all $j,r\in\mathbb{N}$, the base changes of π_j,π_r to the compositum K_jK_r satisfy*

$$(DC)\qquad (\pi_j)_{K_jK_r}\ \simeq\ (\pi_r)_{K_jK_r}.$$

Then there exists a unique cuspidal automorphic representation π of $GL(n,\mathbb{A}_F)$ such that

$$(\pi)_{K_j}\ \simeq\ \pi_j,$$

for all but a finite number of j.

Proof. First we need the following

Lemma 3.6.2 *Let π be a cuspidal automorphic representation of $GL(n,\mathbb{A}_F)$. Then there exist atmost a finite number of idele class characters χ such that*

$$\pi\ \simeq\ \pi\otimes\chi.$$

Proof of Lemma. Suppose π is isomorphic to its self-twist by an idele class character χ. First we note that, if ω is the central character of π, then the central chacter of $\pi \otimes \chi$ is simply $\omega\chi^n$. Thus χ must have order dividing n.

Next we claim that χ must be unramified at *any* finite place v where π is unramified. Indeed, for any such v, $L(s, \pi_v)$ is by definition, of the form $\prod_{j=1}^{n} L(s, \mu_j)$, for some unramified characters μ_j of F_v^*; this L-factor is not 1. On the other hand, if χ_v is ramified, $L(s, \pi_v \otimes \chi_v)$ is none other than $\prod_{j=1}^{n} L(s, \mu_j\chi)$, which equals 1. Hence the claim.

Consequently, if S is the (finite) support of the conductor of π, then the number of possible self-twists is bounded by the number of idele class characters χ of F such that the (finite) support of the conductor of χ is in S. Every such character cuts out, by class field theory, a cyclic extension $F(\chi)/F$ of degree $m \leq n$, which is ramified only at the places in S. But the number of such extensions is, for each $m \leq n$, known to be finite by a well known variant of a classical theorem of Hermite. The assertion now follows.

□

Now we continue with the proof of the Proposition. For each j, let θ_j be a generator of $\mathrm{Gal}(K_j/F)$, and δ_j a character of F cutting out K_j (by class field theory). Note that, for each $i \geq 1$, the pull back to K_i of δ_j by the norm map N_i from K_i to F cuts out the compositum K_iK_j.

We claim that

$$\pi_j \circ \theta_j \simeq \pi_j \quad (\forall j). \tag{3.6.3}$$

For all $j, r \geq$, let $\theta_{j,r}$ denote the automorphism of K_jK_r such that (i) $\theta_{j,r}|_{K_j} = \theta_j$, and (ii) $\theta_{j,r}|_{K_r} = 1$ (where 1 denotes the identity automorphism). It is easy to see that the base change of $\pi_j \circ \theta_j$ to K_jK_r is simply $(\pi_j)_{K_jK_r} \circ \theta_{j,r}$. Applying (DC), we then have

$$(\pi_j \circ \theta_j)_{K_jK_r} \simeq (\pi_r)_{K_jK_r} \circ \theta_{j,r} \simeq (\pi_r)_{K_jK_r} \simeq (\pi_j)_{K_jK_r},$$

since $\theta_{j,r}$ is trivial on K_r. Since K_jK_r is a cyclic extension of K_j of prime degree, we have by Arthur-Clozel (Prop. 2.3.1)

$$\pi_j \circ \theta_j \simeq \pi_j \otimes (\delta_r \circ N_j)^{m_r},$$

for some $m_r \in \{0, 1, \ldots, p-1\}$. For every fixed $r \geq 1$, and for all $k \neq r$, we then have the self-twist identity

$$\pi_j \simeq \pi_j \otimes (\delta_r \circ N_j)^{m_r} (\delta_k \circ N_j)^{-m_k}.$$

Note that $\delta_r \circ N_j$ and $\delta_k \circ N_j$ must be distinct unless their ratio is a power of δ_j. So the Lemma above forces m_r to be 0 for all but a finite number of r. Then, since (3.6.3) is independent of r, the claim follows.

As a result, by applying Prop. 2.3.1 again, we see that, there exists, for each $j \geq 1$, a cuspidal automorphic representation $\pi(j)$ of $\mathrm{GL}(n, \mathbb{A}_F)$ such that

$$\pi_j \simeq (\pi(j))_{K_j}.$$

Such a $\pi(j)$ is of course unique only up to twisting by a power of δ_j.

It is important to note that, for any $r \neq j$, we have the following compatibility for base change in (cyclic) stages:

$$((\pi(j))_{K_j})_{K_jK_r} \simeq ((\pi(j))_{K_r})_{K_jK_r}. \tag{3.6.4}$$

We see this as follows. Let v be a finite place of K_jK_r which is unramified for the data. Denote by u (resp. w, resp. w') the place of F (resp. K_j, resp. K_r) below v. If σ_u denotes the representation of W'_{F_u} associated to $\pi(j)_u$, then

$$\mathrm{res}^{(K_j)_w}_{(K_jK_r)_v}(\mathrm{res}^{F_u}_{(K_j)_w}(\sigma_u)) \simeq \mathrm{res}^{(K_r)_{w'}}_{(K_jK_r)_v}(\mathrm{res}^{F_u}_{(K_r)'_w}(\sigma_u)).$$

Then (2.3.0) implies the local identity (for all such v)

$$((\pi(j)_u)_{(K_j)_w})_{(K_jK_r)_v} \simeq ((\pi(j)_u)_{(K_r)'_w})_{(K_jK_r)_v}.$$

The global isomorphism (3.6.4) follows by the strong multiplicity one theorem.

We can then rewrite (DC) as saying, for all $j, r \geq 1$,

$$((\pi(j))_{K_j})_{K_jK_r} \simeq ((\pi(r))_{K_j})_{K_jK_r}.$$

Applying Prop. 2.3.1, we get

$$(\pi(j))_{K_j} \simeq (\pi(r))_{K_j} \otimes (\delta_r \circ N_j)^{m(r,j)},$$

for some integer $m(r,j)$. We can replace $\pi(r)$ by $\pi(r) \otimes \delta_r^{-m(r,j)}$ and get

$$(\pi(j))_{K_j} \simeq (\pi(r))_{K_j}.$$

Then, by replacing $\pi(j)$ by a power of δ_j, we can arrange for $\pi(j)$ and $\pi(r)$ to be isomorphic. In sum, we have produced, for *every pair* (j,r), a *common descent*, say $\pi(j,r)$, of $\{\pi_j, \pi_r\}$.

Fix $a, b \in \{1, \ldots, p-1\}$, and consider the possible isomorphism

$$\pi(j,r) \simeq \pi(j,r) \otimes \delta_j^a \delta_r^{-b}. \tag{3.6.5}$$

We claim that this cannot happen outside a finite set $S_{a,b}$ of pairs (j,r). To see this fix a pair (i,ℓ) and consider the relationship of $\pi(i,\ell)$ to $\pi(j,r)$. Since $\pi(i,\ell)$ and $\pi(j,\ell)$ have the same base change to K_ℓ, they must differ by twisting by a power of δ_ℓ. Similarly, $\pi(j,\ell)$ and $\pi(j,r)$ differ by a character twist as they have the same base change to K_r. Put together, this shows that $\pi(i,\ell)$ and $\pi(j,r)$ are twists of each other. Then (3.6.5) would imply that

$$\pi(i,\ell) \simeq \pi(i,\ell) \otimes \delta_j^a \delta_r^{-b}.$$

The claim now follows since (by Lemma above) $\pi(i,\ell)$ admits only a finite number of self-twists and since the characters $\delta_j^a \delta_r^{-b}$ are all distinct for distinct pairs (j,r) (as a, b are fixed).

Now choose a pair (j,r) *not belonging* to $S_{a,b}$ for *any* $(a,b) \in (\mathbb{Z}/p)^* \times (\mathbb{Z}/p)^*$, and set

$$\pi = \pi(j,r).$$

We assert that, for all but a finite number of indices m,

$$\pi_{K_m} \simeq \pi_m.$$

It suffices to show that, for any large enough m, $\pi = \pi(j,r)$ is isomorphic to either $\pi(j,m)$ or $\pi(m,r)$. Suppose neither is satisfied. Then there exist $a, b \in \{1, \ldots, p-1\}$ such that

$$\pi(j,m) \simeq \pi(j,r) \otimes \delta_j^a \quad \text{and} \quad \pi(m,r) \simeq \pi(j,r) \otimes \delta_r^b.$$

We also have $\pi(j,m) \simeq \pi(m,r) \otimes \delta_m^c$, for some $c \in \{0,1,\ldots,p-1\}$. Putting these together, we get the self-twisting identity

$$\pi(j,r) \;\simeq\; \pi(j,r) \otimes \delta_j^a \delta_r^{-b} \delta_m^{-c}.$$

By our choice of (j,r), c cannot be 0. For each non-zero c, the set of indices m for which such an identity can hold is finite (again by the Lemma). Hence we get a contradiction for large enough m, which implies that a or b should be 0. The Proposition is now proved.

□

3.7. **Modularity in the general case.** In this section we will complete the proof of Theorem M without any hypothesis on (π,π'). we begin by noting the following (well known)

Lemma 3.7.1 *Let v be a finite place where π_v is supercuspidal. Then there is a finite solvable, normal extension E of F_v such that the base change $\pi_{v,E}$ is in the principal series.*

Proof. Let σ_v denote as before the two-dimensional representation of W'_{F_v} associated to π_v. By [Ku], σ_v is an irreducible of W_{F_v}. The structure of the Weil group implies that σ_v must in fact be of the form $\tau_v \otimes \nu$, where τ_v is an irreducible $\mathbb{C}$-representation of W_{F_v} *factoring through a finite quotient*, and ν a quasi-character. So there is a finite (necessarily solvable) normal extension E/F_v such that the restriction of τ_v to the Weil group of $E(v)$ is trivial. (Take E to be the fixed field of $\overline{F}_v$ under the kernel of τ_v.) Then by the functoriality of base change, $(\pi_v)_{E(v)}$ must be in the principal series, in fact of the form $(1 \boxplus 1) \otimes (\nu \circ N_{E(v)/F})$.

□

It should be noted that by using Krasner's lemma, we can find a solvable Galois extension k'/k of number fields with local extension E/F_v such that $\mathrm{Gal}(k'/k)$ is isomorphic to $\mathrm{Gal}(E/F_v)$ (see [PR], sec.4, Lemma 3, for example). But we *cannot* hope to be able to take k to be F! But we can find (see below) a finite chain of cyclic extensions of F with good properties, but the field on top need not be normal over F.

Proof of Theorem M We may, by Proposition 3.1.1, assume that we are not in one of the special situations (I), (II), (III) treated there.

Given any finite solvable group G, define its *length* $\ell(G)$ to be the length ℓ of any chain $\{1\} = G_0 \subset G_1 \subset \ldots \subset G_\ell = G$, with each G_i normal in G_{i+1} of prime index. Let $S = S(\pi)$ denote the (possibly empty) finite set of finite places where either π_v is not supercuspidal. At each $v \in S$, let $\ell(\pi_v)$ denote the length of a minimal Galois extension $L(v)/F_v$ such that the base change $(\pi_v)_{L(v)}$ is in the principal series. Let $\ell(\pi)$ denote the maximum of $\{\ell(\pi_v) \,|\, v \in S\}$, and let S' denote the subset of S where this maximum is attained. Further, for each v in S', let $p(v)$ denote the maximum over all $L(v)$, of the degree, required to be a prime or 1, of the largest cyclic extension $K(v)$ of F_v contained in $L(v)$. Let $p = p(\pi)$ be the maximum of $p(v)$ over all v in S', and let S'' denote the subset of S' where $p(v) = p$ (and $\ell(\pi_v) = \ell(\pi)$). Note that p is a *prime* unless π is *good* over F, i.e., has no supercuspidal components, in which case $p = 1$.

Let us set

$$r(\pi) \;=\; (\ell(\pi), p(\pi)). \tag{3.7.2}$$

We will order these pairs as follows: $(\ell, p) < (\ell', p)$ if either $\ell < \ell'$ or if $\ell = \ell'$ *and* $p < p'$.

Suppose $r = r(\pi)$ is $(0,1)$. Then, for any quadratic extension K' which is totally complex, the assertion holds by Theorem 3.5.1. Also, given any finite place u, we can find a totally complex, quadratic extension $K'(u)$ such that u splits in $K'(u)$. Applying the descent criterion (Proposition 3.6.1), we can find a common descent having the requisite properties.

Now let $r > (0,1)$, and assume by induction that the Theorem is proved (over all number fields K) for pairs (π_1, π_2) of cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_K)$ with $r(\pi_1, \pi_2) < r$.

Fix, at every place v in S'', a character χ_v of F_v^* cutting out a $K(v)$ (as above) of degree p.

Enumerate the set of finite places where π is unramified, and list them as $\{v_1, v_2, \ldots, v_j, \ldots\}$.

Fix an index $j \geq 1$ (for the moment), and let $S(j)$ be the union of v_j with S''. Let χ_{v_0} denote the trivial character of $F_{v_0}^*$.

Now by the Grunewald-Wang theorem (see [AT], chap. 10, Theorem 5), we can find a global character $\chi(j)$ of C_F of order ℓ whose local restrictions are given by χ_v at every v in $S(j)$. (Note that the set S_0 which occurs in *loc. cit.* is empty in our case as each local degree is p or 1 and cannot be divisible by 4, thus allowing us to find $\chi(j)$ of order p, not just $2p$.) Let K_j be the p-extension of F cut out by $\chi(j)$.

By construction we have, for every $j \geq 1$,

$$r(\pi) \;<\; r. \tag{3.7.3}$$

Thus, by induction, the Theorem holds for π_{K_j} for each j. Note that if the automorphic representation π_{K_j} is not cuspidal for some j, it must be dihedral (with $p = 2$), and by the remark earlier, we may assume that we are not in this case. Put

$$\Pi_j \;=\; \pi_{K_j} \boxtimes \pi'_{K_j}. \tag{3.7.4}$$

The non-dihedrality of π also gives the following

Lemma 3.7.5 *There exist at most a finite number of indices j such that Π_j is not cuspidal.*

Proof. It suffices to prove that, outside a finite set of indices, we have

1. π_{K_j} is not dihedral;
2. π_{K_j} is not the twist of π'_{K_j} by an idele class character of K_j.

Suppose π_{K_j} is dihedral. Then, by Gelbart-Jacquet ([GJ]), the symmetric square of π_{K_j} is Eisensteinian, while by asumption, $\mathrm{sym}^2(\pi)$ is not. This forces p to be 3 and $\mathrm{sym}^2(\pi)$ to be automorphically induced by a character μ of K_j. In other words, the symmetric squre of π admits a self-twist relative to the character of F corresponding to K_j. By Lemma 3.6.2, this can only happen for a finite number of j.

We may then asume that π_{K_j} is not dihedral and consider the second assertion. Suppose π_{K_j} is the twist of π'_{K_j} by an idele class character. Then, for a character

μ of C_{K_j}, $L^{T'}(s, \Pi_j \otimes \mu)$ must have a pole at $s = 1$, where T' is any finite set of places of K_j. By the inductivity of L-functions, the same holds for the (L-function of the) automorphic induction $I^F_{K_j}(\Pi_j)$. Let us choose T' to be the inverse image of the set T of places of F where π or π' or K_j is ramified. Then it is easy to see that we have the identity:

$$L^T(s, I^F_{K_j}(\Pi_j \otimes \mu)) \;=\; L^T(s, \pi \times (\pi' \boxtimes I^F_{K_j}(\mu))).$$

Since T contains all the ramified places, the right hand side L-function is an incomplete form of *any* of the triple product L-functions attached to $(\pi, \pi', I^F_{K_j}(\mu))$. By [Ik1], it cannot have a pole unless π is dihedral. Done.

□

Consequently, after shrinking the index set by removing an appropriate finite subset and renumbering it, we may assume that

Π_j *is cuspidal for every* j.

Next we fix a pair (j, r) of indices and consider the descent criterion (DC) of Proposition 3.6.1. Let w be a finite place where $((\Pi_j)_{K_jK_r})_w$, $((\Pi_r)_{K_jK_r})_w$ and K_iK_j are all unramified. Then, by construction, both these local representations correspond to the restriction (to the Weil group of $(K_jK_r)_w$) of $\sigma_v \otimes \sigma'_v$, where v signifies the place of F below w. (Recall that σ_v, σ'_v are associated to π_v, π'_v respectively.) This leads to the identity

$$L^Y(s, (\Pi_j)_{K_jK_r}) \;=\; L^Y(s, (\Pi_r)_{K_jK_r}),$$

for a large enough finite set X of places, which gives (DC) by the strong multiplicity one theorem.

Applying Proposition 3.6.1, we then get a unique cuspidal descent Π on $\mathrm{GL}(4)/F$ such that, for all but a finite number of indices, we have

$$\Pi_{K_j} \;\simeq\; \Pi_j.$$

Finally, by construction, each (unramified) finite place v_j splits completely in K_j; let w_j be a divisor of v_j in K_j. This implies (by the definition of base change) that, for almost all j, we have

$$L(s, \Pi_{v_j}) \;=\; L(s, (\Pi_j)_{w_j}) \;=\; L(s, (\pi_{K_j})_w \times (\pi'_{K_j})_w) \;=\; L(s, \pi_v \times \pi'_v).$$

Similarly for the ε-factors. Thus Π is a weak (tensor product) lifting of (π, π'). By Proposition 3.2.1, it is also the strong lifting.

□

4. **Applications**

4.1. **A multiplicity one theorem for SL(2).** Denote by $L^2_0(\mathrm{SL}(2, F)\backslash\mathrm{SL}(2, \mathbb{A}_F))$ the subspace of cusp forms in $L^2(\mathrm{SL}(2, F)\backslash\mathrm{SL}(2, \mathbb{A}_F))$. It is acted upon by $\mathrm{SL}(2, \mathbb{A}_F)$ by right translation, and this action is unitary relative to the natural scalar product on $L^2(\mathrm{SL}(2, F)\backslash\mathrm{SL}(2, \mathbb{A}_F))$. The main result of this section is

Theorem 4.1.1 *The representation of* $SL(2, \mathbb{A}_F)$ *on* $L^2_0(\mathrm{SL}(2, F)\backslash\mathrm{SL}(2, \mathbb{A}_F)$ *has multiplicity one.*

This was conjectured by Labesse and Langlands (see [LL], [Lab1]). The Theorem says that the commutant of this representation is abelian, and equivalently, since the space of cusp forms is completely reducible, every irreducible unitary representation of $\mathrm{SL}(2, \mathbb{A}_F)$ oocurring in (the decomposition of) this space appears with multiplicity one.

The stable trace formula for SL(2) was analyzed deeply, and in detail, in [LL]. We refer to this paper for all the background material. The results there show in particular that multiplicity one for SL(2) is a consequence of the following

Theorem 4.1.2 *Let π, π' be unitary, cuspidal automorphic representations of $GL(2, \mathbb{A}_F)$. Suppose we have, for almost all v,*

$$(LL(v)) \qquad \mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi'_v).$$

Then there exists an idele class character χ, which is unique if π is not automorphically induced by a character of a quadratic extension, such that

$$\pi' \simeq \pi \otimes \chi.$$

If π and π' have the same central character, then χ is quadratic.

Here $\mathrm{Ad}(\pi)$ denotes the automorphic representation $\mathrm{sym}^2(\pi) \otimes \omega^{-1}$ of $\mathrm{GL}(3, \mathbb{A}_F)$, where ω is the central character of π.

As noted in the Introduction, in the special case when π and π' are defined by holomorphic eigenforms, with F necessarily totally real, various people have known the proof of this result by making use of the associated ℓ-adic repreentations. But this does not work in general, and the main point here is to give a unified proof using the modularity of $\pi \boxtimes \pi'$.

Before beginning the proof of this Theorem, we would like to indicate the following concrete

Corollary 4.1.3 *Let π, π' be unitary, cuspidal automorphic representations of $GL(2, \mathbb{A}_F)$ of trivial character. Suppose we have at every unramified finite place v,*

$$(Sq(v)) \qquad a_v(\pi)^2 = a_v(\pi')^2,$$

where $a_v(\pi) = tr(A_v(\pi))$ is the v-th Hecke eigenvalue of π. Then there is a unique quadratic idele class character χ such that

$$a_v(\pi') = \chi_v(\varpi_v) a_v(\pi'),$$

for almost all v, where ϖ_v denotes the uniformizer at v. If in addition, the conductors N, N' of π, π' are square-free, then $\chi = 1$.

Theorem 4.1.2 implies Corollary 4.1.3:
Let π, π' be as in the Corollary, and let v be a finite place where both π and π' are unramified. Since by hypothesis, the central characters are trivial, we may write $A_v(\pi) = \mathrm{diag}(\alpha_v, \alpha_v^{-1})$ and $A_v(\pi') = \mathrm{diag}(\alpha'_v, \alpha_v'^{-1})$. Then the Langlands class of $\mathrm{Ad}(\pi_v)$ (resp. $\mathrm{Ad}(\pi'_v)$) is given by $\{\alpha_v^2, 1, \alpha_v^{-2}\}$ (resp. $\{\alpha_v'^2, 1, \alpha_v'^{-2}\}$). Consequently, we have (in $\Re(s) > 1$)

$$\log(L(s, \mathrm{Ad}(\pi_v))) = 1 + \sum_{m \geq 1} \frac{\alpha_v^{2m} + 1 + \alpha_v^{-2m}}{m(Nv)^{ms}}$$

and

$$\log(L(s, \mathrm{Ad}(\pi'_v))) \;=\; 1 \;+\; \sum_{m\geq 1} \frac{{\alpha'_v}^{2m} + 1 + {\alpha'_v}^{-2m}}{m(Nv)^{ms}}.$$

We claim that, for every $m \geq 1$,

$$\alpha_v^{2m} + \alpha_v^{-2m} \;=\; {\alpha'_v}^{2m} + {\alpha'_v}^{-2m}.$$

By induction on m, it suffices (by the binomial formula) to verify

$$(\alpha_v^2 + \alpha_v^{-2})^m \;=\; ({\alpha'_v}^2 + {\alpha'_v}^{-2})^m,$$

which follows immediately from the hypothesis (Sq(v)). Hence the claim.

Since (LL(v)) then holds for almost all v, the Corollary follows by applying Theorem 4.1.2.

Proof of Theorem 4.1.2.

First note that, since (LL(v)) holds for almost all v, the automorphic representations $\mathrm{Ad}(\pi)$ and $\mathrm{Ad}(\pi')$ are isomorphic by the strong multiplicity one theorem.

Suppose π is dihedral, i.e., of the form $I_K^F(\mu)$, for a character μ of C_K, for some quadratic extension K of F. Its central character then identifies with $\mu_0\delta$, where μ_0 is the restriction of μ to C_F and δ the quadratic character of C_F associated to K/F by class field theory. Denote by θ the non-trivial automorphism of K/F. We claim that

$$(ad) \qquad \mathrm{Ad}(\pi) \;=\; \delta \boxplus I_K^F(\mu/(\mu\circ\theta)).$$

Since $(\mu_0)_K = \mu(\mu\circ\theta)$, this is equivalent to the identification of $\mathrm{sym}^2(\pi)$ with $\mu_0 \boxplus I_K^F(\mu^2)$, which is easy to verify at the unramified places. So the claim follows by the strong multiplicity one theorem ([JS2]).

A particular consequence of the claim is that $L(s, \mathrm{Ad}(\pi)\otimes\delta)$ has a pole at $s = 1$. This forces π' also to be dihedral, for otherwise, $L(s, \mathrm{Ad}(\pi')\otimes\delta)$ will be entire ([GJ]). Then δ must in fact occur in the isobaric decomposition of $\mathrm{Ad}(\pi'_K)$, which in turn is a summand of $\pi' \boxtimes \pi'^{\vee}$. Then $L(s, \pi' \boxtimes \pi'^{\vee} \otimes \delta)$ also has a pole at $s = 1$. This implies that

$$\pi' \;\simeq\; \pi' \otimes \delta.$$

In other words, π' is of the form $I_K^F(\mu')$, for a character μ' of C_K.

Now, base changing $\mathrm{Ad}(\pi)$ =$\mathrm{Ad}(\pi')$ to K, and using (ad), we get the identity

$$\mu/(\mu\circ\theta) \boxplus (\mu\circ\theta)/\mu \;\simeq\; \mu'/(\mu'\circ\theta) \boxplus (\mu'\circ\theta)/\mu'.$$

Replacing μ' by $\mu'\circ\theta$ if necessary, we can then deduce that

$$\mu/\mu' \;=\; (\mu/\mu')\circ\theta.$$

In other words, there exists a character χ of C_F such that $\mu' = \mu\chi_K$. Then we have

$$\pi' \;=\; I_K^F(\mu') \;\simeq\; I_K^F(\mu)\otimes\chi \;\simeq\; \pi\otimes\chi,$$

as desired. To verify the middle isomorphism, one first checks it locally almost everywhere and then apply the strong multiplicity one theorem.

So we may assume henceforth in the main case of interest, namely when neither π nor π' is dihedral. Put

$$\Pi \;=\; \pi \boxtimes \pi'.$$

We know that it is an isobaric automorphic representation of $\mathrm{GL}(4, \mathbb{A}_F)$. The main point here is to prove the following

Lemma 4.1.4. Π *is not cuspidal.*

Proof of Lemma. Let S be a finite set of places containing the ramified places (for π and π') and the archimedean ones.

Suppose Π is cuspidal. Then, by [JS2], we know that the Rankin-Selberg L-function $L^S(s, \Pi \times \Pi^\vee)$ must have a *simple pole* at $s = 1$.

On the other hand, at any finite place v, if we denote (as usual) by σ_v (resp. σ'_v) the representation of W'_{F_v}, we get, as $\sigma_v \otimes \sigma_v^\vee \simeq \mathrm{Ad}(\sigma_v) \oplus 1$, the following identity:

$$(\sigma_v \otimes \sigma'_v) \otimes (\sigma_v \otimes \sigma'_v)^\vee \ \simeq \ 1 \oplus \mathrm{Ad}(\sigma_v) \oplus \mathrm{Ad}(\sigma'_v) \oplus \ \mathrm{Ad}(\sigma_v) \otimes \mathrm{Ad}(\sigma'_v).$$

At the unramified places v, we know that $\sigma_v \otimes \sigma'_v$ corresponds to Π_v. So this translates to the L-function identity

$$(Id) \quad L^S(s, \Pi \times \Pi^\vee) \ = \ \zeta_F^S(s) L^S(s, \mathrm{Ad}(\pi)) L^S(s, \mathrm{Ad}(\pi')) L^S(s, \mathrm{Ad}(\pi) \times \mathrm{Ad}(\pi')).$$

But we know that $\mathrm{Ad}(\pi)$ is isomorphic to $\mathrm{Ad}(\pi')$. Moreover, since $\mathrm{Ad}(\pi) = \mathrm{sym}^2(\pi) \otimes \omega^{-1}$, we have

$$\mathrm{Ad}(\pi)^\vee \ \simeq \ \mathrm{sym}^2(\pi^\vee) \otimes \omega \ \simeq \mathrm{Ad}(\pi),$$

as $\mathrm{sym}^2(\pi^\vee)$ is $\mathrm{sym}^2(\pi) \otimes \omega^{-2}$.

Consequently, $L^S(s, \mathrm{Ad}(\pi) \times \mathrm{Ad}(\pi'))$ has a pole at $s = 1$. Also, since π and π' are non-dihedral, $L^S(s, \mathrm{Ad}(\pi)$ and $L^S(s, \mathrm{Ad}(\pi')$ are entire (cf. [GJ]). Then the identity (Id) implies that $L^S(s, \Pi \times \Pi^\vee)$ has at least a double pole at $s = 1$, leading to the desired contradiction.

□

Thus π, π' are non-dihedral cusp forms on $\mathrm{GL}(2)/F$ with $\pi \boxtimes \pi'$ not cuspidal. Then by the cuspidality criterion of Theorem M (see page 9), which was proved in section 3.2 (on pp. 14-15), we see that π' must be isomorphic to $\pi \otimes \chi$ for an idele class character χ. Done.

□

4.2. **Some new functional equations.** Let $\pi_1, \pi_2, \pi_3, \pi_4$ be cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$, and let S be a finite set of places containing the archimedean and ramified places. Let us set

$$L^S(s, \pi_1 \times \pi_2 \times \pi_3 \times \pi_4) \ = \ \prod_{v \notin S} \det(I - (Nv)^{-s} A_v(\pi_1) \otimes A_v(\pi_2) \otimes A_v(\pi_3) \otimes A_v(\pi_4))^{-1},$$

$$L^S(s, \mathrm{sym}^2(\pi_1) \times \pi_2 \times \pi_3) \ = \ \prod_{v \notin S} \det(I - (Nv)^{-s} \mathrm{sym}^2(A_v(\pi_1)) \otimes A_v(\pi_2) \otimes A_v(\pi_3))^{-1},$$

and

$$L^S(s, \mathrm{sym}^3(\pi_1) \times \pi_2) \ = \ \prod_{v \notin S} \det(I - (Nv)^{-s} \mathrm{sym}^3(A_v(\pi_1)) \otimes A_v(\pi_2))^{-1}.$$

These functions of s converge absolutely in some right half plane.

Theorem 4.2.1. *Each of the three (incomplete) Euler products above admit meromorphic continuations to the whole s-plane. In addition, each of them can be completed to a convergent Euler product over all places in* $\Re(s) > 1$ *, admitting meromorphic continuation and a functional equation of the usual form (as conjectured by Langlands). They are non-zero on the line* $s = 1$ *with no pole except*

possibly at the point $s = 1$. The first two L-functions are in fact analytic everywhere except for possible poles at $s = 0, 1$.

Remark: The meromorphic continuation and functional equation of the third of the L-functions above is already known by the work of Shahidi using the theory of Eisenstein series ([Sh2]). But his approach does not seem to be able to give invertibility in $\Re(s) \geq 1$ except for a possible pole at $s = 1$.

Proof. Since by Theorem M, there exist $\pi_1 \boxtimes \pi_2$ and $\pi_3 \boxtimes \pi_4$ in $\mathcal{A}(4, F)$, we may set

$$L(s, \pi_1 \times \pi_2 \times \pi_3 \times \pi_4) \;=\; L(s, (\pi_1 \boxtimes \pi_2) \times (\pi_3 \boxtimes \pi_4)).$$

By the definition of $\boxtimes$, the unramified local factors of the L-function on the right agree with those of $L^S(s, \pi_1 \times \pi_2 \times \pi_3 \times \pi_4)$. This way we get a complete Euler product in $\Re(s) > 1$, and the remaining assertions follow immediately for the first L-function from the corresponding ones, due to Jacquet, Piatetski-Shapiro and Shalika ([JPSS]), Moeglin-Waldspurger ([MW2]) Shahidi [Sh6], for the Rankin-Selberg L-functions on $\mathrm{GL}(4)\times\mathrm{GL}(4)$.

We complete the second and third L-functions by respectively setting

$$L(s, \mathrm{sym}^2(\pi_1) \times \pi_2 \times \pi_3) \;=\; L(s, \mathrm{sym}^2(\pi_1) \times (\pi_2 \boxtimes \pi_3))$$

and

$$L(s, \mathrm{sym}^3(\pi_1) \times \pi_2) \;=\; \frac{L(s, \mathrm{sym}^2(\pi_1) \times (\pi_1 \boxtimes \pi_2))}{L(s, (\pi_1 \boxtimes \pi_2) \otimes \omega_1},$$

where ω_1 is the central character of π_1.

The verification of the fact that the unramified factors of these L-functions are the right ones is an easy exercise, which is left to the reader. Since the Rankin-Selberg L-functions on $\mathrm{GL}(n)\times\mathrm{GL}(m)$ are non-zero in $\{\Re(s) \geq 1\}$ and are analytic everywhere except for possible poles at $s = 0, 1$, all the assertions of Theorem 4.2.1 follow except for knowing that the third L-function is non-zero at $s = 1$. So we will be done if we prove

Lemma 4.2.2.

$$-\mathrm{ord}_{s=1} L(s, \pi_1 \boxtimes \pi_2 \otimes \omega_1) \;\leq\; -\mathrm{ord}_{s=1} L(s, \mathrm{sym}^2(\pi_1) \times (\pi_1 \boxtimes \pi_2)).$$

Suppose $L(s, \pi_1 \boxtimes \pi_2 \otimes \omega_1)$ has a pole at $s = 1$. Since π_1, π_2 are cuspidal, the pole has to be a simple pole, and besides, this can happen iff π_2 is isomorphic to the contragredient of $\pi_1 \otimes \omega_1$. It then follows, since $\pi_1 \boxtimes \pi_1^\vee = \mathrm{Ad}(\pi_1) \boxplus 1$, that

$$(Fac) \qquad L(s, \mathrm{sym}^2(\pi_1) \times (\pi_1 \boxtimes \pi_2)) \;=\; L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1)) L(s, \mathrm{Ad}(\pi_1)).$$

We have seen earlier that $\mathrm{Ad}(\pi_1)$ is self-dual. Hence $L(s, \mathrm{Ad}(\pi_1) \times \mathrm{Ad}(\pi_1))$ has a pole at $s = 1$. We claim that $L(s, \mathrm{Ad}(\pi_1))$ is holomorphic at $s = 1$, which is clear if π is non-dihedral, as $\mathrm{Ad}(\pi_1)$ is then cuspidal by [GJ]. So suppose π is of the form $I_K^F(\mu)$ for a character of (the idele classes of) a quadratic extension K of F, with non-trivial automorphism θ over F. Then, by the identity (ad) appearing in the proof of Theorem 4.1.2, $L(s, \mathrm{Ad}(\pi_1))$ can have a pole at $s = 1$ iff $\mu/(\mu \circ \theta)$ is the pull back by norm to K of either the trivial character of C_F or the quadratic character δ (attached to K). In either case, we will have $\mu = \mu \circ \theta$, contradicting the cuspidality of π_1.

Consequently, the right (and hence the left) hand side of (Fac) has a pole at $s = 1$. This proves the Lemma and completes the proof of the theorem.

□

Remark 4.2.3. It is now clear how we can also define, in an analogous way, the epsilon factors, both locally and globally, for the L-functions under consideration.

4.3. **Root numbers and representations of orthogonal type.** Let k be a non-archimedean local field of characteristic zero. Our first aim is to prove the following, which is not clear even if one knew the local Langlands conjecture for GL(4).

Proposition 4.3.1. Let $n \leq 4$, $\{\eta_j | j \leq n\}$ supercuspidals of $\mathrm{GL}(2,k)$, with corresponding 2-dimensional irreducibles $\{\tau_j | j \leq n\}$. Then we have

$$L(s, \eta_1 \times \ldots \times \eta_n) \,=\, L(s, \tau_1 \otimes \ldots \otimes \tau_n)$$

and

$$\varepsilon(s, \eta_1 \times \ldots \times \eta_n) \,=\, \varepsilon(s, \tau_1 \otimes \ldots \otimes \tau_n).$$

Proof of Proposition. For $n = 1$, this is contained in [Ku]. It is also well known for $n = 2$; see Prop. 4.2 of [PR], for example. So assume $n \geq 3$. If either of the η_j is the local automorphic induction of a (quasi)character μ of E^*, for some quadratic extension E/k, then the Proposition is the same as Prop. 4.2 of *loc. cit.* Even otherwise, we can modify the argument there and appeal to the global Rankin-Selberg product to get what we want. We give the details for completeness. We start by appealing to Lemma 3, sec. 4 of [PR], and find a number field F with $k = F_u$ for some place u, and irreducible 2-dimensional representations σ_j of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ with solvable image such that their respective restrictions to the decomposition group at u identify with τ_j. Solvability allows us to apply the theorem of Langlands ([La5]) and Tunnell ([Tu]), and find corresponding cuspidal automorphic representations π_j with $\pi_{j,u} \simeq \eta_j$. Then we choose a finite order character μ of C_F which is highly ramified at all the ramified places except u and which is 1 at u. Comparing the functional equations of $L(s, \pi_1 \times \ldots \times \pi_n)$ and $L(s, \sigma_1 \otimes \ldots \otimes \sigma_n)$, we proceed as in Proposition 3.2.1 and deduce the assertions above.

□

Remark 4.3.2. In chapter 3 we asociated, to each pair (π, π') of cupidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$ an irreducible admissible representation Π of $\mathrm{GL}(4, \mathbb{A}_F)$ and an isobaric automorphic representation $\pi \boxtimes \pi'$ such that $\Pi_v \simeq (\pi \boxtimes \pi')_v$ for almost all v. Using Proposition 4.3.3 we can conclude that such an identity holds at *every* v.

To clarify this remark, we will digress a bit about the local Langlands conjecture for GL(4), which predicts a canonical bijection, preserving local factors, between the set $\mathcal{A}^0(4,k)$ of irreducible, supercuspidal representations η of $\mathrm{GL}(4,k)$ and the set $\mathcal{R}^0(4,k)$ of irreducible, continuous, 4-dimensional $\mathbb{C}$-representations τ of W_k. There is a numerical bijection between the sets ([He4], compatible with twisting by unramified characters. A theorem of M. Harris ([Ha]) gives a (geometrically defined) bijection $\tau \to \eta$, which gives moreover an equality of epsilon factors of pairs

$$\varepsilon(s, \tau \otimes \tau') \,=\, \varepsilon(s, \eta \times \eta'),$$

for all irreducibles τ', with corresponding η', of dimension m, for all $m \leq 4$, **provided** the residual characteristic is prime to $2m$. On the other hand, a recent result of Jeff Chen ([Ch]) asserts that any η in $\mathcal{A}^0(4,k)$ is uniquely determined by

the collection of epsilon factors $\varepsilon(s, \eta \times \eta')$, as η' ranges over $\mathcal{A}^0(m,k)$ for $m = 1$ and $m = 2$. Consequently, as noted in [PR], sec. 4, the local Langlands conjecture follows for GL(4) in the odd residual characteristic case by combining these results of [Ha] and [Ch]. In the case of even residual characteristic, one known by [He2], one knows the bijection for representations of "cyclic type", i.e., those which admit a quartic self-twist. Moreover, Prop. 4.1 of [PR] shows how to extend this and treat also the case of representations admitting just a quadratic self-twist.

Now let $\mathrm{GO}(4,\mathbb{C})$ denote the subgroup of $\mathrm{GL}(4,\mathbb{C})$ consisting of orthogonal similitudes with similitude factor μ. Let $\mathrm{GSO}(4,\mathbb{C})$ denote the kernel of the homomorphism $\mathrm{GO}(4,\mathbb{C}) \to \{\pm 1\}$, $g \to \mu(g)^{-2}\det(g)$. In this section we will be concerned with the subset $\mathcal{R}^1(4,k)$ of τ in $\mathcal{R}^0(4,k)$ whose image lie in $\mathrm{GSO}(4,\mathbb{C})$. It is well known that there is a short exact sequence

$$1 \to \mathbb{C}^* \to \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GSO}(4,\mathbb{C}) \to 1,$$

inducing a surjection

$$\mathrm{Hom}(W_k, \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})) \to \mathrm{Hom}(W_k, \mathrm{GSO}(4,\mathbb{C})).$$

See, for example, the proof of Lemma 4 in sec. 5 of [PR]. Consequently, every τ in $\mathcal{R}^1(4,k)$ is a tensor product $\tau_1 \otimes \tau_2$, with each τ_j an irreducible of dimension 2. The pair (τ_1, τ_2) is not unique, as it can be replaced by $(\tau_1 \otimes \nu, \tau_2 \otimes \nu^{-1})$, but this is the only ambiguity.

Motivated by this, we define $\mathcal{A}^1(4,k)$ to be the irreducible supercuspidal representations η of $\mathrm{GL}(4,k)$ which are of the form $\eta_1 \boxtimes \eta_2$ (see sec. 3.7 for a definition). In what follows, we will make use of the local Langlands correspondence for GL(2) and GL(3), established respectively by Kutzko ([Ku]) and Henniart ([He1]).

Proposition 4.3.3. *There is a unique bijection $\tau \to \eta$ between $\mathcal{R}^1(4,k)$ and $\mathcal{A}^1(4,k)$, compatible with twisting by characters and taking duals, such that the following identity of epsilon factors holds for all $\tau' \in \mathcal{R}^0(m,k)$, with corresponding $\eta' \in \mathcal{A}^0(m,k)$, for $m = 1$ and $m = 2$:*

$$\varepsilon(s, \tau \otimes \tau') = \varepsilon(s, \eta \times \eta').$$

Proof. Let $\tau \in \mathcal{R}^1(4,k)$. Then we know by the remark above that it is of the form $\tau_1 \otimes \tau_2$, with τ_1, τ_2 irreducible 2-dimensionals of W_k. Let η_1, η_2 be the respective supercuspidals of $\mathrm{GL}(2,k)$ given by the local Langlands correspondence. Put

$$\eta = \eta_1 \boxtimes \eta_2.$$

Suppose $\eta = \eta_1 \boxtimes \eta_2$ is not supercuspidal. Then in its isobaric sum decomposition, there must occur a supercuspidal, say β, of $\mathrm{GL}(m,k)$, for some $m < 4$. (Since η is a representation of W_k, there will be no summand of Steinberg type.) Then $L(s, \beta \times \beta^\vee)$, and hence $L(s, \eta \times \beta^\vee)$ will have a pole at $s = 0$ (see [JPSS], [JS2]).

Now let λ denote the irreducible m-dimensional representation of W_k attached to β by the local Langlands correspondence, which we can do by [Ku] and [He1] as $m \leq 3$. Then, by the Proposition above, $L(s, \tau \otimes \lambda^\vee)$ equals $L(s, \eta \times \beta^\vee)$, and hence must have a pole at $s = 0$ as well. This cannot hapen as τ is an irreducible of dimension 4, and λ is one of dimension < 4. This proves that η is supercuspidal, and we get a "reciprocity map"

$$r : \mathcal{R}^1(4,k) \to \mathcal{A}^1(4,k).$$

The ε-factor identity of Prop. 4.3.1, when used in conjunction with Jeff Chen's theorem ([Ch]), shows the injectivity of r. It is also easy to see that r is compatible with taking duals and twisting by characters.

The surjectivity of r is shown by reversing the construction of $\tau \to \eta$. The ireducibility of τ is checked by using Prop. 4.3.2.

□

Let $\pi_1, \pi_2, \pi_3, \pi_4$ be cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$. Define its *root number* to be

$$W(\pi_1 \times \pi_2 \times \pi_3 \times \pi_4) \,=\, \prod_v W(\pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \times \pi_{4,v}),$$

where the *local root numbers* are defined by

$$W(\pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \times \pi_{4,v}) \,=\, \varepsilon(\frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \times \pi_{4,v}).$$

Note that the functional equation implies that the root number is ± 1 in the self-dual situation.

Theorem 4.3.4. *Let π_j, $j \leq 4$, be as above. Assume that the central character of each π_j is trivial. Then we have*

$$W(\pi_1 \times \pi_2 \times \pi_3 \times \pi_4) \,=\, 1.$$

When all the π_j's are isomorphic, say to π, the root number is $W(\mathrm{Sym}^2(\pi) \times \mathrm{Sym}^2(\pi))W(\mathrm{Sym}^2(\pi))^2 W(1)$, and this theorem follows in that case from Proposition 6.1 of [PR].

Proof. This is an immediate consequence of Proposition 4.3.1 in conjunction with the method of [PR] (see also [Ro]). We give the argument for completeness.

It suffices to show that the local root number at *any* place v is 1. Clearly, for each $j \leq 4$, the associated representation $\sigma_{j,v}$ of W'_{F_v} is self-dual with determinant 1 as π_j has trivial central character. Put

$$\beta_v \,=\, \sigma_{1,v} \otimes \sigma_{2,v} \otimes \sigma_{3,v} \otimes \sigma_{4,v}.$$

By Prop. 4.3.1, we are reduced to showing that $W(\beta_v)$ (with the obvious definition) is 1. Being the tensor product of an even number of symplectic representations, β_v is orthogonal, and by Deligne ([De]) the root number of $\tilde{\beta}_v := \beta_v \ominus 4[1]$ is given by its second Stieffel-Whitney class $w_2(\tilde{\beta}_v)$. But since $\mathrm{Sp}(2, \mathbb{C})$ is simply connected, and since the image of β_v in $\mathrm{O}(4, \mathbb{C})$ factors through a 4-fold product of $\mathrm{Sp}(4, \mathbb{C})$, this image is simply connected. It follows that $\tilde{\beta}_v$ lifts to the Spin group, which is a 2-fold cover of $\mathrm{SO}(4, \mathbb{C})$. Consequently, $w_2(\tilde{\beta}_v)$ is trivial, from which it follows that $W(\beta_v) \,=\, W(\tilde{\beta}_v)W(1)^4$ is 1.

□

4.4. **Triple product L-functions revisited.** Let π_1, π_2, π_3 be cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$ of respective central characters $\omega_1, \omega_2, \omega_3$. Recall from section 3.1 the definitions of the three candidates for the triple product L-functions and ε-factors associated to (π_1, π_2, π_3). The object of this section is to extract the following precise statement from Theorem M.

Theorem 4.4.1. *Let π_j, $j \leq 3$, be as above. Then we have, at* **every** *place v,*

$$L_1(s, \pi_{1,v} \times \pi_{2,v}, \pi_{3,v}) \overset{(a)}{=} L(s, \pi_{1,v} \times \pi_{2,v}, \pi_{3,v}) \overset{(b)}{=} L_2(s, \pi_{1,v} \times \pi_{2,v}, \pi_{3,v}),$$

and

$$\varepsilon_1(s, \pi_{1,v} \times \pi_{2,v}, \pi_{3,v}) \overset{(a)}{=} \varepsilon(s, \pi_{1,v} \times \pi_{2,v}, \pi_{3,v}) \overset{(b)}{=} \varepsilon_2(s, \pi_{1,v} \times \pi_{2,v}, \pi_{3,v})$$

Proof. Since we know this (see section 3.3) at archimedean places, we may restrict our attention to finite places; fix such a place u. Let p be the residual characteristic of v. Recall from the discussion in section 3.3 that (a) (resp. (b)) is known (i) when all the representations are unramified, *and* (ii) when one of the representations is not supercuspidal (resp. when all the representations are tempered).

One has the following (at any place v) by the works of Shahidi:

$$\gamma(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) = \gamma_2(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}). \tag{4.4.2}$$

In view of the remark above, the only case to check is when one of the representations, say $\pi_{1,v}$, is a subquotient of a principal series representation defined by quasi-characters μ_1, μ_2 of F_v^*. Then one has by the multiplicativity of local factors ([Sh4]), the identity

$$\gamma_2(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) = \gamma_2(s, \mu_1 \otimes \pi_{2,v} \times \pi_{3,v})\gamma_2(s, \mu_{\otimes}\pi_{2,v} \times \pi_{3,v}).$$

The analog for $\gamma(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ holds by the decomposition of $\sigma_{1,v} \otimes \sigma_{2,v} \otimes \sigma_{3,v}$. Done.

Our first object is to establish a weak analog of (4.4.2) for the γ_1-factor in the place of the γ_2-factor.

Lemma 4.4.3. *Let $S(p)$ be the set of places of F above p. Then we have*

$$\prod_{v \in S(p)} \gamma(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) = \prod_{v \in S(p)} \gamma_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}).$$

Remark 4.4.4. In [Ik1], page 229, corollary to Lemma 2.2, one finds an argument to prove Thm. 4.4.1 (a) in the special case when *each* $\pi_{j,u}$ is dihedral, being associated to a character $\chi_{j,u}$ of a quadratic extension $K_{j,u}$ of F_u. But in the course of the argument, there is an assertion that there is a quadratic extension K_j of F with local extension $K_{j,u}/F_u$, together with an idele class character χ of K, such that (1) χ agrees with $\chi_{j,u}$ on $K_{j,u}$ (which is fine) *and* that (2) the cuspidal automorphic representation η_j of $\mathrm{GL}(2, \mathbb{A}_F)$ defined by χ_j has principal series components at *all* the places v outside u. We are unable to make sure that (2) can be chieved. (One cannot in general control the behavior of χ_j at an infinite number of places!) However, the argument can be made to work with a slight refinement as follows. We can find K_j, χ_j satisfying (1) and also (2'): χ_j is trivial at every place in $S(p) - \{u\}$; this is possible by the Grunewald-Wang theorem ([AT]). By applying Lemma 4.4.3 above, and using Lemma 2.1 of [Ik1], the desired equality then follows.

Proof of Lemma 4.4.3. Let T be the set of finite places where at least one of the representations $\{\pi_1, \pi_2, \pi_3\}$ is ramified. Then, comparing the functional equations of $L(s, \pi_1 \times \pi_2 \times \pi_3)$ and $L_1(s, \pi_1 \times \pi_2 \times \pi_3)$, we get the equality

$$\prod_{v \in T} \gamma(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}) = \prod_{v \in T} \gamma_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v}).$$

When v is not in $S(p)$, its norm is a power of a prime ℓ distinct from p, and the corresponding local factors are inverses of polynomials in ℓ^{-s}. Since the identity above holds for all s, it is easy to separate the contribution from the primes in $S(p)$ and those outside.

□

Proof of Theorem 4.4.1 (contd.). Now we need the following

Proposition 4.4.5. *Let v be a finite place, and let L_* denote L, L_1 or L_2. Then, for any quasi-character ν of which is sufficiently ramified, we have*

$$L_*(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \nu) = 1.$$

Proof of Proposition 4.4.5. First we consider when $L_* = L$. Since by definition $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \nu)$ is the same as $L(s, \sigma_{1,v} \otimes \sigma_{2,v} \otimes \sigma_{3,v} \otimes \nu)$, it suffices, by the additivity of the L-factor, to show that, for any irreducible summand τ of $\sigma_{1,v} \otimes \sigma_{2,v} \otimes \sigma_{3,v}$, viewed as a representation of $W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$, we have $L(s, \tau \otimes \nu)$ is 1 for ν of large enough conductor. Any irreducible τ is of the form $\beta \otimes S^k$, where β is an irreducible of W_{F_v} and S^j denotes the symmetric j-th power representation of $\mathrm{SL}(2, \mathbb{C})$. When β is one-dimensional, $L(s, \tau \otimes \nu)$ equals $L(s, \beta\nu|.|^{j/2})$, which is trivial (by Tate's thesis) if ν is sufficiently ramified. If β is higher dimensional, then $L(s, \tau \otimes \nu)$ is a product, for suitable half-integers t, of factors of the form $L(s, \beta \otimes \nu|.|^t)$, which are 1 for *any* ν (because β is irreducible of dimension > 1).

Next consider the case $L_* = L_i$, with $i = 1, 2$. We may asume that at least one of the $\pi_{j,v}$, say $\pi_{3,v}$, is tempered. Suppose $\pi_{1,v}$ is not supercuspidal. Then by using Proposition 3.3.6 for L_1, and (4.4.2) for L_2 together with the temperedness of $\pi_{3,v}$, we deduce that (for any ν)

$$\frac{L_1(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \nu)}{L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \nu)} \quad \text{is} \quad \text{entire.}$$

The assertions (a) and (b) of the Theorem about L-factors follows in this case by the triviality of the denominator for sufficiently ramified ν. The identities for the ε-factors also follows because the respective γ-factors are the same.

So we may take all the $\pi_{j,v}$ to be supercuspidal. We will now use a slight variant of the inductive argument utilized in the proof of section 3.7. Let $\ell(\pi_{1,v})$, $\ell(\pi_1)$, $p(v)$, $p = p(\pi_1)$, $r = r(\pi_1) = (\ell(\pi_1), p(\pi_1))$ be as in section 3.7, together with the (lexicographic) ordering $<$ introduced there. We will use induction on r. If $r = (0, 1)$, we are done, and so take that $(0, 1) < r$ and assume by induction that the Proposition holds for L_1 at all places for any triple (π, π', π'') of cuspidal automorphic representations o $\mathrm{GL}(2)$ over a number field if $r(\pi) < r$. Fix at every place v in S'', a character χ_v cutting out a $K(v)$ of degree p such that the base change of $\pi_{1,v}$ to $K(v)$ has ℓ equal to $\ell(\pi_1)/p$. Choose, using the Grunewald-Wang theorem, a global character χ of order p, cutting out a cyclic extension K/F with local extensions $K(v)/F_v$. Then $r(\pi_{1,K}$ is less than r and we may use induction in conjunction with the functional equations of $L_i(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \nu)$ and $L_i(s, \pi_{1,K} \times \pi_{2,K} \times \pi_{3,K} \otimes \nu)$ to conclude that, for any idele class character ν of F which is sufficiently ramified at each of the places in S'', we have (for $i = 1, 2$)

$$\prod_{v \in S''} \gamma_i(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \nu) = 1.$$

As remarked earlier, we may assume that, for every $j \leq 3$ and every $v \in S''$, the local representation $\pi_{j,v}$ is supercuspidal. We may then apply Lemma 2.1 of [Ik1] to conclude that $L_i(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \nu)$ has no pole in common with $L_i(1-s, \pi_{1,v}^\vee \times \pi_{2,v}^\vee \times \pi_{3,v}^\vee \otimes \nu^{-1})$. The proposition follows for L_i.

□

Remark 4.4.6. One can also prove Proposition 4.4.5 for L_2 by arguing as in the proof of Lemma 5 of [PR]. Moreover, Shahidi has informed us that he can now prove a very general statement of this sort for L_2.

Proof of Theorem 4.4.1 (contd.). Fix any finite place u of residual characteristic p, and let $S(p)$ be as in Lemma 4.4.3. Choose an idele class character ν of F such that (i) ν_v is trivial, and (ii) ν_v is highly ramified at each v in $S(p)$. Then, comparing the functional equations of $L_1(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \nu)$ and $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \nu)$ and using (4.4.2), Lemma 4.4.3 and Proposition 4.4, we get

$$\gamma_1(s, \pi_{1,u} \times \pi_{2,u} \times \pi_{3,u} \otimes \nu) = \gamma(s, \pi_{1,u} \times \pi_{2,u} \times \pi_{3,u} \otimes \nu). \tag{4.4.7}$$

Again, we can reduce to the case when each $\pi_{j,u}$ is supercuspidal and so, applying Lemma 2.1 of [Ik1] and the obvious analog for L_2, we deduce the assertions (a) and (b) of the theorem for the L-factors. The identities for the ε-factors also follow by (4.4.7).

□

4.5. **The Tate conjecture for 4-fold products of modular curves.** Let Y denotes the moduli scheme over $\mathbb{Q}$ parametrizing elliptic curves E with level infinity structure, i.e., equipped with an isomorphism of the Tate module $T_f(E) = \lim_n E[n]$ with $\hat{\mathbb{Z}}^2$. It comes with a natural right action of $\mathrm{GL}(2, \mathbb{A}^f)$, and a smooth compactification X ([KM]). For every compact, open subgroup K of $\mathrm{GL}(2, \mathbb{A}^f)$, let Y_K (resp. $X_K = Y_K \cup Y_K^\infty$) denote Y/K (resp. X/K). Then X_K is a smooth projective curve over $\mathbb{Q}$, and one has the identifications

$$Y_K(\mathbb{C}) = \mathrm{GL}(2, \mathbb{Q}) \backslash \mathcal{H}^\pm / K,$$

where $\mathcal{H}^\pm$ denotes $\mathbb{C} - \mathbb{R}$, and

$$Y_K^\infty = \mathbb{P}^1(\hat{\mathbb{Z}})/K.$$

A cofinal system of compact open subgroups is provided by $\{K_1(N) | N \geq 1\}$, where

$$K_1(N) = \{k = (k_{ij}) \in \mathrm{GL}(2, \hat{\mathbb{Z}}) \,|\, k_{21}, k_{22} - 1 \in N\hat{\mathbb{Z}}\}.$$

For every $N \geq 1$, we will also be interested in

$$K_0(N) = \{k = (k_{ij}) \in \mathrm{GL}(2, \hat{\mathbb{Z}}) \,|\, k_{21} \in N\hat{\mathbb{Z}}\}.$$

We will write $X_1(N), X_0(N)$ instead of $X_{K_1(N)}, X_{K_0(N)}$ respectively.

Let V be a smooth projective variety over a number field k. For every $j \geq 0$, denote by $C^j(V/k)$ the $\mathbb{Q}$-vector space spanned by codimension j algebraic cycles on V over k modulo homological equivalence. Then, for every prime ℓ, one has an ℓ-adic cycle class map

$$C^j(V/k) \longrightarrow H_{et}^{2j}(V_{\overline{k}}, \mathbb{Q}_\ell)(j),$$

which is injective and lands in the group of codimension j *Tate cycles* over k, namely

$$\mathrm{Ta}_\ell^j(V/k) = H_{et}^{2j}(V_{\overline{k}}, \mathbb{Q}_\ell)(j)^{\mathrm{Gal}(\overline{k}/k)}.$$

Let S be a finite set of places containing the ramified places for the $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$-module $H^{2j}_{\mathrm{et}}(V_{\overline{k}}, \mathbb{Q}_\ell)$, the archimedean places, and also the places above ℓ. The associated L-function is

$$L^{(2j)}(s, V/k) \;=\; \prod_{v\,:\,\mathrm{finite}} L_v^{(2j)}(s, V/k),$$

where

$$L_v^{(2j)}(s, V/k) \;=\; \det(I - \mathrm{Fr}_v T | H^{2j}_{\mathrm{et}}(V_{\overline{k}}, \mathbb{Q}_\ell)^{I_v})^{-1}|_{T=(Nv)^{-s}}.$$

Here Fr_v denotes the geometric Frobenius at v and I_v the inertia group at v. In computing $L_v^{(2j)}(s, V/k)$, one takes ℓ to be prime to v. For each ℓ, let $S = S(\ell)$ be the finite set of places consisting of the ramified places for the $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$-module $H^{2j}_{\mathrm{et}}(V_{\overline{k}}, \mathbb{Q}_\ell)$, together with the places above ℓ. One knows by Deligne's proof of the Weil conjectures that the incomplete L-function $L^{(2j),S}(s, V/k)$, obtained from $L^{(2j)}(s, V/k)$ by deleting the factors over S, converges absolutely in $\Re(s) > j + 1$. One also has a definition of $L_v^{(2j)}(s, V/k)$ for any archimedean place v (see [Ra3], sec.5.5, for example). Write $L_\infty^{(2j)}(s, V/k)$ for the product $\prod_{v|\infty} L_v^{(2j)}(s, V/k)$, and put

$$\tilde{L}^{(2j)}(s, V/k) := L_\infty^{(2j)}(s, V/k) L^{(2j)}(s, V/k).$$

One also has epsilon factors $\varepsilon_v^{(2j)}(s, V/k)$ for each v, and the global one

$$\varepsilon^{(2j)}(s, V/k) \;=\; \prod_{\mathrm{all}\ v} \varepsilon_v^{(2j)}(s, V/k),$$

which is an invertible holomorphic function.

Theorem 4.5.1. *Fix positive integers N_j, $1 \le j \le 4$, and put*

$$V \;=\; X_1(N_1) \times X_2(N_2) \times X_3(N_3) \times X_4(N_4).$$

Let k be a finite solvable, normal extension of $\mathbb{Q}$, and let ℓ be a prime. Then the following hold:

(a) $L^{(4)}(s, V/k)$ admits a meromorphic continuation to the whole s-plane, has a convergent Euler product in $\{Re(s) > 3\}$, and satisfies the exact functional equation:

$$\tilde{L}^{(4)}(s, V/k) \;=\; \varepsilon^{(4)}(s, V/k) \tilde{L}^{(4)}(5 - s, V/k).$$

(b) $L^{(4)}(s, V/k)$ has no pole anywhere except possibly at the "edge" point $s = 3$.

(c) If we let $-\mathrm{ord}_{s=3}$ stand for the order of pole at $s = 3$,

$$-\mathrm{ord}_{s=3} L^{(4)}(s, V/k) \;=\; \dim_{\mathbb{Q}_\ell} \mathrm{Ta}_\ell^2(V/k).$$

(d) Suppose some N_j is square-free. Then

$$-\mathrm{ord}_{s=3} L^{(4),S}(s, V/k) \;=\; \dim_{\mathbb{Q}} C^2(V/k).$$

In this case, for any number field k, $\dim_{\mathbb{Q}_\ell} \mathrm{Ta}_\ell^2(V/k)$ and $\dim_{\mathbb{Q}} C^2(V/k)$ are equal.

Parts (a) and (b) are as predicted by the Hasse-Weil hypothesis, while parts (c) and (d) verify certain conjectures of Tate (see [Ta2]; [Ra3], chap.5).

Proof. First we recall some well known basic facts about the cohomology of the modular curves $X_1(N)$. For $j \in \{0, 1, 2\}$, put (for any prime ℓ)

$$W_\ell^j(N) := H^j_{\mathrm{et}}(X_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell).$$

For any field extension E of $\mathbb{Q}$, denote by $\mathcal{H}_E(N)$ the E-algebra of Hecke correspondences at level N. This algebra is semisimple and acts on $W_\ell^j(N) \otimes_{\mathbb{Q}} E$, commuting

with the Galois action. If $\pi = \pi_\infty \otimes \pi_f$ is an irreducible automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$, with π_f rational over $\overline{\mathbb{Q}}$, then the $K_1(N)$-invariants in the space of π_f over $\overline{\mathbb{Q}}$ is naturally an $\mathcal{H}_{\overline{\mathbb{Q}}}(N)$-module. Conversely, any irreducible $\mathcal{H}_{\overline{Q}}(N)$-module occurring in $W_\ell^j(N)$ is a summand of some $\pi_f^{K_1(N)}$. Denote by $\mathrm{Coh}^j(N)$ the set of irreducible automorphic representations π contributing to the cohomology in degree j with $\pi_f^{K_1(N)}$ non-zero. Then one knows that π is one-dimensional for $j = 0, 2$, and that for $j = 1$ it is cuspidal with π_∞ in the lowest discrete series, corresponding, in the classical language, to a holomorphic newform of weight 2 and level N. Decomposing according to the Hecke algebra, one gets

$$W_\ell^j(N) \otimes_{\mathbb{Q}_\ell} \overline{Q}_\ell \;\simeq\; \oplus_{\pi \in \mathrm{Coh^j(N)}} W_\ell^j(\pi)^{m(\pi_f, N)}, \tag{4.5.2}$$

where $W_\ell^j(\pi)$ is an irreducible $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of dimension 2 (resp. 1) when $j = 1$ (resp. $j = 0, 2$), and $m(\pi_f, N)$ the dimension of the $K_1(N)$-invariants in the space of π_f. When a π occurs in degree 1, then any of its Galois conjugate $\pi^\tau := \pi_\infty \otimes \pi_f^\tau$ also occurs. One knows the following at *any* place v by the Eichler-Shimura theory and its refinement due to Igusa, Deligne, Langlands and Carayol([Ca]):

$$L_v(s, W_\ell^0(\pi)) \;=\; L(s, \pi_v) \;=\; L_v(s+1, W_\ell^2(\pi)), \tag{4.5.3}$$

and

$$L_v(s, W_\ell^1(\pi)) \;=\; L(s - 1/2, \pi_v).$$

In fact this also holds for one-dimensional twists. The analogous identities hold for the ε-factors as well, implying in particular the identification of the conductors.

Now let us turn our attention to the variety V at hand. In the following i will denote a any 4-tuple (i_1, i_2, i_3, i_4) of integers in $\{0, 1, 2\}$ with $\sum_{j=1}^4 i_j = 4$, and for each such i, π will denote $(\pi_1, \pi_2, \pi_3, \pi_4)$ with $\pi_j \in \mathrm{Coh}^{i_j}(N_j)$ for each j. Let $m(i, \pi)$ signify $\prod_{j=1}^4 m(\pi_{j,f}, N_j)$.Applying the Künneth formula in conjunction with the decomposition (4.5.2), we get

$$H^4(V_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \;\simeq\; \oplus_i \, \oplus_\pi \; H_\ell^i(\pi)^{m(i,\pi)}, \tag{4.5.4}$$

where

$$H_\ell^i(\pi) \;\simeq\; W_\ell^{i_1}(\pi_1) \otimes W_\ell^{i_2}(\pi_2) \otimes W_\ell^{i_3}(\pi_3) \otimes W_\ell^{i_4}(\pi_4).$$

Applying Proposition 4.3.1 in conjunction with (4.5.3), we get moreover,

$$L(s, H_\ell^i(\pi)) \;=\; L(s-2, \pi_1 \times \pi_2 \times \pi_3 \times \pi_4). \tag{4.5.5}$$

Similarly for the ε-factors.

Since the cycle class maps are functorial for the action of correspondences, we can split the problem and prove the assertions of the theorem for the image of the codimension 2 cycles in $H_\ell^i(\pi)$, for every i and π. Let $\mathrm{Ta}_\ell^i(\pi)_k$ denote the space of Tate classes over k in $H_\ell^i(\pi)$.

Suppose two of the indices, say i_3, i_4, are zero. Then $H_\ell^i(\pi)$ is $W_\ell^2(\pi_1) \otimes W_\ell^2(\pi_2) \otimes W_\ell^0(\pi_3) \otimes W_\ell^0(\pi_4)$, which is one-dimensional, in fact of the form $\mu \otimes \overline{\mathbb{Q}}_\ell(-2)$ with μ of finite order. Then for *any* number field k, we have $L(s, H_\ell^i(\pi)) = L(s-2, \mu)$. The assertions (a), (b) follow by Hecke. The order of pole at $s = 3$ is 1 or 0 depending n whether or not μ is trivial when restricted to $\mathrm{Gal}(\overline{Q}/k)$. When μ is trivial, the corresponding algebraic cycle is given by the intersection of the the π-components of the threefolds $\{P\} \times X_2(N)_2) \times X_3(N_3) \times X_4(N_4)$ and $X_1(N_1) \times \{Q\} \times X_3(N_3) \times$

$X_4(N_4)$, for points P, Q on $X_1(N_1), X_2(N_2)$ respectively. So (d) also follows in this case.

So we may assume from now on that $i = (1,1,1,1)$ and consequently that each π_j is cuspidal. In view of (4.5.5), parts (a) and (b) hold by Theorem 4.2.1. Note that the determinant of each $W^1_\ell(\pi_j)$ is necessarily of the form $\omega_j \otimes \overline{\mathbb{Q}}_\ell(-1)$, where ω_j is of finite order corresponding to the central character of π_j by class field theory.

Suppose all the π_j are dihedral, defined by a Hecke character χ_j of a (necessarily imaginary) quadratic field K_j. Then $H^i_\ell(\pi)$ is the $\overline{\mathbb{Q}}_\ell$-realization of a CM motive (in the category of pure motives for absolute Hodge cycles), and the assertion (c) is well known (see [DMOS]). The conductor M_j, say, of each π_j is the conductor of an abelian variety factor $A(\pi_j)$ over $\mathbb{Q}$, up to isogeny, of the Jacobian of the corresponding modular curve. One knows by a theorem of Fontaine ([Fon]) that there is no abelian variety over $\mathbb{Q}$ of good reduction everywhere. On the other hand, since π_j is automorphically induced by χ_j, it has no local component which is special. Then by the description of th conductor of π_j in [Ge], for example, we see that M_j, and hence N_j which it divides, cannot be square-free. So the hypothesis of part (d) precludes the case when *all* the π_j are dihedral.

Thus we may, and we will, assume that some π_j, say π_1, is *non-dihedral*.

Lemma 4.5.6. *Let $i = (1,1,1,1)$ and $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$, with each π_j cuspidal automorphic of weight 2 at infinity and π_1 non-dihedral. Fix any number field k. Then the dimension of $Ta^i_\ell(\pi)_k$ is at most 2. It is non-zero iff we can find, after possibly renumbering $\{\pi_2, \pi_3, \pi_4\}$, a character μ of $Gal(\overline{\mathbb{Q}}/k)$ such that*

(i) as $Gal(\overline{\mathbb{Q}}/k)$-modules,

$$W^1_\ell(\pi_2) \simeq W^1_\ell(\pi_1) \otimes \mu$$

and

(ii) $(\mu\omega_1)^{-1}$ occurs in $W^1_\ell(\pi_3) \otimes W^1_\ell(\pi_4)$ over k.

Moreover, when this happens, the dimension of $Ta^i_\ell(\pi)$ is 1 unless we are in one of the following situations:

(a) Over k, $W^1_\ell(\pi_3))$ is irreducible and non-dihedral, with its symmetric square being isomorphic to $S^2(W^1_\ell(\pi_1)) \otimes \omega_3\omega_1^{-1}$;

(b) $W^1_\ell(\pi_3)$ and $W^1_\ell(\pi_4)$ are both reducible over k, and dim $Hom(\mu\omega_1^{-1}, W^1_\ell(\pi_3) \otimes W^1_\ell(\pi_4)$ is 2.

Proof. Since π_1 is non-dihedral, we know by a result of Ribet ([Ri1]) that $W^1_\ell(\pi_1)$ is irreducible upon restriction to any open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It follows easily that there are no Tate classes over k unless (i) holds after a possible renumbering. There will be a Tate cycle iff (ii) also holds. Now suppose there is a Tate class.

If π_3 is irreducible, (ii) will hold iff π_4 is also irreducible and

$$(ii') \qquad W^1_\ell(\pi_4) \simeq W^1_\ell(\pi_3) \otimes (\mu\omega_1\omega_3)^{-1}.$$

We claim that, since π_1 is non-dihedral, $S^2(W^1_\ell(\pi_1))$ must irreducible under restriction to any open subgroup. Indeed, if the claim is false, then over a number field the symmetric square would admit a one-dimensional and would force $W^1_\ell(\pi_1)$ would have to be dihedral (over the extension field), contradicting the openness of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{GL}(W^1_\ell(\pi_1))$. Hence the only way there could be more than 1 Tate class over k is to have (a), and this is not possible if π_3 is dihedral. Suppose

π_3 is non-dihedral. Then each π_j is non-dihedral and $S^2(W_\ell^1(\pi_j))$ is irreducible (by the above argument) for *any* j. Then the number of (independent) such classes can evidently not be more than 2.

Finally, consider the case when (i), (ii), hold, *and* π_3, π_4 are both reducible. Looking at the Hodge-Tate types we see that the dimension of $\mathrm{Ta}_\ell^i(\pi)$ can never exceed 2. It is then also clear that to have dimension 2, we need (b).

Remark 4.5.7. The reason for requiring in Theorem 4.5.1 that some N_j is square-free is just to rule out the case when all the π_j are dihedral, in which case the number of Tate classes can at the extreme be 6, and only four of those can be shown to be algebraic by the argument below. When one of them is non-dihedral, the numbers of tate and algebraic cycles in $H_\ell^i(\pi)$ coincide over every number field, not necessarily solvable. The reason for restricting to solvable extensions is to be able to appeal to base change in conjunction with Theorem 4.2.1 and deduce the standard properties of the L-function.

Proof of Theorem 4.5.1 (contd.). Let k be a number field. We need to match the number of Tate classes with the order of pole. By using the standard results on Hecke L-functions, we may assume that one of the π_j, say π_1, is non-dihedral. In view of (4.5.5), part (c) of the Theorem will follow once we establish the following analog of Lemma 4.5.6:

Lemma 4.5.8 *Let $i = (1,1,1,1)$ and $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$, with each π_j cuspidal automorphic of weight 2 at infinity and π_1 non-dihedral. Fix any number field k, and denote $(\forall j)$ by $\pi_{j,k}$ the base change of π to k The order of pole, $m(\pi_k)$ say, of $L(s, \pi_k) := L(s, \pi_1 \times \pi_2 \times \pi_3 \times \pi_4)$ at $s = 1$ is at most 2. It is non-zero iff we can find, after possibly renumbering $\{\pi_2, \pi_3, \pi_4\}$, an idele class character μ of k such that*

(i) $\pi_{2,k} \simeq \pi_{1,k} \otimes \mu$,

and

(ii) $(\mu_k \omega_{1,k})^{-1}$ *occurs in the isobaric sum decomposition of* $\pi_{3,k}) \boxtimes \pi_{4,k}$.

Moreover, when this happens, $m(\pi_k)$ is 1 unless we are in one of the following situations:

(a) $\pi_{3,k}$ *is cuspidal and non-dihedral, with its symmetric square being isomorphic to* $S^2((\pi_1)) \otimes \omega_{3,k}\omega_{1,k}^{-1}$;

(b) $\pi_{3,k}$ *and* $\pi_{4,k}$ *are both Eisensteinian over k, and the multiplicity of* $(\mu_k \omega_{1,k}^{-1}$ *in* $\pi_{3,k} \boxtimes \pi_{4,k}$ *is 2.*

Proof of Lemma. The conditions (i) and (ii) are easy to verify if every π_j is twist equivalent to π_1. So suppose not. Then, after a renumbering, we may assume that $\pi_{1,k} \boxtimes \pi_{3,k}$ is cuspidal. Using Theorem M we can write $L(s, \pi_k)$ as $L(s, (\pi_1 \boxtimes \pi_3) \times (\pi_2 \boxtimes \pi_4))$. So by [JS2], it has a pole at $s = 1$ iff we have

$$\pi_{2,k} \boxtimes \pi_{4,k} \;\simeq\; (\pi_{1,k} \boxtimes \pi_{3,k}) \;\simeq\; \pi_{1,k} \boxtimes \pi_{3,k} \otimes (\omega_{1,k}\omega_{3,k})^{-1}.$$

Up to a renumbering, this can happen iff (i) and (ii) hold.

If $\pi_{3,k}$ is cuspidal, then (ii) will hold iff $\pi_{4,k}$ is also cuspidal and

$$(ii') \qquad \pi_{4,k} \;\simeq\; \pi_{3,k} \otimes (\mu_k \omega_{1,k} \omega_{3,k})^{-1}.$$

We claim that $S^2(\pi_{1,k})$ is cuspidal. Suppose not. Then by [GJ], $\pi_{1,k}$ will be dihedral. This would imply, by (4.5.3) and the functoriality of base change, that $W_\ell^1(\pi_1)$ is dihedral when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$, contradicting the openness of the

image of Galois ([Ri1]). Hence the claim. Applying [JS2], we then see that $m(\pi_k)$ could be more than 1 iff we have (a), and this is evidently not possible is π_3 is dihedral. Then each π_j is non-dihedral and $m(\pi_k)$ cannot possibly be more than 2. when π_3, π_4 are both reducible, we see by looking at the infinity type that $m(\pi_k)$ can never exceed 2, and that it can be 2 iff (b) holds.

□

It remains to show (d). In view of (c), it suffices to show, over any number field k, that $\mathrm{Ta}_\ell^i(\pi)$ consists of algebraic classes. Suppose there is a non-zero Tate class. Then (i) holds and this is represented by a twisting correspondence R_μ on $X_1(N_1)_k \times X_1(N_2)_k$ (see [Ri2], [Mu]). If $\pi_{3,k}$ is also cuspidal, then (ii') holds and this is represented by another twisting correspondence R_ν on $X_1(N_3) \times X_1(N_4)$, with $\nu = (\mu\omega_1\omega_3)^{-1}$. We get a codimension 2 algebraic cycle on V_k whose π_k-component has cycle class in $\mathrm{Ta}_\ell^i(\pi)$ by taking the intersection γ of $R_\mu \times X_1(N_3) \times X_1(N_4)$ and $X_1(N_1) \times X_1(N_2) \times R_\nu$. Now suppose there are two independent Tate classes. If $\pi_{3,k}$ is cuspidal, then we would have the symmetric square twisting equivalence of (a), and this will give us a twisting correspondence R_λ directly on the 4-fold V, with $\lambda = \omega_3\omega^{-1}$. This cycle is evidently not multiple of γ. So we are done in this case. If we are in the situation of (b), the abelian varieties $A(\pi_3)$ and $A(\pi_4)$ corresponding to π_3 and π_4 will necessarily be of CM type, and we will have two independent Tate classes in $\mathrm{Hom}(A(\pi_3), A(\pi_4))$ over k which are algebraic by Faltings; they correspond to divisors δ_1, δ_2 on $X_1(N_3) \times X_1(N_4)$ rational over k. If we take ξ_j, for $j = 1, 2$ to be $R_\mu \times \delta_j$, they define the needed cycles on V.

□

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