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AN EXERCISE CONCERNING THE SELFDUAL CUSP FORMS ON $GL(3)$

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ABSTRACT. Using L-functions and various known results, we provide a proof of the following:
Let F be a number field and Π a cuspidal automorphic form on $\mathrm{GL}(3)/F$ which is selfdual. Then, up to replacing Π by a quadratic twist, it can be realized as the adjoint of a cusp form π on $\mathrm{GL}(2)/F$, with π unramified at any prime where Π is. We also investigate the properties of π when Π is regular and algebraic.

Key words : Selfdual representations; automorphic forms; symmetric square; adjoint

Introduction

The object of this Note is to supply a proof of the following result, which is in the folklore, and deduce a Corollary. There is no pretension to anything creative here, and all that is involved is a synthesis of results due to various people. It is an exercise as the title indicates, but a non-trivial one as some of the needed facts are not easily found and one has to resort to some stop-gap arguments. The approach here is via L-functions.

Theorem A *Let F be a number field, and Π a cuspidal, selfdual automorphic representation of $GL_3(\mathbb{A}_F)$. Then there exists a non-dihedral cusp form π on $GL(2)/F$, and an idele class character ν of F with $\nu^2 = 1$, such that*

$$\Pi \simeq Ad(\pi) \otimes \nu. \tag{1}$$

The form π is unique up to a character twist, while ν is simply the central character of Π. The central character ω of π may be chosen to be unitary. Moreover, we may choose π such that, *for any finite place v, π_v is unramified, resp. an unramified twist of Steinberg, when $\Pi_v \otimes \nu_v$ is unramified, resp. an unramified twist of Steinberg.*

Here $Ad(\pi)$ denotes the Adjoint of π, a selfdual automorphic form on $\mathrm{GL}(3)/F$, defined to be $\mathrm{sym}^2(\pi)\otimes\omega^{-1}$, where $\mathrm{sym}^2(\pi)$ is the symmetric square of π, defined by Gelbart and Jacquet in [GeJ]. As π is non-dihedral, $\mathrm{Ad}(\pi)$ is cuspidal. Note that Theorem A remains valid for any cusp form Π on $\mathrm{GL}(3)/F$ which satisfies $\Pi^\vee \simeq \Pi \otimes |\cdot|^t$ for some t, the reason being that we may replace Π by $\Pi \otimes |\cdot|^{t/2}$, which is selfdual.

Corollary B *Let F, Π be as in Theorem, with associated (π, ν). Then at any archimedean place w, Π_w has a regular parameter iff π_w does. If Π is algebraic, then π can be chosen to be of type A. Moreover, when Π is regular and algebraic, hence cohomological, π can be chosen to have that property.*

Theorem A has been known to experts for a while. It is a consequence of a comparison of the stable trace formula for $\mathrm{SL}(2)/F$ with the twisted trace formula for $\mathrm{PGL}(3)/F$ (relative to transpose inverse); this fundamental idea of Langlands has been carried out in detail by Flicker in a series of papers, establishing (1). It is also a special case of Arthur's recent major work relating selfdual automorphic representations of $\mathrm{GL}(n)$ with those of suitable classical groups, again comparing appropriate trace formulae. In this Note we deduce Theorem A in a different way, via L-functions, by appealing to the backwards lifting (descent) of Ginzburg, Rallis and Soudry, as well as the forward transfer, for generic cusp forms, from odd orthogonal groups to $\mathrm{GL}(n)$, due to Cogdell, Kim, Piatetski-Shapiro, and Shahidi.

This Note has been around as a preprint for some years, and in the meanwhile has found a bit of use elsewhere, e.g., in the recent work of F. Calegari on *even* Galois representations, and we thank Venkataramana for suggesting that it be published in this special issue. We would like to acknowledge some support from the NSF, and thank the referee for making helpful comments on an earlier version which led to an improvement of the exposition.

1 Proof of Theorem A - Part I

Let Π be a selfdual cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$. Then its central character η is necessarily quadratic (or trivial), and replacing Π by its twist $\Pi \otimes \eta$, we may assume that Π has trivial central character. Fix a finite set S of places of F containing the ramified (for Π) and archimedean places. Given any Euler product $L(s) = \prod_v L_v(s)$, we will write $L^S(s)$ for the incomplete product $\prod_{v \notin S} L_v(s)$. The selfduality of Π results in a pole at $s = 1$ of the Rankin- Selberg L-function (on the left hand side of the following factorization):

$$L^S(s, \Pi \times \Pi) = L^S(s, \Pi; \mathrm{sym}^2) L^S(s, \Pi; \Lambda^2), \tag{2}$$

where the L-functions on the right are the (incomplete) symmetric and exterior square L-functions of Π. Moreover, since we are on $\mathrm{GL}(3)$ and with Π having trivial central character, one has the identity

$$L^S(s,\Pi;\Lambda^2) = L^S(s,\Pi), \tag{3}$$

which can be checked factor by factor explicitly. Indeed, at any $v \notin S$, if the unordered triple (Langlands class) associated to Π_v is $\{\alpha_v, \beta_v, \gamma_v\}$, then we have

$$\Lambda^2(\{\alpha_v,\beta_v,\gamma_v\}) = \{\alpha_v\beta_v, \alpha_v\gamma_v, \beta_v\gamma_v\} = \{\gamma_v^{-1}, \beta_v^{-1}, \alpha_v^{-1}\},$$

since $\alpha_v\beta_v\gamma_v = 1$. Now (3) follows because $\Pi \simeq \Pi^\vee$

The utility of (3) is that it shows, by the cuspidality of Π, the entireness of $L^S(s,\Pi;\Lambda^2)$ with no zero at $s=1$. So we have, thanks to (2) and (3),

$$\mathrm{ord}_{s=1} L^S(s,\Pi;\mathrm{sym}^2) \;=\; 1. \tag{4}$$

Consequently, the global parameter $\phi = \phi(\Pi)$ of Π, which *a priori* takes values in $\mathrm{GL}(3,\mathbb{C})$, lands in $\mathrm{O}(3,\mathbb{C})$. It in fact lands in $\mathrm{SO}(3,\mathbb{C})$ since the central character (which corresponds to the determinant of ϕ) is trivial. Since $\mathrm{SO}(3,\mathbb{C})$ is the L-group of $\mathrm{SL}(2)$, a general conjecture of Langlands predicts the existence of a cusp form on $\mathrm{SL}(2)/F$ which transfers to Π. We may now apply the descent theorem of Ginzburg, Rallis and Soudry ([GRS], [Sou]), and indeed find a cuspidal, globally generic automorphic representation π_0 of $\mathrm{SL}_2(\mathbb{A}_F)$. Furthermore, if r denotes the standard (3- dimensional) representation of the dual group of $\mathrm{SL}(2)$, which is $\mathrm{PGL}_2(\mathbb{C})$, the following holds:

Proposition C *The descent $\Pi \mapsto \pi_0$ satisfies the following:*

(a) At any place v, we have the identity of local gamma factors:

$$\gamma(s,\Pi_v) \;=\; \gamma(s,\pi_{0,v};r).$$

(b) If S_∞ is the set of archimedean places of F,

$$\otimes_{w\in S_\infty}\sigma_w(\Pi) \;\simeq\; \otimes_{w\in S_\infty} r(\sigma_w(\pi_0)),$$

where $\sigma_w(\Pi)$, resp. $\sigma_w(\pi 0)$, denotes the archimedean parameter of Π_w, resp. $\pi_{0,w}$, i.e., the associated representation of the Weil group W_w into $GL(3,C)$, resp. $PGL(2,C)$.

(c) If v is a non-archimedean place of F where Π is unramified or an unramified twist of Steinberg, then so is $\pi_{0,v}$, and conversely. Moreover, at such a place,

$$L(s,\Pi_v) = L(s,\pi_{0,v};r).$$

We do not need the following, but when Π_v is supercuspidal, which due to selfduality can happen here only in residual characteristic 2, one can show that π_v is supercuspidal. In the reverse direction, when π_v is supercuspidal, Π_v is either supercuspidal or irreducibly induced from a parabolic. One can see these using the elementary arguments below or by appealing to the general recent results in [JiSou].

2 Proof of Proposition C

By the work of Cogdell, Kim, Piatetski-Shapiro ad Shahidi [CKPSS], we can transfer π_0 back to a cusp form Π' on $\mathrm{GL}(3)/F$ such that the arrow $\pi_{0,v} \to \Pi'_v$ is functorial at all but a finite number of unramified places v of F, compatible with the descent of [GRS], [Sou]. So Π' and Π are equivalent almost everywhere, hence isomorphic by the strong multiplicity one theorem. In other words, the composition of the local parameters of π_0 with the natural embedding

$$\mathrm{PGL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C}) \hookrightarrow \mathrm{GL}(3,\mathbb{C}) \tag{5}$$

are the same as the parameters of Π at all v outside a finite set S. Comparing the functional equations of $L(s,\Pi)$ and $L(s,\pi_0;r)$ and using stability of γ-factors [CKPSS], which allows us to twist by a character ν highly ramified at all the finite places of S such that $\gamma_v(s,\Pi\otimes\nu)$ is an invertible holomorphic function everywhere, one gets

$$\prod_{w\in S_\infty} \gamma(s,\Pi_w) \sim \prod_{w\in S_\infty} \gamma(s,\pi_{0,w}),$$

where $\sim$ means the quotient of the two sides is invertible. One gets as a consequence,

$$\prod_{w\in S_\infty} L(s,\Pi_w)L(1-s,\pi_{0,w};r) \sim \prod_{w\in S_\infty} L(1-s,\Pi_w)L(s,\pi_{0,w};r).$$

One knows that the local L-factors of Π do not have any pole in $\Re(s) \geq 1/2$ (one knows even better- cf. [LRS]), which shows, since none of the local factors (of either kind) has any zero, that the poles of $\prod_{w\in S_\infty} L(s,\Pi_w)$ are contained in those of $\prod_{w\in S_\infty} L(s,\pi_{0,w};r)$; these must be all the poles of the latter as the maximal number is reached. This yields (b).

If v is a finite place in S, one may use stability again and twist by a character which is 1 at v but is highly ramified at all other finite places in S to deduce (a).

Now let v be a finite place where π_v is unramified. Then the γ-identity of (a) implies, in conjunction with the fact that $L(s,\Pi_v)$ has no pole in $\Re(s) \geq 1/2$ [LRS], one sees that the three poles, counted with multiplicity, of $L(s,\Pi_v)$ must be contained in, and hence comprise of the entirety of, the set of poles of $L(s,\pi_{0,v};r)$. Suppose $\pi_{0,v}$ is supercuspidal. Then we can construct a global cuspidal automorphic representation π_1 of $\mathrm{SL}_2(\mathbb{A}_F)$ whose v-component is

$\pi_{0,v}$. More precisely, we can globalize (as in [PR], Lemma 3) the local projective Weil group representation $\sigma_{0,v}$ attached to the supercuspidal and get a global representation σ_1 of W_F of solvable type into $\mathrm{PGL}(2,\mathbb{C})$ whose restriction at v is $\sigma_{0,v}$; the automorphic representation π_1 is an irreducible constituent of the restriction to $\mathrm{SL}_2(\mathbb{A}_F)$ of the representation of $\mathrm{GL}_2(\mathbb{A}_F)$ attached by Langlands and Tunnell to a lift of σ_1 to $W_F \to \mathrm{GL}(2,\mathbb{C})$. Now comparing the functional equation of $L(s,\pi_1;r)$ with that of the Artin L-function $L(s,r(\sigma_1))$, we may identify $\gamma(s,\pi_{0,v};r)$ with $\gamma(s,r(\sigma_1)_v)$. Note that $r(\sigma_1)$ is either irreducible, which can only hapen if the residual characteristic p is 2, since when p is odd, σ_1 is dihedral and hence has a reducible adjoint, or else $r(\sigma_1)$ is reducible, but not containing the trivial character (by Schur) since σ_1 is irreducible. When it is of type $(2,1)$, Π_v will evidently be ramified, and it is of type $(1,1,1)$ iff π_v is induced from more than one quadratic extension of F, forcing Π_v to again be ramified. Thus, in either case, $\pi_{0,v}$ cannot be supercuspidal. Now suppose $\pi_{0,v}$ is a twist of Steinberg. Then we can choose a totally real number field K with a place u such that $K_u = F_v$, and globalize $\pi_{0,v}$ to a cuspidal automorphic representation π_1 of $\mathrm{SL}(2,\mathbb{A}_F)$ whose archimedean components are in the holomorphic discrete series. One can then attach, using Taylor or Blasius-Rogawski, an ℓ-adic representation ρ_1 such that $\rho_{1,u}$ is a Weil-Deligne representation attached to $\pi_{0,v}$. We can conclude that $r(\rho_{1,u})$ is also a twist of Steinberg, implying that $L(s,\pi_{0,v})$ cannot have three poles. So we get a contradiction, and so the only possibility is for $\pi_{0,v}$ to be a principal series representation attached to a character of F_v^*, which we can write as $\mu|\cdot|^z$, for some $z \in \mathbb{C}$, with μ a ramified character of finite order. To show that $L(s,\pi_{0,v})$ does not have three poles (counted with multiplicity), it suffices to show that same for $L(s,I(\mu);r)$, where $I(\mu)$ is the principal series attached to μ, and this is so because the two L-functions are translates of each other. Then the parameter of $I(\mu)$ is the image in $\mathrm{PGL}(2,\mathbb{C})$ of a reducible 2-dimensional representation $\mu_1 \oplus \mu_2$ of $\mathrm{Gal}(\overline{F}_v/F_v)$, so that $\mu = \mu_1\mu_2^{-1}$. We may choose the μ_j to be of finite orders as well. Now we may globalize and obtain an irreducible 2-dimensional representation τ of $\mathrm{Gal}(\overline{F}/F)$ (of dihedral type) such that its restriction to $\mathrm{Gal}(\overline{F}_v/F_v)$ is $\mu_1 \oplus \mu_2$. Then by restricting the corresponding cusp form on $\mathrm{GL}(2)/F$ to $\mathrm{SL}(2)/F$ we obtain a cuspidal automorphic representation η of $\mathrm{SL}_2(\mathbb{A}_F)$ such that $\eta_v \simeq I(\mu)$. Now comparing the functional equations of $L(s,\eta;r)$ and $L(s,r(\tau))$, we may identify $L(s,I(\mu);r)$ with $L(s,r(\tau)_v)$, implying that the former has only a single pole. Thus π_v is forced to be unramified when Π_v is unramified.

It remains to consider when Π_v is an unramified twist of Steinberg. Arguing as above w can rule out all the cases except when π_v is a twist by μ, say, of Steinberg. Globalizing as above to a regular holomorphic cusp form π_1 on $\mathrm{SL}(2)/K$ over a totally real number field K with a place u such that $K_u = F_v$ and $\pi_{1,u} \simeq \pi_{0,v}$. Then $L(s,\pi_{0,v};r)$ identifies with $L(s,r(\rho_{1,u}))$, with ρ_1 the Galois representation attached to π_1. But the latter L-factor can have the same number of poles as $L(s,\Pi_v)$ iff μ is unramified.

The converse direction, namely that Π_v is unramified, resp. an unramified twist of Steinberg, when $\pi_{0,v}$ has that property, is also evident from above.

This finishes the proof of (c), and hence the Proposition.

3 Proof of Theorem A - Part II

We begin by noting that we can find a generic cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_F)$ whose restriction to $\mathrm{SL}(2, \mathbb{A}_F)$ contains π_0. This can be done by appealing to Labesse and Langlands [LL]. But we want to refine their construction in such a way that we keep track of what happens at the finite primes in order that we do not introduce new ramification. Here is what we do. First choose a character ω_1 of $Z(\mathbb{A}_F)$, where Z denotes the center of $\mathrm{GL}(2)$, such that ω_1 is trivial on $Z_\infty^+ Z(F)$ and agrees with the restriction of π_0 to $Z(\mathbb{A}_F)\mathrm{SL}_2(\mathbb{A}_F)$. We may choose ω_1 to be ramified only where π_0 is. The pair (π_0, ω_1) defines a representation π_1 of the group $H := \mathrm{SL}_2(\mathbb{A}_F)Z(\mathbb{A}_F)$, such that the central character ω_1 of π_1 is trivial on Z_∞^+ and $Z(F)$. In particular, ω_1 is a finite order character, unramified where π_0 is. If Π has conductor N, then by Proposition C, π_0 is unramified outside N, and so the same holds for π_1. Note that $H(\mathbb{A}_F)$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{A}_F)$ with a countable quotient group. Now induce π_1 to $\mathrm{GL}_2(\mathbb{A}_F)$, and choose (as follows) a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ occurring in the induced representation, which is necessarily globally generic. Denote by ω the central character of π, which is of finite order and unramified outside N. Let $K(M)$ denote a principal congruence subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,f})$ such that π_1 has a fixed vector under $K'(M) := K(M) \cap \mathrm{SL}_2(\mathbb{A}_{F,f})$. Then the induced representation will, by Frobenius reciprocity, have at least one constituent which will have a vector fixed under $K(M)$, and we will choose such a π. In particular, thanks to Proposition C (and the construction of the extension π of π_0), given any finite place v, π_v is unramified whenever Π_v is. Suppose next that v divides M. Then π_1, and hence π, is unramified there, since π_0 is so.

Let $\mathrm{sym}^2(\pi)$ denote the symmetric square transfer of π to $\mathrm{GL}(3)/F$ [GeJ], and put

$$\mathrm{Ad}(\pi) := \mathrm{sym}^2(\pi) \otimes \omega^{-1},$$

which is a selfdual and is a cusp form when π is not dihedral. Note that by the construction of π (and Proposition C) we have, with S denoting the finite set of places dividing N and the archimedean places,

$$L^S(s, \Pi) = L^S(s, Ad(\pi)), \tag{6a}$$

which by the strong multiplicity one for $\mathrm{GL}(n)$ furnishes a global isomorphism

$$\Pi \simeq \mathrm{Ad}(\pi), \tag{6b}$$

and if π' is another candidate on $\mathrm{GL}(2)/F$, then by the *multiplicity one theorem for* $SL(2)$ [Ram], we must have

$$\pi' \simeq \pi \otimes \mu,$$

for an idele class character μ of F .

As noted earlier, we can take π to be unramified where Π is, and moreover, by replacing π by a character twist if necessary, we may take its central character ω to be uniary. Moreover, when Π_v is an unramified twist of Steinberg, we may, thanks to Proposition C, choose π_v to also be an unramified twist of Steinberg when Π_v is.

This completes the proof of Theorem A.

4 Proof of Corollary B

Let w be an archimedean place of F. Then, since Π is the adjoint transfer of π, which is known to be functorial at every place, we have in particular the isomorphism of archimedean parameters on the Weil group W_w:

$$\sigma_w(\Pi) \simeq \mathrm{Ad}(\sigma_w(\pi)). \tag{7}$$

Evidently, if $\sigma_w(\Pi)$ is regular at w, i.e., has $\mathbb{C}^*$ acting on it with multiplicity one, then the same necessarily holds, thanks to (7), for $\sigma_w(\pi)$.

Next recall that a cusp form η of $\mathrm{GL}_n(\mathbb{A}_F)$ is said to be *algebraic* (cf. [C], Section 1.2.3), or of type A_0, if for every $w \in S_\infty$, and for any character χ of $\mathbb{C}^*$ appearing in the restriction of $\sigma_w(\eta)$ to $\mathbb{C}^*$, we have, for suitable integers p_w, q_w , $\chi(z) = z^{p_w+(n-1)/2}\overline{z}^{q_w+(n-1)/2}$, for all $z \in \mathbb{C}^*$; it is of type A if $p_w, q_w$are required only to be rational.

In our case, since the central character ω of π is of finite order by construction, the restriction of $\sigma_w(\pi)$ to $\mathbb{C}^*$ is, for any $w \mid \infty$, of the form $\mu \oplus \mu^{-1}$. Hence

$$\mathrm{Ad}(\sigma_w(\pi))|_{\mathbb{C}^*} \simeq \mu^2 \oplus 1 \oplus \mu^{-2}. \tag{8}$$

As Π is algebraic, we have

$$\mu^2(z) = z^{p_w+1}\overline{z}^{q_w+1}, \forall z \in \mathbb{C}^*, \tag{9}$$

for some $p_w, q_w \in \mathbb{Z}$. It is evident that π is of type A when Π is.

By the archimedean purity theorem for algebraic cusp forms on $\mathrm{GL}(n)$ (cf. [C], p. 112), we see that $p_w + q_w$ is constant for all the characters of $\mathbb{C}^*$ appearing in $\sigma_w(\Pi)$, and it is also independent of w in S_∞. Since the trivial character also occurs in $\sigma_w(\Pi)$ by (8), we must have

$$p_w + q_w = 0, \quad \forall w \mid \infty. \tag{10}$$

In other words, $\sigma_w(\Pi) \otimes |\cdot|^{-1}$ is *tempered* at each $w \in S_\infty$.

Now suppose Π is regular and algebraic. Then Π_w must be an isobaric sum $\eta_{k_w} \boxplus 1$, with η_{k_w} being the base change to $\mathrm{GL}_2(F_w)$ from $\mathrm{GL}_2(\mathbb{R})$ of a discrete series representation $\mathcal{D}_{k_w}$ of $\mathrm{GL}_2(\mathbb{R})$ of weight $k_w \geq 2$. (If F_w is real, then the base change is the identity). We get

$$\left(\sigma_w(\Pi) \otimes |\cdot|^{-1}\right)|_{\mathbb{C}^*} \simeq \left(\frac{z}{\sqrt{z\overline{z}}}\right)^{k_w-1} \oplus \left(\frac{z}{\sqrt{z\overline{z}}}\right)^{1-k_w} \oplus 1, \tag{11}$$

with $k_w \geq 2$. Comparing (8) with (11), we see that $k_w - 1$ must be even, hence of the form $2(m-1)$, for an integer m. Since $k_w \geq 2$, m is also ≥ 2. It follows that π_w is the base change of a discrete series representation $\mathcal{D}_m$, showing that it is algebraic. Since this holds at every archimedean place w, π is algebraic. In sum, π can be chosen to be regular algebraic, hence cohomological, whenever Π has that property, and moreover, the converse direction is clear. This completes the proof of Corollary B.

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