Icosahedral Fibres of the Symmetric Cube and Algebraicity

Dinakar Ramakrishnan

To Freydoon Shahidi, on the occasion of his sixtieth birthday
Tavalodet Mubarak!

ABSTRACT. For any number field F, call a cusp form $\pi = \pi_\infty \otimes \pi_f$ on $\mathrm{GL}(2)/F$ *special icosahedral*, or just *s-icosahedral* for short, if π is not solvable polyhedral, and for a suitable "conjugate" cusp form π' on $\mathrm{GL}(2)/F$, $\mathrm{sym}^3(\pi)$ is isomorphic to $\mathrm{sym}^3(\pi')$, and the symmetric fifth power L-series of π equals the Rankin-Selberg L-function $L(s, \mathrm{sym}^2(\pi') \times \pi)$ (up to a finite number of Euler factors). Then the point of this Note is to obtain the following result:

Let π be *s-icosahedral* (of trivial central character). Then π_f is algebraic without local components of Steinberg type, π_∞ is of Galois type, and π_v is tempered everywhere. Moreover, if π' is also of trivial central character, it is *s-icosahedral*, and the field of rationality $\mathbb{Q}(\pi_f)$ (of π_f) is $K := \mathbb{Q}[\sqrt{5}]$, with π'_f being the Galois conjugate of π_f under the non-trivial automorphism of K.

There is an analogue in the case of non-trivial central character ω, with the conclusion that π_f is algebraic when ω is, and when ω has finite order, $\mathbb{Q}(\pi_f)$ is contained in a cyclotomic field.

1. Introduction

Let us begin with some motivation and consider a continuous irreducible representation ρ with trivial determinant of the absolute Galois group $\mathcal{G}_F$ (of a number field F) into $\mathrm{GL}(2, \mathbb{C})$ which is *icosahedral*, i.e., whose image in $\mathrm{PGL}(2, \mathbb{C})$ is the alternating group A_5. Then it is well known that ρ is rational over $K = \mathbb{Q}[\sqrt{5}]$, but not equivalent to its Galois conjugate ρ' under the non-trivial automorphism of K. Moreover, one has (see [**Kim2**], [**Wan**], for example): (i) $\mathrm{sym}^3(\rho) \simeq \mathrm{sym}^3(\rho')$, and (ii) $\mathrm{sym}^5(\rho) \simeq \mathrm{sym}^2(\rho') \otimes \rho$. These two are the signature properties of such representations, and they lend themselves to natural automorphic analogues.

Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ with central character ω. For every $m \geq 1$ one has its *symmetric m-th power L-function* $L(s, \pi; \mathrm{sym}^m)$, which is an Euler product over the places v of F, with the v-factors

(for finite unramified v of norm q_v) being given by

$$L_v(s, \pi; \mathrm{sym}^m) \; = \; \prod_{j=0}^{m}(1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1},$$

where the unordered pair $\{\alpha_v, \beta_v\}$ defines the diagonal conjugacy class in $\mathrm{GL}_2(\mathbb{C})$ attached to π_v. Even at a ramified (resp. archimedean) place v, one has by the local Langlands correspondence a 2-dimensional representation σ_v of the extended Weil group $W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$ (resp. of the Weil group W_{F_v}), and the v-factor of the symmetric m-th power L-function is associated to $\mathrm{sym}^m(\sigma_v)$. A special case of the *principle of functoriality* of Langlands asserts that there is, for each m, an (isobaric) automorphic representation $\mathrm{sym}^m(\pi)$ of $\mathrm{GL}(m + 1, \mathbb{A})$ whose standard (degree $m+1$) L-function $L(s, \mathrm{sym}^m(\pi))$ agrees, at least at the primes not dividing $\mathcal{N}$, with $L(s, \pi; \mathrm{sym}^m)$. This was known to be true for $m = 2$ long ago by the work of Gelbart and Jacquet ([**GJ**]), and more recently for $m = 3, 4$ by the deep works of Shahidi and Kim ([**KS2, Kim1, KS1**]). Write $A^{2j}(\pi)$ for $\mathrm{sym}^{2j}(\pi) \otimes \omega^{-j}$ when the latter is automorphic; it is also customary to denote $A^2(\pi)$ by $\mathrm{Ad}(\pi)$. We will say that π is *solvable polyhedral* if $\mathrm{sym}^m(\pi)$ is Eisensteinian for some $m \leq 4$.

Suppose π' is another cusp form on $\mathrm{GL}(2)/F$, say of the same central character as π, such that $\mathrm{sym}^2(\pi) \simeq \mathrm{sym}^2(\pi')$. Then one knows ([**Ram1**]) that π and π' must be abelian twists of each other. One could ask if the same conclusion holds if $\mathrm{sym}^3(\pi)$ is isomorphic to $\mathrm{sym}^3(\pi')$. The answer is in the negative in that case, and a counterexample would be furnished by a π associated to a 2-dimensional icosahedral Galois representation ρ (of trivial determinant), i.e., with its image in $\mathrm{PGL}(2, \mathbb{C})$ being isomorphic to the alternating group A_5. Indeed, as remarked above, ρ would be defined over $\mathbb{Q}[\sqrt{5}]$ and π' would be associated to the Galois conjugate ρ' of ρ (under $a + b\sqrt{5} \mapsto a - b\sqrt{5}$). However, the *even* Galois representations are not (at all) known to be modular. Nevertheless, cusp forms of such icosahedral type are of great interest to study, for themselves and for understanding the fibres of the symmetric cube transfer. What we do is give a definition of an s-icosahedral cusp form which does not depend on any conjecture, and which is robust enough to furnish consequences which one would usually know only when there is an associated Galois representation.

Let us call a cusp form π on $\mathrm{GL}(2)/F$ (of central character ω) *s-icosahedral* if we have

- (sI-1) π is not solvable polyhedral;
- (sI-2) $\mathrm{sym}^3(\pi) \simeq \mathrm{sym}^3(\pi')$, for a cusp form π' on $\mathrm{GL}(2)/F$; and
- (sI-3) the following identity of L-functions holds (outside a finite set S of places of F containing the archimedean and ramified places):

$$L^S(s, \pi; \mathrm{sym}^5) \; = \; L^S(s, \mathrm{Ad}(\pi') \times \pi \otimes \omega^2),$$

   with π' as in (sI-2).

Observe that if (π, π') are associated as above, then $(\pi, \pi' \otimes \nu)$ will also be associated if ν is a cubic character, but this ambiguity can be eliminated by requiring that the ratio of the central characters ω, ω', of π, π' respectively, is not cubic. Note also that the property of being s-icosahedral is invariant under twisting $\pi \mapsto \pi \otimes \chi$ by idèle class characters χ, with $\pi' \mapsto \pi' \otimes \chi$, in particular under taking contragredients $\pi \mapsto \pi^\vee = \pi \otimes \omega^{-1}$.

Given cusp forms π_1, π_2 on $\mathrm{GL}(n_1)/F$, $\mathrm{GL}(n_2)/F$ respectively, one has good analytic properties ([**Sha2, Sha1, JPSS, JS**]) of the associated Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$. When $(n_1, n_2) = (2, 2)$ ([**Ram1**]) and $(n_1, n_2) = (3, 2)$ ([**KS2**]), one also knows the existence of an isobaric automorphic form $\pi_1 \boxtimes \pi_2$ on $\mathrm{GL}(n_1 n_2)$. Thus the hypothesis (sI-3) implies that $\mathrm{sym}^5(\pi)$ is modular; in fact, $\mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2$ represents it at every finite place (and at infinity), as seen by the standard stability results due to Shahidi and others (see section 2 below).

What we want to do in this Note is to look at the situation where one does not know of the existence of a corresponding Galois representation ρ, like when π is a suitable Maass form on $\mathrm{GL}(2)/\mathbb{Q}$, and see what arithmetic properties one can still deduce *unconditionally*. However, it is a *Catch 22* situation because it is not easy, either by the trace formula or by other means, to construct such Maass forms of Laplacian eigenvalue $1/4$ except by starting with a Galois representation. Nevertheless, here is what we prove:

Theorem A *Let F be a number field and $\pi = \pi_\infty \otimes \pi_f$ a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with algebraic central character ω. Suppose π is s-icosahedral, i.e., satisfies (sI-1), (sI-2) and (sI-3) relative to a cusp form π' on $GL(2)/F$ of central character ω'. Then we have the following:*

   (a) *π_f is algebraic with no local components of Steinberg type*
   (b) *If ω is of finite order, π_f is rational over a cyclotomic extension of $\mathbb{Q}$, and π_∞ is of Galois type*
   (c) *π is tempered, i.e., satisfies the Ramanujan hypothesis.*
   (d) *If $\omega = \omega' = 1$, we have:*
      (1) *π_f is rational over $\mathbb{Q}[\sqrt{5}]$, but not over $\mathbb{Q}$;*
      (2) *π' is unique, with π'_f being the Galois conjugate of π_f under the non-trivial automorphism τ of $\mathbb{Q}[\sqrt{5}]$;   and*
      (3) *π' is also s-icosahedral and of Galois type.*

Concerning (b), let us recall that π_∞ is said to be of *Galois type* if at every archimedean place v, the associated 2-dimensional representation σ_v of the Weil group W_{F_v} is trivial upon restriction to $\mathbb{C}^*$.

One writes $\mathbb{Q}(\pi_f)$ for the field of rationality of π_f ([**Clo**], [**Wal**]). We will use the algebraic parametrization at any v which is equivariant for the action of $\mathrm{Aut}\,\mathbb{C}$ on the two sides of the local Langlands correspondence ([**Hen2**]. In particular, for v unramified, it corresponds to the *Tate parametrization* of [**Clo**], which was introduced so as not to worry about spurious square-roots of $q_v = Nv$; the normalization is unitary in [**Wal**].

In [**Ram2**], we introduced a notion of quasi-icosahedrality, based on a conditional assumption that certain symmetric powers of π are automorphic. More precisely, an irreducible cuspidal automorphic representation π of $\mathrm{GL}(2, \mathbb{A}_F)$ is called *quasi-icosahedral* iff we have

   (i) $\mathrm{sym}^m(\pi)$ is automorphic for every $m \leq 6$;
   (ii) $\mathrm{sym}^m(\pi)$ is cuspidal for every $m \leq 4$; and
   (iii) $\mathrm{sym}^6(\pi)$ is not cuspidal.

The key result of [**Ram2**] (see part (b) of Theorem A$'$ of section 2) is that, for every such quasi-icosahedral π, there exists another cusp form π' of $\mathrm{GL}(2)/F$ such that

the symmetric fifth power of such a quasi-icosahedral cusp form π is necessarily a character twist of the functorial product $\mathrm{Ad}(\pi') \boxtimes \pi$. From this we obtain

Proposition B   *Let π be an s-icosahedral cusp form on $GL(2)/F$ of central character ω. Assume that $sym^6(\pi)$ is automorphic. Then π is quasi-icosahedral.*

It will be left to the astute reader to figure out the conditions on a quasi-icosahedral π which will make it s-icosahedral. We do not pursue this any further here because we want to stay in the unconditional realm here.

Sometimes in Mathematics, when one makes a right prediction and puts it in a workable framework, the proof is not hard to find; Theorem A here is an instance of that and the proof mostly requires just bookkeeping. Nevertheless, we hope that the conclusion is of some interest. It should be mentioned that given the definition, it is not surprising that an s-icosahedral π is algebraic, but what is nice is that when $\omega = 1$, π is even rational over $\mathbb{Q}(\sqrt{5})$.

Finally, one can easily construct non-singleton fibres of the symmetric cube transfer by taking, for any cusp form π on GL(2), the collection $\{\pi \otimes \chi \,|\, \chi^3 = 1\}$. (By [**KS1**], $\mathrm{sym}^3(\pi)$ will be cuspidal iff π is not dihedral or tetrahedral.) These fibres are not terribly interesting, however, especially compared to the icosahedral ones.

This article is dedicated to Freydoon Shahidi, from whom I have learnt a lot about automorphic L-functions and the Langlands-Shahidi method, and who has been a longtime friend and in effect a *baradar e bozorgtar*.

2. The symmetric cube constraint

Let F be a number field with adèle ring $\mathbb{A}_F$. For each place v, denote by F_v the corresponding local completion of F, and for v finite, by $\mathfrak{O}_v$ the ring of integers of F_v with uniformizer ϖ_v of norm q_v. Throughout this article, ω (resp. ω') denote the central character of a cusp form π (resp. π' on $GL(2)/F$.

We will use, without mention, the notations and conventions of [**Ram2**], especially section 1 therein.

The symmetric cube condition (sI-2) (see Introduction) imposes strong constraints on how the local components π_v and π'_v are related at all v. We spell these out at the unramified v in the second part of the Lemma below, and show that, not surprisingly, when the central character ω is algebraic, π_v and π'_v are algebraically related. The first part show the effect of π satisfying both (sI-1) and (sI-2) on its "mirror" π'.

Lemma 2.1   *Let π be a cusp form on $GL(2)/F$ which is s-icosahedral relative to π'. Then*

(A) *π' satisfies (sI-j) for $j = 1, 2$, and moreover,*

$$(\omega')^3 \,=\, \omega^3.$$

In particular, ω' is algebraic iff ω is.

(B) *Let v be any finite place where π, π' are unramified, and denote by $\{a, b\}$ with $b = wa^{-1}$ (resp. $\{c, d\}$ with $d = w'c^{-1}$) the unordered pair of complex numbers associated to π_v (resp. π'_v), where $w = \omega_v(\varpi), w' = \omega'_v(\varpi)$. Write*

$$w = zw', \text{ with } z^3 = 1.$$

Then one of the following cases occurs (up to interchanging a and b):
- (1) $\{c, d\} = \{za, zb\}$; $\mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi'_v)$
- (2) $\{c, d\} = \{\mu za, \mu^{-1} zb\}$, $\mu^4 = 1$, $a^2 = \mu w$; $\mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi'_v)$
- (3) $\{c, d\} = \{\zeta za, \zeta^{-1} zb\}$, $\zeta^5 = 1$, $a^2 = \zeta w$

In particular, when ω is algebraic, π_v and π'_v are both algebraic in cases (2) and (3). Moreover, if ω has finite order, $\mathbb{Q}(\pi_v)$ is contained, in these cases, in a finite abelian extension K of $\mathbb{Q}$, independent of v, containing $\mathbb{Q}(\omega)$.

Remark 2.2 In section 3 we will show, by also appealing to (sI-3), which is not used in Lemma 2.1, that (i) π_v is algebraic in *all* cases, and (ii) $a^4 = 1$ in case (2) of part (B) above, if $w = 1$. It is perhaps useful to note that if (sI-3) is not satisfied by π, one may take $F = \mathbb{Q}$ and π' to be a cubic twist of π, and such a π is expected to be transcendental if it is generated, for example, by a Maass form φ of weight 0 relative to a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ acting on the upper half plane with Laplacian eigenvalue $\lambda > 1/4$.

Proof of Lemma 2.1 (A) The fact that π and π' have isomorphic symmetric cubes implies immediately that $\omega'^6 = \omega^6$, which is not sufficient for us. We will first show that π' is not solvable polyhedral. First, since π is not solvable polyhedral, $\mathrm{sym}^j(\pi)$ is cuspidal for $j \leq 4$ (cf. [**KS1**]). By (sI-2), $\mathrm{sym}^3(\pi')$ is also cuspidal. Suppose $\mathrm{sym}^4(\pi')$ is not cuspidal. Then, by the criterion of Kim and Shahidi ([**KS1**]), $\mathrm{sym}^3(\pi')$ must be monomial, which forces it to admit a non-trivial self-twist by a quadratic character ν, say. Then by (sI-2), $\mathrm{sym}^3(\pi)$ also admits a self-twist by ν, implying that $\mathrm{sym}^4(\pi)$ is not cuspidal, contradicting (sI-1). Hence π' satisfies both (sI-j) for $j \leq 2$.

Next we appeal to Kim's theorem ([**Kim1**]) giving the automorphy in $\mathrm{GL}(6)/F$ of the exterior square $\Lambda^2(\Pi)$ of any cusp form Π on $\mathrm{GL}(4)/F$. Applying this with $\Pi = \mathrm{sym}^3(\eta)$ for a cusp form η on $\mathrm{GL}(2)/F$ with central character ν, we obtain the (well known) isobaric decomposition

$$(2.3) \qquad \Lambda^2(\mathrm{sym}^3(\eta)) = \left(\mathrm{sym}^4(\eta) \otimes \nu\right) \boxplus \nu^3.$$

To prove this, we first note that by [**Kim1**], the L-functions agree at almost all places, and then appeal to the strong multiplicity one theorem for global isobaric representations, due to Jacquet and Shalika ([**JS**]). Now applying this to $\eta = \pi, \pi'$, we get by (sI-2), the following equivalence of isobaric sums:

$$(2.4) \qquad \left(\mathrm{sym}^4(\pi) \otimes \omega\right) \boxplus \omega^3 = \left(\mathrm{sym}^4(\pi') \otimes \omega'\right) \boxplus \omega'^3.$$

Since $\mathrm{sym}^4(\pi)$ and $\mathrm{sym}^4(\pi')$ are both cuspidal, we are forced to have

$$(\omega')^3 = \omega^3,$$

as claimed.

(B)   Preserving the notations used in part (B) of the Lemma, and noting that
$$b = wa^{-1},\, d = w'c^{-1},$$
we get by (sI-2), the equality of sets:

(2.5) $\{a^3, a^2b, ab^2, b^3\} \;=\; \{c^3, c^2d, cd^2, d^3\}.$

Clearly,

(2.6)
$$a^2b = wa,\; ab^2 = w^2a^{-1}, b^3 = w^3a^{-3}\;\text{ and }\; c^2d = w'c,\; cd^2 = w'^2c^{-1},\; d^3 = w'^3c^{-3}.$$

Note that, since $w' = z^2w$, we have

(2.7) $c = \alpha a \;\implies\; d = w'\alpha^{-1}a^{-1} = z^2\alpha^{-1}b,$

and

(2.8) $c = \beta b \;\implies\; d = w'\beta^{-1}b^{-1} = z^2\beta^{-1}a.$

A priori, a^3 has four possibilities to satisfy (2.5). However, by interchanging c and d, we are reduced to considering only *two main cases*:

Case **I:** $\mathbf{a^3 = c^3}$

In this case, since $\omega'^3 = \omega^3$ and $w' = z^2w$, (2.5) and (2.6) yield, after dividing by w,

(2.9) $\{a, wa^{-1}\} \;=\; \{-z^2c, zwc^{-1}\}.$

If $a = -z^2c$, then $c = za$ and $d = zb$ (by (2.7)), putting us in the situation (1). Since we have
$$\{c, d\} \;=\; \{a, b\} \cdot z,$$
π_v is (at most) a cubic twist of π'_v; it follows that $\mathrm{Ad}(\pi_v)$ and $\mathrm{Ad}(\pi'_v)$ are isomorphic.

So we may assume $a = zwc^{-1}$. Then we get (using (2.8))

(2.10) $c = zwa^{-1} = zb,\;\text{ and }\; d = za,$

again landing us in (1).

Case **II:** $\mathbf{a^3 = w'c}$

In this case, $c = w'^{-1}a^3$, and we obtain from (2.5) and (2.6),

(2.11) $\{wa, w^2a^{-1}\} \;=\; \{w^{-3}a^9, w^6a^{-9}\}.$

If $wa = w^6a^{-9}$, then $a^{10} = w^5$, so that

(2.12) $a^2 \;=\; \zeta w,\;\text{ with }\; \zeta^5 = 1.$

Hence $a^3 = \zeta wa$, and so

(2.13) $c = w'^{-1}\zeta wa \;=\; \zeta za.$

It follows that

(2.14) $d = w'c^{-1} = (z^2w)\zeta^{-1}z^{-1}a^{-1} \;=\; \zeta zb.$

Thus we are in situation (3).

It is left to consider when $wa = w^{-3}a^9$. In this case, $a^8 = w^4$, so that

(2.15) $a^2 \;=\; \mu w,\;\text{ with }\; \mu^4 = 1.$

Hence $a^3 = \mu wa$, and arguing as above, we deduce

(2.16) $$c = \mu za, \text{ and } d = \mu^{-1} zb.$$

This puts us in situation (2).

We still need to show that $\mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi_v')$ in this case as well. To this end, note that since $a^2 = \mu w$ and $c = \mu za$,

(2.17) $$c = (a^2 w^{-1})za = w^{-1}za^3.$$

On the other hand, raising $c = \mu za$ to the fourth power, we get, since $\mu^4 = 1$,

(2.18) $$c^4 = za^4 = za(a^3) = za(wz^{-1}c),$$

which in turn yields

(2.19) $$c^3 = wza.$$

Furthermore,

$$c^2 = w^{-2}z^2a^6 = w^{-2}z(za^4)a^2 = w^{-2}zc^4a^2,$$

implying

(2.20) $$c^{-2} = w^{-2}za^2, \text{ and } c^2 = w^2z^2a^{-2}.$$

The unramified representation $\mathrm{Ad}(\pi_v)$ is described by the unordered triple

(2.21) $$\{ab^{-1}, 1, a^{-1}b\} = \{w^{-1}a^2, 1, wa^{-2}\}.$$

Similarly, $\mathrm{Ad}(\pi_v')$ is given by the unordered triple

$$\{cd^{-1}, 1, c^{-1}d\} = \{zw^{-1}c^2, 1, z^{-1}wc^{-2}\},$$

which, by (2.20), is the same as

(2.22) $$\{wa^{-2}, 1, w^{-1}a^2\}.$$

The assertion follows.

$\square$

3. The nicety of $\mathrm{sym}^5(\pi)$ for π s-icosahedral

As mentioned in the Introduction, the condition (s-I3) implies that for an s-icosahedral π of central character ω, the automorphic representation $\mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2$ of $\mathrm{GL}_6(\mathbb{A}_F)$, whose existence is given by [**KS2**], represents $\mathrm{sym}^5(\pi)$ at all places outside a finite set S of places containing the archimedean and ramified places. In fact we have the following strengthening:

Proposition 3.1 *Let π be an s-icosahedral cusp form on $\mathrm{GL}(2)/F$ with central character ω.*

(a) *At every finite place v,*

(3.2) $$L(s, \pi_v; \mathrm{sym}^5) = L(s, \mathrm{Ad}(\pi_v') \boxtimes \pi_v \otimes \omega_v^2)$$

and

(3.3) $$\varepsilon(s, \pi_v; \mathrm{sym}^5) = \varepsilon(s, \mathrm{Ad}(\pi_v') \boxtimes \pi_v \otimes \omega_v^2)$$

(b) *If Σ_∞ denotes the set of archimedean places of F, we have*

(3.4) $$L(s, \pi_\infty; \mathrm{sym}^5) = L(s, \mathrm{Ad}(\pi_\infty') \boxtimes \pi_\infty \otimes \omega_\infty^2),$$

where π_∞ (resp. π_∞') denotes $\otimes_{v \in \Sigma_\infty} \pi_v$ (resp. $\otimes_{v \in \Sigma_\infty} \pi_v'$).

(c) *If π, π' are not dihedral twists of each other, i.e., if their symmetric squares are not twist equivalent, then $\mathrm{sym}^5(\pi)$, defined to be $\mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2$, is cuspidal, and hence $L(s, \pi; \mathrm{sym}^5)$ is entire in this case.*

Remark Of course we are dealing with a very special class of cusp forms π here. For the status of results in the general case (which is much more complicated), see [**Sha2**], [**Sha3**] and [**KS1**].

Proof. At any place v, we have the well known factorization (using Clebsch-Gordan):

$$(3.5) \qquad L(s, \mathrm{sym}^4(\pi_v) \times \pi_v) = L(s, \pi; \mathrm{sym}^5)L(s, \mathrm{sym}^3(\pi_v) \otimes \omega_v).$$

Similarly for the ε-factors, making use of the local Langlands conjecture for $\mathrm{GL}(n)$, now known by the works of Harris-Taylor [**HT**] and Henniart [**Hen1**]. So it holds for the γ-factors as well. We claim that one has, for a sufficiently ramified character ν_v of F_v^*,

$$(3.6) \qquad L(s, \pi; \mathrm{sym}^5 \otimes \nu_v) = 1.$$

Indeed, by the standard stability results for the Rankin-Selberg L-functions, we may choose a highly ramified character ν_v such that both $L(s, \mathrm{sym}^4(\pi_v) \times \pi_v \otimes \nu_v)$ and $L(s, \mathrm{sym}^3(\pi_v) \otimes \omega_v \nu_v)$ are both 1. One way to see this is to use the fact that the local Langlands correspondence preserves the L-functions of pairs, and make use of the well known result on the Galois side. The claim now follows, thanks to (3.5). We can deduce analogously the same statement for the γ-factors and ε-factors. For a representation theoretic proof of the stability of γ-factors (which yields what we want for the ε-factors because stability holds for the L-factors), see for example [**CPSS**], where a general situation is treated.

Consider the global L-functions $L_1(s) := L(s, \mathrm{sym}^4(\pi) \times \pi)$ and

$$(3.7) \qquad L_2(s) := L(s, \mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2)L(s, \mathrm{sym}^3(\pi) \otimes \omega).$$

They both have functional equations and analytic continuations, and moreover, thanks to (s-I3) and (3.5), they are the same local factors out side a finite set of places containing Σ_∞ and the ramified finite places. First choose a global character ν such that at each finite place in S, ν_v is sufficiently ramified so that the ν_v twists of $L_{1,v}(s)$ and $L_{2,v}(s)$ are both 1; we can do this by the (standard) argument above. Comparing functional equations, and using the fact that the archimedean factors of $L_j(s), j = 1, 2$, have no poles to the right of $\Re(s) > \frac{1}{2}$, we get

$$L_{1,\infty}(s) = L_{2,\infty}(s),$$

which furnishes (3.4).

Now pick any finite place v in S and choose a ν such that for every $u \in S, u \neq v$, ν_u is sufficiently ramified. Comparing the functional equations again, we get (3.2); here we use the invertibility of the ε-factors and the shape of the local factors, as well as the very weak Ramanujan at v which separates the poles of $L_{j,v}(s)$ from its dual. After that we may, in the same way, deduce (3.3) as well. We have now proved parts (a) and (b) of the Proposition.

Part (c) of the Proposition follows easily from the general cuspidality criterion ([**RW**]) for the Kim-Shahidi functorial transfer from $\mathrm{GL}(2) \times \mathrm{GL}(3)$ to $\mathrm{GL}(6)$. In fact, already by Theorem 1.2 of [**Wan**], given that π is not solvable polyhedral, which is our hypothesis, the only way $\mathrm{Ad}(\pi') \boxtimes \pi \otimes \omega^2$ can fail to be cuspidal is for

$\operatorname{sym}^2(\pi)$ and $\operatorname{sym}^2(\pi')$ to be twists of each other by a character. But this has been ruled out by the hypothesis, and we are done. $\qquad\square$

4. Proof of part (a) of Theorem A

Let π be as in Theorem A, attached to (π', ν), with ω algebraic. We will show the following:
(**4.1**)

- (i) At each place v, π_v is algebraic.
- (ii) The global representation π_f is rational over a finite extension of $\mathbb{Q}$ (containing $\mathbb{Q}(\omega)$.

To deduce (ii) from (i), we need to show in addition that $\mathbb{Q}(\pi_v)$ has degree bounded independent of the place v.

4.1. When π_v and π_v' are unramified. Let v be a finite place where π_v and π_v' are unramified. Preserving the notations of section 1, let us recall that the only case where we do not yet know the algebraicity of π_v is case (1) of Lemma 2.1, part (B). We treat this case now, making use of (sI-3).

Lemma 4.1.1 *Let v be a finite place where π_v is unramified. Suppose we have $\operatorname{Ad}(\pi_v)) \simeq \operatorname{Ad}(\pi_v')$, which is satisfied in cases (1), (2) of Lemma 2.1, part (B). Then (sI-3) implies that one of the following identities holds:*

$$a^2 = w, \ a^4 = w^2, \ a^6 = w^3.$$

In particular, π_v is algebraic if ω_v is algebraic. Moreover, if $w = 1$, then $a^m = 1$ with $m \in \{4, 6\}$.

Remark 4.1.2 The reason for also including here the case (2) of Lemma 2.1, part (B), is the following. We knew earlier that in this case, $a^2 = \mu w$, with $\mu^4 = 1$, so that when $w = 1$, $a^4 = \pm 1$. But now, using Lemma 4.1.1 in addition, we rule out the (potentially troublesome) possibility $a^4 = -1$ (when $w = 1$), which will be important to us later.

Proof of Lemma 4.1.1. Since $\operatorname{Ad}(\pi_v')$ is by assumption (in this Lemma) isomorphic to $\operatorname{Ad}(\pi_v)$, it is described by the triple $\{w^{-1}a^2, 1, wa^{-2}\}$. So $\operatorname{Ad}(\pi_v') \otimes \omega_v^2$ is associated to $\{wa^2, w^2, w^3a^{-2}\}$. Thus $\operatorname{Ad}(\pi_v') \otimes \pi_v \otimes \omega_v^2$ is given by the sextuple

$$\{wa^3, w^2a, w^2a, w^3a^{-1}, w^3a^{-1}, w^4a^{-3}\}.$$

On the other hand, $\operatorname{sym}^5(\pi_v)$ is attached to the sextuple

$$\{a^5, wa^3, w^2a, w^3a^{-1}, w^4a^{-3}, w^5a^{-5}\}.$$

Comparing these two tuples, we get

$$\{wa^3, w^2a, w^3a^{-1}, w^4a^{-3}\} = \{a^5, wa^3, w^4a^{-3}, w^5a^{-5}\}.$$

Looking at the possibilities for w^2a, we see that *either* $a^4 = w^2$ (which subsumes both the possibilities $w^2a = a^5$ and $w^2a = w^4a^{-3}$), *or* $a^2 = w$ (when $w^2a = wa^3$), *or* $a^6 = w^3$ (when $w^2a = w^5a^{-5}$).

$\qquad\square$

Thanks to the Lemma, we see that at all the unramified places v, $\mathbb{Q}(\pi_v)$ is contained in a finite solvable extension of $\mathbb{Q}(\omega)$, with its degree over $\mathbb{Q}(\omega)$ bounded

independent of v (see Lemma 2.1 for cases (2) through (4). And since ω is by hypothesis (of Theorem A) algebraic, i.e., of type A_0, $\mathbb{Q}(\omega)$ is a number field; in fact it is a CM field or totally real. Consequently, we have proved the following:

Lemma 4.1.3 *Let π be as in Theorem A, and let S be the union of the archimedean places and the finite places where π_v is ramified. Let π^S denote (as usual) the restricted tensor product of the local components π_v as v runs over all the places outside S. Then π^S is algebraic, rational over a finite extension K of $\mathbb{Q}(\omega)$. In fact, K is contained in a compositum of cyclotomic and Kummer extensions over $\mathbb{Q}(\omega)$.*

4.2. Proof of non-occurrence of π_v of Steinberg type. Suppose π_v is a special representation. Then there is a character λ of F_v^* such that the associated 2-dimensional representation σ_v of the extended Weil group $W_{F_v}' = W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$ is of the form $\lambda \otimes st$ where st denotes the standard representation of $\mathrm{SL}(2, \mathbb{C})$. From (sI-2), it follows that π_v' is also necessarily special, with its associated 2-dimensional representation σ_v' of W_{F_v}' being of the form $\lambda' \otimes st$. Then $\mathrm{Ad}(\pi_v')$ has parameter $\mathrm{Ad}(\sigma_v') \simeq 1 \otimes \mathrm{sym}^2(st)$, so that $\mathrm{Ad}(\pi_v') \otimes \pi_v \otimes \omega^2$ has the parameter

$$(4.2.1) \qquad\qquad \lambda^3 \otimes \left(\mathrm{sym}^3(st) \oplus st\right).$$

On the other hand, $\mathrm{sym}^5(\pi_v)$ has the parameter

$$(4.2.2) \qquad\qquad \lambda^5 \otimes \mathrm{sym}^5(st).$$

But the representations (4.2.1) and (4.2.2) cannot be isomorphic, contradicting (sI-3). Thus π_v cannot be of Steinberg type. $\qquad\qquad\square$

4.3. When π_v is unramified, but π_v' is ramified. Thanks to (sI-2) and (sI-3), $\mathrm{sym}^3(\pi_v')$ and $\mathrm{Ad}(\pi_v')$ must both be unramified in this case. Suppose π_v' is a principal series representation. So we may write

$$\pi_v' \simeq \mu_1 \boxplus \mu_2, \ \omega' = \mu_1 \mu_2,$$

with $\mu_1^3, \mu_2^3, \mu_1^2 \mu_2, \mu_1 \mu_2^2, \mu_1 \mu_2^{-1}$ unramified. Then μ_1 and μ_2 must themselves be unramified, forcing π_v' to be in the unramified principal series.

The only remaining possibility is for π_v' to be a supercuspidal representation with (irreducible) parameter $\sigma_v = \mathrm{Ind}_E^{F_v}(\chi)$, for a character χ of the multiplicative group of a quadratic extension E of F_v. If θ denotes the non-trivial automorphism of E/F_v, then recall that the irreducibility of σ_v implies that $\chi \circ \theta \neq \chi$. We have

$$\mathrm{sym}^2(\sigma_v) \simeq \mathrm{Ind}_E^{F_v}(\chi^2) \oplus \chi_0,$$

where χ_0 is the restriction of χ to F_v^*. The determinant of σ_v, which corresponds to ω_v', is then $\chi_0 \nu$, where ν is the quadratic character of W_{F_v} corresponding to E/F_v. Since $\mathrm{Ad}(\pi_v') = \mathrm{sym}^2(\sigma_v) \otimes (\chi_0 \nu)^{-1}$ is unramified, we must have ν, i.e., E/F_v, unramified, and

$$\chi^2 \circ \theta = \chi^2,$$

which implies

$$\chi = \lambda(\mu \circ N_{E/F_v}), \quad \text{with} \quad \lambda^2 = 1, (\lambda \circ \theta) \neq \lambda,$$

with μ a character of F_v^*. Then $\chi_0 = \lambda_0 \mu^2$ and

$$\mathrm{Ad}(\sigma_v) \simeq \lambda_0 \oplus \lambda_0 \nu \oplus \nu.$$

Moreover, λ_0 must be unramified as well. If $\lambda_0 = 1$, then $\mathrm{Ad}(\sigma_v)$ contains the trivial representation $\mathbf{1}$, and therefore

$$\dim_{\mathbb{C}} \left(\mathrm{Hom}_{W_{F_v}} (\mathbf{1}, \mathrm{End}(\sigma_v)) \right) = 2,$$

since $\mathrm{End}(\sigma_v) \simeq \mathrm{Ad}(\sigma_v) \oplus \mathbf{1}$. This contradicts, by Schur's Lemma, the irreducibility of σ_v. So we must have $\lambda_0 \neq 1$. But then, as λ_0 is an unramified quadratic character, it must coincide with ν, making $\lambda_0 \nu = 1$. Again, we get $\mathbf{1} \subset \mathrm{Ad}(\sigma_v)$, leading to a contradiction.

So we conclude that π_v' must be unramified when π_v is.

4.4. When π_v is ramified. Suppose π_v is a ramified principal series representation attached to the characters μ_1, μ_2 of F_v^*, with $\omega_v = \mu_1 \mu_2$. Then π_v' is also necessarily a ramified principal series representation, attached to characters μ_1', μ_2', with $\omega_v' = \mu_1' \mu_2'$. The criteria (sI-1) and (sI-2) give conditions relating various powers of these characters and of ν_v. The situation is similar to the unramified case, and we conclude algebraicity as before. Again, $\mathbb{Q}(\pi_v)$ is a finite extension of Kummer type over $\mathbb{Q}(\omega)$.

It remains to consider the case when π_v is supercuspidal, in which case the constraints force π_v' to also be supercuspidal. It is well known that a supercuspidal representation with algebraic central character ω_v is algebraic, in fact rational over $\mathbb{Q}(\omega_v)$. Indeed, as mentioned in the Introduction, we are using the algebraic parametrization ([**Hen2**]), which is equivariant for the action of $\mathrm{Aut}(\mathbb{C})$. So it suffices to verify this for the parameter σ_v of π_v. Since we are over a local field, the image G_v of σ_v is necessarily solvable, and in particular, the image of G_v in $\mathrm{PGL}(2, \mathbb{C})$ must be dihedral, tetrahedral, or octahedral. The assertion about rationality follows from the known results on the irreducible representations of coverings of D_{2n}, A_4 and S_4.

So we have now proved part (a) of Theorem A. $\square$

5. Proof of part (b) of Theorem A

Here we are assuming that ω is of finite order. Then so is ω', and the arguments of sections 1 and 2 imply immediately that $\mathbb{Q}(\pi_f)$ is a cyclotomic field. To be precise, this is clear outside places v where π_v is not square-integrable, and the assertion in the supercuspidal case holds because it is rational over $\mathbb{Q}(\omega)$, which is cyclotomic.

We need to show that at any archimedean place v, π_v is of Galois type. Let σ_v, resp. σ_v', denote the 2-dimensional representation of the Weil group W_{F_v} associated to π_v, resp. π_v'. First suppose that the restriction of σ_v to $\mathbb{C}^*$ is a one-dimensional twist of $(z/|z|)^m \oplus (z/|z|)^{-m}$ with $m > 0$; this happens when either v is real and π_v is a discrete series representation (of lowest weight m), or v is complex and π_v is the base change of a discrete series representation of $\mathrm{GL}(2, F_{\mathbb{R}})$. In either case, (sI-2) implies that π_v is also of the same form. But then we see that the restriction to $\mathbb{C}^*$ of the parameter of $\mathrm{Ad}(\pi_v') \otimes \pi_v$ is a one-dimensional twist of

$$\left((z/|z|)^{2m} \oplus 1 \oplus (z/|z|)^{-2m} \right) \otimes \left((z/|z|)^m \oplus (z/|z|)^{-m} \right),$$

which is

$$(z/|z|)^{3m} \oplus (z/|z|)^m \oplus \left((z/|z|)^m \oplus (z/|z|)^{-m} \oplus (z/|z|)^{-m} \oplus (z/|z|)^{-3m} \right).$$

Since $m \neq 0$, this representation cannot possibly be a one-dimensional twist of the restriction to $\mathbb{C}^*$ of $\mathrm{sym}^5(\sigma_v)$, which is evidently multiplicity-free. Note that we would have no contradiction if we had allowed $m = 0$, which corresponds to the Galois type situation.

Thus we may assume, by the classification of unitary representations, that π_v is a unitary character twist of a spherical representation $\sigma_v^0 = \mu_1 \otimes \mu_2$, where μ_1, μ_2 are unramified characters of F_v^*. Then, since ω_v has finite order by assumption, the restriction of σ_v^0 to $\mathbb{C}^*$ is necessarily of the form $|\cdot|^s \oplus |\cdot|^{-s}$. Applying (sI-2), we see that the restriction of σ_v' to $\mathbb{C}^*$ will also need to be a unitary character twist of $\sigma_v^0|_{\mathbb{C}^*}$. Then $(\mathrm{Ad}(\sigma_v') \otimes \sigma_v)|_{\mathbb{C}^*}$ cannot, unlike $\mathrm{sym}^5(\sigma_v)|_{\mathbb{C}^*}$, contain a unitary character twist of $|\cdot|^{5s}$, unless $s = 0$. So we get a contradiction to (sI-3) if $s \neq 0$. Hence s must be zero, and since ω_v has finite order, σ_v is a finite order twist of a σ_v^0 whose restriction to $\mathbb{C}^*$ is $1 \oplus 1$. In other words, π_v is of Galois type.

$\square$

6. Temperedness of π

It suffices to prove temperedness at each place v. When v is archimedean, we have already shown that π_v is even of Galois type, so we may assume that v is finite. Suppose π_v is a principal series representation with parameter σ_v. If π_v is non-tempered, i.e., violates the Ramanujan hypothesis, we must have, since $\pi_v^\vee \simeq \overline{\pi}_v$,

$$\sigma_v \simeq \lambda_v \otimes \left(|\cdot|^t \oplus |\cdot|^{-s} \right), \quad \text{with} \quad \lambda_v^2 = \omega_v, \, t \in \mathbb{R}_+^*.$$

As ω_v has finite order, λ_v does as well. As in the archimedean spherical case, (sI-2) implies that the parameter of π_v' is necessarily of the form

$$\sigma_v' \simeq \lambda_v' \otimes \left(|\cdot|^t \oplus |\cdot|^{-s} \right), \quad \text{with} \quad {\lambda_v'}^2 = \omega_v', \, t \in \mathbb{R}_+^*,$$

for the same t; here we have used the fact that ν_v has finite order. It follows (since $t > 0$) that

$$\mathrm{Hom}\left(|\cdot|^{5t}, \mathrm{Ad}(\sigma_v') \otimes \sigma_v \otimes \omega_v^2 \right) = 0,$$

while

$$\mathrm{Hom}\left(|\cdot|^{5t}, \mathrm{sym}^5(\sigma_v) \right) \neq 0.$$

This contradicts (sI-3) and so t must be zero. In other words, π_v must be tempered.

Finally, as is well known, if π_v is a discrete series representation with unitary central character, it is necessarily tempered.

This proves part (c) of Theorem A.

$\square$

7. Proof of part (d) of Theorem A

Proposition 7.1 *Let π, π' be as in Theorem A, with $\omega = \omega' = 1$. Then at any finite place v, π_v is either a unitary principal series or a supercuspidal representation, with π_v' of the same type, and furthermore,*

$$\mathbb{Q}(\pi_v) \subset \mathbb{Q}[\sqrt{5}].$$

More precisely, we have

(a) *When π_v, π_v' are in the unitary principal series,*

(ai) $\mathbb{Q}(\pi_v) = \mathbb{Q}$, *if* $\mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi_v')$;

(aii) $\mathbb{Q}(\pi_v) = \mathbb{Q}[\sqrt{5}]$, *if* $\mathrm{Ad}(\pi_v) \not\simeq \mathrm{Ad}(\pi_v')$.

(b) *When π_v is supercuspidal,* $\mathbb{Q}(\pi_v) = \mathbb{Q}$.

Corollary 7.2 *Let π, π' be as above (in Proposition 7.1). Then we have $\mathbb{Q}(\pi_f) \subset \mathbb{Q}[\sqrt{5}]$, and moreover,*

$$\mathbb{Q}(\pi_f) = \mathbb{Q}[\sqrt{5}] \Leftrightarrow \pi \not\simeq \pi'.$$

Proposition 7.1 $\implies$ **Corollary 7.2**:

Since $\mathbb{Q}(\pi_f)$ is the compositum of all the $\mathbb{Q}(\pi_v)$ as v runs over finite places (cf. [**Clo**], Prop. 3.1, for example), we have

(7.2) $$\mathbb{Q}(\pi_f) \subset \mathbb{Q}[\sqrt{5}].$$

First suppose $\mathrm{Ad}(\pi_f) \not\simeq \mathrm{Ad}(\pi_f')$. Then at some finite place u, say, $\mathrm{Ad}(\pi_u)$ and $\mathrm{Ad}(\pi_u')$ need to be non-isomorphic, forcing, by (aii) of Prop. 7.1,

$$\mathbb{Q}(\pi_f) = \mathbb{Q}(\pi_u) = \mathbb{Q}[\sqrt{5}].$$

On the other hand, if $\mathrm{Ad}(\pi_f)$ and $\mathrm{Ad}(\pi_f')$ are isomorphic, their local components are isomorphic as well, and (ai) of Prop. 7.1 then implies that

$$\mathbb{Q}(\pi_f) = \mathbb{Q}.$$

Now we *claim* that for our π, π',

(7.3) $$\mathrm{Ad}(\pi_f) \simeq \mathrm{Ad}(\pi_f') \implies \pi_f \simeq \pi_f'.$$

Indeed, by the *multiplicity one theorem for $SL(2)$* (cf. [**Ram1**]), we have

(7.4) $$\pi_f' \simeq \pi_f \otimes \nu_f,$$

for some idèle class character ν of F. Since $\omega = \omega' = 1$, we must have

$$\nu^2 = 1.$$

Anyhow, (7.4) implies

$$\mathrm{sym}^3(\pi_f') \simeq \mathrm{sym}^3(\pi_f) \otimes \nu_f^3,$$

Since π and π' have, by (sI-2), isomorphic symmetric cubes, we must then have

$$\nu^3 = 1,$$

or else, $\mathrm{sym}^3(\pi)$ will need to admit a non-trivial self-twist, which is not possible as π is not solvable polyhedral. So it follows that $\nu = 1$, *proving the claim.*

This also finishes the proof of the Corollary assuming Prop. 7.1.

$\square$

Proof of Proposition 7.1

Since we are assuming here that $\omega = \omega' = 1$, we first claim that at the places v where π_v is supercuspidal, $\mathbb{Q}(\pi_v)$ is just $\mathbb{Q}$. Indeed, as we are using the algebraic parametrization, we can transfer the problem to the field of rationality of the associated irreducible 2-dimensional representation σ_v of W_{F_v}. This is a known fact, and we briefly sketch an argument. Let θ be an automorphism of $\mathbb{C}$. It suffices to check that $\varepsilon(s, \pi_v^\theta \otimes \chi^\theta)$ equals $\varepsilon(s, \sigma_v^\theta \otimes \chi^\theta)$, for all characters χ; we are identifying, by class field theory, the characters χ of F_v^* with the corresponding ones of W_{F_v}. (We will also identify the central character ω_v with $\det(\sigma_v)$.) If we look at the root number $W(\sigma_v \otimes \chi) = \varepsilon(1/2, \sigma_v \otimes \chi)$, then the ratio $W(\sigma_v^\theta \otimes \chi^\theta)/W(\omega_v^\tau \otimes \chi^\theta)$

equals $(W(\sigma_v \otimes \chi)/W(\omega_v \otimes \chi))^\tau$. An analogous property holds on the GL(2) side, as seen by the integral representation of the local zeta functions, using the Whittaker model, which is compatible with the θ-action ([**Wal**]. The conductors also correspond, and one gets the Galois equivariance of $\pi_v \mapsto \sigma_v$. Next, note that we may write σ_v as an unramified abelian twist of an irreducible 2-dimensional representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ which will have solvable, finite image. As we saw at the end of section 3, such a σ_v is rational over $\mathbb{Q}(\omega_v)$. Hence the claim.

Moreover, we know that π_v will not be of Steinberg type, nor non-tempered. So let us focus on the finite places v where π_v is in the unitary principal series.

First let v be an unramified place for π. Then $w = w' = z = \delta = 1$ by hypothesis, and combining Lemma 2.1 with Proposition 4.1.1, we get the following:

Lemma 7.5 *When π_v, π'_v are unramified with $\omega_v = \omega'_v = 1$, one of the following happens (up to exchanging a with $b = a^{-1}$ and c with $d = c^{-1}$):*

   (i) $a^m = 1$, *for* $m \in \{4, 6\}$, *and* $c = \pm a$; $\mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi'_v)$
   (ii) $c = a^3, a = c^{-3}, a^{10} = 1$; $\mathrm{Ad}(\pi_v) \not\simeq \mathrm{Ad}(\pi'_v)$

In the first case, $\mathrm{Ad}(\pi'_v)$ and $\mathrm{Ad}(\pi_v)$ are isomorphic, and $\mathbb{Q}(\pi_v)$ is contained in either $\mathbb{Q}(\zeta_4)$ or $\mathbb{Q}(\zeta_6)$, where by ζ_n we mean a primitive n-th root of unity in $\mathbb{C}$. In case (ii), it is clear that $\mathbb{Q}(\pi_v)$ is contained in $\mathbb{Q}(\zeta_{10})$. On the other hand, since π_v is selfdual, the trace and determinant of the conjugacy class of π_v are both real, and so π_f is rational over the real subfield of $\mathbb{Q}(\zeta_n)$ for appropriate n. In other words, when π_v, π'_v are unramified with trivial central character,

$$\mathbb{Q}(\pi_v) = \mathbb{Q} \quad \text{when} \quad \mathrm{Ad}(\pi_v) \simeq \mathrm{Ad}(\pi'_v),$$

and

$$\mathbb{Q}(\pi_v) = \mathbb{Q}(\sqrt{5}) \quad \text{when} \quad \mathrm{Ad}(\pi_v) \not\simeq \mathrm{Ad}(\pi'_v),$$

It remains to consider the case when π_v is a ramified, tempered principal series representation with $\omega_v = 1$. Then its parameter decomposes as

$$\sigma_v \simeq \mu \oplus \mu^{-1},$$

for a ramified character μ of F_v^*. We have, thanks to the algebraic normalization of parameters,

$$\mathbb{Q}(\pi_v) = \mathbb{Q}(\mu)^+,$$

where the right hand side is the totally real subfield of the cyclotomic field $\mathbb{Q}(\mu)$. Again, the reason is that $\mathbb{Q}(\pi_v)$ is *a priori* contained in $\mathbb{Q}(\mu)$, and the selfduality of π_v makes it rational over the real subfield.

When π_v is such a ramified principal series representation, π'_v is also forced to be of similar type, thanks to (sI-2), with parameter $\sigma'_v = \mu' \oplus \mu'^{-1}$. Furthermore, we need to have

$$\mu^3 \oplus \mu \oplus \mu^{-1} \oplus \mu^{-3} \simeq \mu'^3 \oplus \mu' \oplus \mu'^{-1} \oplus \mu'^{-3}$$

This implies that either $\pi_v \simeq \pi'_v$ or, up to interchanging the roles of μ' and μ'^{-1}, $\mu' = \mu^3$. In the former case, arguing as in the unramified case above, we deduce that either $\mu^4 = 1$ or $\mu^6 = 1$. So $\mathbb{Q}(\pi_v) = \mathbb{Q}$ in this case. So we may assume that $\mu' = \mu^3$. Then (sI-3) yields

$$\mu^5 \oplus \mu^3 \oplus \mu \oplus \mu^{-1} \oplus \mu^{-3} \oplus \mu^{-5} \simeq \left(\mu^6 \oplus 1 \oplus \mu^{-6}\right) \otimes \left(\mu \oplus \mu^{-1}\right).$$

Hence

$$\mu^3 \oplus \mu^{-3} \simeq \mu^7 \oplus \mu^{-7}.$$

In other words, we have

$$\mu^4 = 1 \quad \text{or} \quad \mu^{10} = 1.$$

Thus $\mathbb{Q}(\mu)$ is generated over $\mathbb{Q}$ by an m-th root of unity ζ_m with $m \in \{4, 10\}$. Consequently, since $\mathbb{Q}(\zeta_4)^+ = \mathbb{Q}$ and $\mathbb{Q}(\zeta_{10})^+ = \mathbb{Q}(\sqrt{5})$,

$$\mathbb{Q}(\pi_v) = \mathbb{Q}(\zeta_4)^+ = \mathbb{Q} \quad \text{or} \quad \mathbb{Q}(\pi_v) = \mathbb{Q}(\sqrt{5}).$$

This finishes the proof of Proposition 7.1.

$$\square$$

Let τ be the non-trivial automorphism of $\mathbb{Q}(\sqrt{5})$. At each finite v, let π_v^τ denote the τ-conjugate representation of π_v, which makes sense because $\mathbb{Q}(\pi_v) \subset \mathbb{Q}(\sqrt{5})$. Similarly, let π_f^τ denote the τ-conjugate of π_f. Then π_f^τ is an admissible irreducible, generic representation because π_f has those properties. However, it is not at all clear if π_f^τ is automorphic. In the present case, we can deduce the following:

Proposition 7.6 *Let π be an s-icosahedral cusp form (with trivial central character) on $GL(2)/F$ relative to π', also of trivial central character. Then we have*

(i) *π' is unique, satisfying*

$$\pi_f^\tau \simeq \pi_f'.$$

(ii) *π_f is not rational over $\mathbb{Q}$.*

Proof.   We will first prove (ii) by contradiction. Suppose π_f is rational over $\mathbb{Q}$. Then by Corollary 7.2, we must have $\pi_f' \simeq \pi_f$. Hence

$$\text{sym}^2(\pi_f') \otimes \pi_f \simeq \text{sym}^3(\pi_f) \boxplus \pi_f.$$

This means that (by (sI-3),

$$L(s, \pi_f; \text{sym}^5) = L(s, \text{sym}^3(\pi_f))L(s, \pi_f)$$

On the other hand, by Clebsch-Gordan,

$$L(s, \text{sym}^4(\pi_f) \times \pi_f) = L(s, \pi_f; \text{sym}^5)L(s, \text{sym}^3(\pi_f))$$

So

$$L(s, \text{sym}^4(\pi_f) \times \pi_f) = L(s, \text{sym}^3(\pi_f))^2 L(s, \pi_f).$$

This leads to the identity

$$L(s, \text{sym}^4(\pi_f) \times (\pi_f \boxtimes \pi_f)) = L(s, \text{sym}^3(\pi_f) \times \pi_f)L(s, \pi_f \boxtimes \pi_f).$$

The rightmost L-function has a pole at $s = 1$ since π is selfdual, and one knows by Shahidi that $L(s, \text{sym}^3(\pi_f) \times \pi_f)$ does not vanish at $s = 1$. It then forces the left hand side L-function to have a pole at $s = 1$. But this can't be, because $\text{sym}^4(\pi)$ is, thanks to the condition (sI-1), a cusp form on $GL(5)/F$ (see [**KS1**]) and $\pi \boxtimes \pi$ is an automorphic form on $GL(4)/F$. This contradiction proves that π_f cannot be rational over $\mathbb{Q}$.

To deduce (i), it suffices to prove, locally at each finite place v, that

$$\pi_v' \simeq \pi_v^\tau.$$

Let v be a finite place where π_v is in the principal series with parameter $\sigma_v = \mu \oplus \mu^{-1}$, where μ may or may not be ramified. Then π_v' will also be of the same

form, say with parameter $\sigma'_v = \mu' \oplus {\mu'}^{-1}$. Thanks to (sI-2), we must have, up to permuting μ' and ${\mu'}^{-1}$,

$$\mu' = \mu^m, \quad \text{where} \quad m \in \{1, 3\}.$$

When $\mu' = \mu$, $\pi'_v \simeq \pi_v$, and as we have seen above, $\mu^m = 1$ with $m \in \{4, 6\}$, implying that $\mathbb{Q}(\pi_v) = \mathbb{Q}$. So let us assume that $\mu' = \mu^3$. Then (again as above) $\mu^4 = 1$ or $\mu^10 = 1$. In the former case, we get $\pi'_v \simeq \pi_v$, while in the latter case, either $\mu^2 = 1$, again implying $\pi'_v \simeq \pi_v$, or else, $\mu^{10} = 1$, but $\mu^2 \neq 1$. Let $\tilde{\tau}$ be the automorphism of $M := \mathbb{Q}(\zeta_{10})$ given by $\zeta_{10} \mapsto \zeta_{10}^3$. The totally real subfield of M is $K = \mathbb{Q}(\sqrt{5})$ and $\tilde{\tau}$ restricts to the non-trivial automorphism τ of K. It follows that

$$\pi'_v \simeq \pi_v^\tau,$$

which holds at every principal series place v. In fact it holds at every finite place because at the supercuspidal places, $\mathbb{Q}(\pi_v) = \mathbb{Q}$ due to π_v having trivial central character. Consequently, $\pi'_v \simeq \pi_v^\tau$ at each finite place v. Moreover, π' is unique by the strong multiplicity one theorem.

This proves Proposition 7.6.

$\square$

We claim that π' is s-icosahedral with $(\pi')' \simeq \pi$. Indeed, (sI-2) is a reflexive condition, and it also proves that π' is not dihedral or tetrahedral. We get (sI-3) (for π') by applying τ (to the (sI-3) for π) and making use of π'_f being π_f^τ. Furthermore, suppose $\text{sym}^4(\pi')$ is Eisensteinian. Then by [**KS1**], $\text{sym}^3(\pi')$ must admit a quadratic self-twist. Then so must $\text{sym}^3(\pi)$ by (sI-2), resulting in the non-cuspidality of $\text{sym}^4(\pi)$. But this contradicts (sI-1), and the claim follows.

Finally, since π' is s-icosahedral, it also follows, by the reasoning we employed for π, that π' is also of Galois type at infinity.

This proves part (d) of Theorem A.

$\square$

Remark 7.7 It should be noted that as π is of Galois type at infinity with trivial central character, it cannot correspond, for F totally real, to a holomorphic Hilbert modular form. So there is no reason at all, given the state of our current knowledge, to assert that its Galois conjugate π^τ should be modular (though it is expected). However, in our special case, it is automorphic because it happens to be isomorphic to the shadow cusp form π'.

References

[Clo]   L. Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 77–159. Academic Press, Boston, MA, 1990.

[CPSS] J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi. Stability of γ-factors for quasi-split groups. *J. Inst. Math. Jussieu* **7** (2008), 27–66.

[GJ]   S. Gelbart and H. Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)* **11** (1978), 471–542.

[HT]   M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[Hen1] G. Henniart. Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. *Invent. Math.* **139** (2000), 439–455.

[Hen2]   G. Henniart. Une caractérisation de la correspondance de Langlands locale pour GL(n). *Bull. Soc. Math. France* **130** (2002), 587–602.

[JPSS]   H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.* **105** (1983), 367–464.

[JS]   H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic forms. II. *Amer. J. Math.* **103** (1981), 777–815.

[Kim1]   H. H. Kim. Functoriality for the exterior square of GL$_4$ and the symmetric fourth of GL$_2$. *J. Amer. Math. Soc.* **16** (2003), 139–183 (electronic). With Appendix 1 by Dinakar Ramakrishnan and Appendix 2 by Kim and Peter Sarnak.

[Kim2]   H. H. Kim. An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type. *Invent. Math.* **156** (2004), 495–502.

[KS1]   H. H. Kim and F. Shahidi. Cuspidality of symmetric powers with applications. *Duke Math. J.* **112** (2002), 177–197.

[KS2]   H. H. Kim and F. Shahidi. Functorial products for GL$_2$ × GL$_3$ and the symmetric cube for GL$_2$. *Ann. of Math. (2)* **155** (2002), 837–893. With an appendix by Colin J. Bushnell and Guy Henniart.

[Ram1]   D. Ramakrishnan. Modularity of the Rankin-Selberg *L*-series, and multiplicity one for SL(2). *Ann. of Math. (2)* **152** (2000), 45–111.

[Ram2]   D. Ramakrishnan. Remarks on the symmetric powers of cusp forms on GL(2). In *Automorphic Forms and L-functions I: Global Aspects, Proceedings of a Workshop in honor of Steve Gelbert*, pages 237–256. IMCP/CONM, AMS Series, 2009.

[RW]   D. Ramakrishnan and S. Wang. A cuspidality criterion for the functorial product on GL(2) × GL(3) with a cohomological application. *Int. Math. Res. Not.* (2004), 1355–1394.

[Sha1]   F. Shahidi. On certain *L*-functions. *Amer. J. Math.* **103** (1981), 297–355.

[Sha2]   F. Shahidi. On the Ramanujan conjecture and finiteness of poles for certain *L*-functions. *Ann. of Math. (2)* **127** (1988), 547–584.

[Sha3]   F. Shahidi. Symmetric power *L*-functions for GL(2). In *Elliptic curves and related topics*, volume 4 of *CRM Proc. Lecture Notes*, pages 159–182. Amer. Math. Soc., Providence, RI, 1994.

[Wal]   J.-L. Waldspurger. Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2). *Compositio Math.* **54** (1985), 121–171.

[Wan]   S. Wang. On the symmetric powers of cusp forms on GL(2) of icosahedral type. *Int. Math. Res. Not.* (2003), 2373–2390.

253-37 CALTECH, PASADENA, CA 91125, USA
E-mail address: dinakar@caltech.edu