MODULAR CURVES, MODULAR SURFACES, AND MODULAR FOURFOLDS

Dinakar Ramakrishnan

To Jacob Murre

1 Introduction

We begin with some general remarks. Let X be a smooth projective variety of dimension n over a field k. For any positive integer $p < n$, it is of interest to understand, modulo a natural equivalence, the algebraic cycles $Y = \sum_j m_j Y_j$ lying on X, with each Y_j closed and irreducible of codimension p, together with codimension $p+1$ algebraic cycles $Z_j = \sum_i r_{ij} Z_{ij}$ lying on Y_j, for all j. There is a natural setting in which to study such a chain $(X \supset Y_j \supset Z_{ij})_{ij}$ of cycles, namely when the following hold:

(a) Each Z_j is, as a divisor on Y_j, linearly equivalent to zero, i.e., of the form $\mathrm{div}(f_j)$ for a function f_j on Y_j;

(b) The formal sum $\sum_j m_j Z_j$ is zero as a codimension $p+1$ cycle on X.

Those satisfying (a), (b) form a group $\mathcal{Z}^{p+1}(X, 1)$. An easy way to construct elements of this group is to take a codimension $p-1$ subvariety W of X, with a pair of (non-zero) functions (φ, ψ) on W, and take the formal sum $\sum_j (Y_j, T_{Y_j}(\varphi, \psi))$, where $\{Y_j\}$ is the finite set of codimension p subvarieties where φ or ψ has a zero or a pole, and

$$T_{Y_j}(\varphi, \psi) \;=\; (-1)^{\mathrm{ord}_j(\varphi)\mathrm{ord}_j(\psi)} \left(\frac{\varphi^{\mathrm{ord}_j(\psi)}}{\psi^{\mathrm{ord}_j(\varphi)}}\right)|_{Y_j},$$

the *Tame symbol* of (φ, ψ) at Y_j, where ord_j denotes the order at Y_j. It is a fact that $\sum_j T_{Y_j}(\varphi, \psi)$ is zero as a codimension $p+1$ cycle on X. Let

$CH^{p+1}(X,1)$ denote the quotient of $\mathcal{Z}^{p+1}(X,1)$ by the subgroup generated by such elements. This is a basic example of Bloch's higher Chow group ([B]). For any abelian group A, let $A_{\mathbb{Q}}$ denote $A \otimes \mathbb{Q}$. Then it may be worthwhile to point out the isomorphisms

$$CH^{p+1}(X,1)_{\mathbb{Q}} \simeq H^{2p+1}_{\mathcal{M}}(X, \mathbb{Q}(p+1)) \simeq \mathrm{Gr}^{p+1}_{\gamma} K_1(X) \otimes \mathbb{Q},$$

where $H^*_{\mathcal{M}}(X, \mathbb{Q}(**))$ denotes the bigraded *motivic cohomology* of X, and Gr^r_γ denotes the r-th graded piece defined by the gamma filtration on $K_*(X)$.

There is another way to construct classes in this group, and that is to make use of the product map

$$CH^p(X) \otimes k^* \to CH^{p+1}(X,1),$$

the image of this being generated by the *decomposable classes* $\{(Y,\alpha)\}$, where each Y is a codimension p subvariety and α a non-zero scalar.

It should also be mentioned, for motivation, that when k is a number field, Beilinson predicts that the elements in $CH^{p+1}(X,1)_{\mathbb{Q}}$ which come from a regular proper model $\mathcal{X}$ over $\mathbb{Z}$ span a $\mathbb{Q}$-vector space of dimension equal to the order of vanishing at $s=p$ of the associated L-function $L(s, H^{2p}(X))$, which we will denote by $L(s,X)$ if there is no confusion. By the expected functional equation, the order of pole at $s=p+1$ of $L(s,X)$, denoted $r_{\mathrm{an}}(X)$, will be the difference of the order of pole of the Gamma factor $L_\infty(s,X)$ at $s=p$ and the order of zero of $L(s,X)$ at $s=p$. The celebrated conjecture of Tate asserts that $r_{\mathrm{an}}(X)$ is the dimension of $\mathrm{Im}\Big(CH^p(X) \to H^{2p}_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\Big)$ for any prime ℓ. One of the main objects here is to sketch a proof of the Tate conjecture in codimension 2 for Hilbert modular fourfolds, and also deduce the Hodge conjecture under a hypothesis.

Going back to $CH^{p+1}(X,1)$, the first case of interest is when X is a surface and $p=1$. When X is the Jacobian $J_0(37)$ of the modular curve $X_0(37)$, Bloch constructed a non-trivial example $\beta \in CH^2(X,1)_{\mathbb{Q}}$ by using the curve and the fact that $J_0(37)$ is isogenous to a product of two elliptic curves over $\mathbb{Q}$. This was generalized by Beilinson ([Be]; see also [Sch]) to a product of two modular curves by going up to a (ramified) cover $X_0(N) \times X_0(N)$ and by taking $\{Y_j\}$ to be the union of the diagonal Δ and the curves $X_0(N) \times \{P\}$ and $\{Q\} \times X_0(N)$, where P, Q are cusps; the existence of the functions f_j came from the Manin-Drinfeld theorem saying that the difference of any two cusps is torsion in the Jacobian. Later the author generalized this ([Ra1,2]) to the case of Hilbert modular surfaces X by using a class of curves on X called the Hirzebruch-Zagier cycles, carefully chosen to have appropriate intersection properties; in general these curves meet in CM points or cusps.

The second main goal of this article is to describe briefly the ideas behind an ongoing project of the author involving the construction of ($\mathbb{Q}$-rational) classes in $CH^3(X,1)_{\mathbb{Q}}$ for certain modular fourfolds $X/\mathbb{Q}$. We will restrict ourselves to Hilbert modular fourfolds defined by a biquadratic, totally real field F. Very roughly, the basic idea is to use suitable translates Y_j of embedded Hilbert modular surfaces (coming from the three quadratic subfields), choose Z_j to be made up of (translates of) embedded modular curves which are homologically trivial, hence rationally trivial (as the Y_j are simply connected), and also use the fact that the Tate and Hodge conjectures are known (by the author [Ra4]) for codimension 2 cycles on X (for square-free level), as well as the knowledge (*loc. cit.*) that a basis of $\mathbb{Q}$-rational cycles modulo homological equivalence (in the middle dimension) on X, and on the embedded Hilbert modular surfaces, is given by appropriate Hirzebruch-Zagier cycles and their twists. We will first construct decomposable classes and then indicate some candidates for the indecomposable part.

The ultimate goal is to understand this phenomenon for Shimura varieties X, of which modular curves and Hilbert modular varieties are examples. It is known that the most interesting (cuspidal) part of the cohomology of X is in the middle dimension, which leads us to consider such X of dimension $n = 2m$ and take $p = m$. When there are Shimura subvarieties Y of X of dimension m with $H^1(Y) = 0$, like for Siegel modular varieties, one can hope to construct promising classes in $CH^{m+1}(X,1)_{\mathbb{Q}}$. This will be taken up elsewhere.

This article is dedicated to Jaap Murre, from whom I have learnt a lot over the years – about algebraic cycles and about the (conjectural) Chow-Künneth decompositions, though they exist for simple reasons in the cases considered here. We have a long term collaboration as well on the zero cycles on abelian surfaces. I would also like to acknowledge a helpful conversation I had with Spencer Bloch about $CH^*(X,1)$ in the modular setting (see section 14). I thank the referee and Mladen Dimitrov for spotting various typos on an earlier version, and for making suggestions for improvement of exposition. Finally, I am pleased to acknowledge the support of the National Science Foundation through the grant DMS-0402044.

2 Notations

Let X be a smooth projective fourfold over a number field k. Set:

$V_B = H^4(X(\mathbb{C}), \mathbb{Q})$

$Hg^2(X) = V_B \cap H^{2,2}(X(\mathbb{C}))$

$r_{\mathrm{Hg}} = \dim_{\mathbb{Q}} Hg^2(X)$

$\mathcal{G}_k = \mathrm{Gal}(\overline{k}/k)$

ℓ: a prime

$V_\ell = H^4(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$: $\mathcal{G}_k$-module

$r_{\mathrm{alg},k} = \dim \left(\mathrm{Im}(CH^2(X)_{\mathbb{Q}} \to V_\ell(2))\right)$

$\mathcal{T}_{\ell,k} = V_\ell(2)^{\mathcal{G}_k}$

$r_{\ell,k} = \dim_{\mathbb{Q}_\ell} \mathcal{T}_{\ell,k}$

$S_{\mathrm{bad}} := \{\mathcal{P} \mid V_\ell \neq V_\ell^{I_{\mathcal{P}}}\} \cup \{\mathcal{P} \mid \ell\}$

S: a finite set of places $\supset S_{\mathrm{bad}}$

$Fr_{\mathcal{P}}$: geometric Frobenius at $\mathcal{P}$, $\forall \mathcal{P} \notin S_{\mathrm{bad}}$ with norm $N(\mathcal{P})$

$L(s, X) = \prod_{\mathcal{P} \notin S} \det(I - Fr_{\mathcal{P}} T | V_\ell)^{-1}_{|_{T=N(\mathcal{P})^{-s}}}$

By Deligne's proof of the Weil conjectures, the inverse roots of $Fr_{\mathcal{P}}$ on V_ℓ are, for $\mathcal{P} \notin S$, of absolute value $N(\mathcal{P})^2$, implying that the L-function $L(s, X)$ converges absolutely in $\{\Re(s) > 3\}$. The boundary point $s = 3$, where $L(s, X)$ could be divergent, is called the Tate point. The fourfolds of interest to us will admit meromorphic continuation to the whole s-plane and satisfy a functional equation relating s to $5 - s$. Put

$$r_{\mathrm{an},k} \;=\; -\mathrm{ord}_{s=3} L(s, X).$$

Tate's conjecture is that this *analytic rank* equals the *algebraic rank* $r_{\mathrm{alg},k}$ of codimension 2 algebraic cycles on X modulo (ℓ-adic) homological equivalence. It is also expected that these two ranks are the same as the *ℓ-adic cycle rank* $r_{\ell,k}$, and one always has $r_{\mathrm{alg},k} \leq r_{\ell,k}$.

3 Hilbert Modular fourfolds

Let K be a quartic, Galois, totally real number field with embedding $K \hookrightarrow \mathbb{R}^4$ given by the archimedean places. Fix a square-free ideal $\mathcal{N}$ in the ring $\mathcal{O}_K$ of integers of K, and write Γ for the congruence subgroup $\Gamma_1(\mathcal{N}) \subset \mathrm{SL}(2, \mathcal{O}_K)$ of level $\mathcal{N}$. Then there is a natural embedding

$$\Gamma \hookrightarrow \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}),\ \gamma \to (\gamma^\sigma)_{\sigma \in \mathrm{Hom}(K,\mathbb{R})}.$$

Using this one gets an action of Γ on the four-fold product of the upper half plane $\mathcal{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The quotient $Y = \Gamma\backslash\mathcal{H}^4$ is a coarse moduli space of polarized abelian fourfolds A with Γ-structure, with $\mathrm{End}(A) \hookleftarrow \mathcal{O} \subset K$. It is a quasi-projective variety, with Baily-Borel-Satake compactification Y^*, and a smooth toroidal compactification $X := \tilde{Y} = Y \cup \tilde{Y}^\infty$, all defined over $\mathbb{Q}$.

For simplicity of exposition, we have used here the classical formalism. Later, we will need to work with the adelic version $S_{C_1(\mathcal{N})}$ relative to the standard compact open subgroup $C_1(\mathcal{N})$ of $G(\mathbb{A}_{K,f})$, where $\mathbb{A}_{K,f}$ denotes the ring of finite adeles of K; one has $C_1(\mathcal{N}) \cap \mathrm{GL}(2,F) = \Gamma_1(\mathcal{N})$. Moreover,

$$S_{C_1(\mathcal{N})}(\mathbb{C}) = \mathrm{GL}(2,K)\backslash(\mathbb{C}-\mathbb{R})^4 \times \mathrm{GL}(2,\mathbb{A}_{K,f})/C_1(\mathcal{N}),$$

which is finitely connected, and $\Gamma_1(\mathcal{N})\backslash\mathcal{H}^4$ occurs as an étale quotient of the connected component. (If the object is to realize $\Gamma_1(\mathcal{N})\backslash\mathcal{H}^4$ as *exactly* the connected component, one needs to consider instead the Shimura variety associated to the $\mathbb{Q}$-group G with $G(\mathbb{Q}) = \{g \in \mathrm{GL}(2,K) \,|\, \det(g) \in \mathbb{Q}^*\}$.) The Shimura variety $S_{C_1(\mathcal{N})}$ is defined over $\mathbb{Q}$, and the same holds for its Baily-Borel-Satake compactification $S^*_{C_1(\mathcal{N})}$. One can also choose a smooth toroidal compactification $X = \tilde{S}_{C_1(\mathcal{N})}$ over $\mathbb{Q}$. Again we will use Y, resp. $\tilde{Y}^\infty$, to denote $S_{C_1(\mathcal{N})}$, resp. the boundary, so that $X = Y \cup \tilde{Y}^\infty$.

4 Results on Cycles of codimension 2 on X

Let X be the smooth toroidal compactification over $\mathbb{Q}$ of a Hilbert modular fourfold of *square-free* level $\mathcal{N}$ as above, relative to a quartic Galois extension K of $\mathbb{Q}$.

Theorem A ([Ra4])

(i) The Tate classes in $V_\ell(2)$ are algebraic. In fact, $r_{\ell,k} = r_{\mathrm{alg},k} = r_{\mathrm{an},k}$.

(ii) *If $\mathcal{N}$ is a proper ideal, the Hodge classes in $V_B(2)$ are algebraic when they are not pull-backs of classes from the full level, and moreover, they are not all generated by intersections of divisors.*

The next few sections indicate a proof of this, while at the same time developing the theory and setting the stage for what is to come afterwards.

Now let K be *biquadratic* so that $\mathrm{Gal}(K/\mathbb{Q}) = \{1, \sigma_1, \sigma_2, \sigma_3 = \sigma_1\sigma_2\}$, with $\sigma_j^2 = 1$ for each j. Let $F_j \subset K$ be the real quadratic field obtained as the fixed field of σ_j. For every $g \in \mathrm{GL}_2^+(K)$, let $Y_{F_j,g}$ denote the closure in X of the image of $g\mathcal{H}^2$, which identifies with the Hilbert modular surface attached to F_j and the congruence subgroup $g^{-1}\Gamma g \cap \mathrm{SL}(2, \mathcal{O}_{F_j})$. It is the natural analogue of the Hirzebruch-Zagier cycle on a Hilbert modular surface.

The proof of Theorem A has as a consequence the following:

Theorem B *Let K be biquadratic with intermediate quadratic fields F_1, F_2, F_3. Define $N > 0$ by $N\mathbb{Z} = \mathcal{N} \cap \mathbb{Z}$, and assume that the modular curve $X_0(N)$ has genus > 0. Then there exist $g_1, g_2 \in \mathrm{GL}_2^+(K)$ such that Y_{F_1,g_1}, Y_{F_2,g_2} span a 2-dimensional subspace of $r_{\mathrm{alg},\mathbb{Q}}$. Consequently,*

$$\dim_{\mathbb{Q}} CH^2(X) \geq 2.$$

By the product structure, one gets non-trivial, decomposable classes in $CH^3(X, 1)$. A refinement will be discussed in section 11.

5 Contribution from the boundary

By the *decomposition theorem*, there is a short exact sequence

$$0 \to IH^4(Y^*) \to H^4(X) \to H^4_{\tilde{Y}^\infty}(X) \to 0,$$

both as Galois modules and as $\mathbb{Q}$-Hodge structures. Here IH^* is the Goresky-MacPherson's middle intersection cohomology. We obtain, for $\alpha = \{\ell, k\}$, an, alg,

$$r_\alpha(X) = r_\alpha(Y^*) + r_\alpha^\infty,$$

where $r_\alpha(Y^*)$, resp. r_α^∞, is the α-rank associated to $IH^4(Y^*)$, resp. $H^4_{\tilde{Y}^\infty}(X)$.

The cohomology with supports in $\tilde{Y}^\infty$ has a nice description:

$$H^*_{\tilde{Y}^\infty}(X) \simeq \oplus_{\sigma:\mathrm{cusp}} H^*_{D_\sigma}(X),$$

where D_σ is a divisor with normal crossings (DNC) with smooth irreducible components D_σ^i. If $D_\sigma^{i,j}$ denotes $D_\sigma^i \cap D_\sigma^j$, there is an exact sequence

$$\sum_i H^2(D_\sigma^i)(1) \to H^4_{D_\sigma}(X)(2) \to \sum_{i\neq j} H^0(D_\sigma^{i,j})$$

Since D_σ^i is toric, its H^2 is generated by divisors. This implies the following string of equalities for large k:

$$r^\infty_{\mathrm{alg,k}} = r^\infty_{\ell,k} = r^\infty_{\mathrm{an},k} = r^\infty_{\mathrm{Hg}}.$$

All but the last equality on the right remain in force for any number field k.

Hence the problem reduces to understanding the $r_\alpha(Y^*)$ for various α and explicating their relationships with each other.

6 The action of Hecke correspondences

If $g \in \mathrm{GL}_2^+(K)$, there are two maps $Y_{\Gamma_g} \to Y_\Gamma$, with $\Gamma_g = \Gamma \cap g^{-1}\Gamma g$, inducing an algebraic correspondence T_g, which extends to Y_Γ^*. The algebra $\mathcal{H}$ of such Hecke correspondences acts semisimply on cohomology. This leads to a $\mathcal{H} \times \mathcal{G}_\mathbb{Q}$-equivariant decomposition

$$IH^4(Y^*) \simeq V_{\mathrm{res}} \oplus V_{\mathrm{cusp}}$$

where the submotive V_{res} is algebraic, and the cuspidal submotive V_{cusp} is the "interesting" part (see below).

To be precise, the *residual part* V_{res} is spanned by the intersections of Chern classes of certain universal line bundles $\mathcal{L}_{ij}, 1 \leq i, j \leq 4$ occurring at every level $\mathcal{N}$. In the complex realization they are defined by the $\mathrm{SL}(2,\mathbb{R})^4$-invariant differential forms $\eta_i \wedge \eta_j$, $1 \leq i \neq j \leq 4$ on $\mathcal{H}^4$, where for each j, $\eta_j = \mathrm{pr}_j^*(dz_j \wedge d\bar{z}_j)$, with $\mathrm{pr}_j : \mathcal{H}^4 \to \mathcal{H}$ being the j-th projection.

Remark: In our case one knows enough about the Galois modules which occur in $H^\cdot(X)$ to be able to get a direct sum decomposition:

$$H^\cdot(X) \simeq IH^\cdot(Y^*) \oplus H^\cdot_{\tilde{Y}^\infty}(X)$$

$IH^\cdot(Y^*)$ can be cut out as a direct summand by an algebraic cycle modulo homological equivalence. (This has recently been done for general Shimura varieties by A. Nair ([N]).) The reason is that the Hecke correspondences act on an inverse limit of a family of toroidal compactifications of Y, though not on any individual one. However, $IH^{\cdot}(Y^*)$ is not immediately a Chow motive, since the algebra of Hecke correspondences modulo rational equivalence is not semisimple.

The Künneth components of Δ in $IH^8(Y^* \times Y^*)$ are algebraic, since it is known that $IH^1(Y^*) = IH^3(Y^*) = 0$ and $IH^2(Y^*)$ is algebraic, being purely of Hodge type $(1,1)$.

There is a further $\mathcal{H} \times \mathcal{G}_{\mathbb{Q}}$-equivariant decomposition:

$$V_{\text{cusp}} \;=\; \oplus_{\varphi} V(\varphi)^{m(\varphi)},$$

where φ runs over holomorphic Hilbert modular cusp forms of level $\mathcal{N}$, which have (diagonal) weight 2, and $m(\varphi)$ is a certain multiplicity which is 1 if φ is a *newform*, i.e., not a cusp form of level a proper divisor of $\mathcal{N}$.

7 The submotives of rank 16

It is now necessary to understand $V(\varphi)$ for a Hilbert modular newform φ of weight 2 and level $\mathcal{N}$. It is easy to see that $V_B(\varphi)$ is 16-dimensional, generated over $\mathbb{C}$ by the differential forms $\varphi(z)dz_I \wedge d\overline{z}_J$ of degree 4 for partitions $\{1,2,3,4\} = I \cup J$.

Let π be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$ of trivial central character associated to φ. We will write $V(\pi)$ instead of $V(\varphi)$. By R.L. Taylor and Blasius-Rogawski, one can associate a 2-dimensional irreducible representation $W_\ell(\pi)$ of $\mathcal{G}_K$ such that $L(s, W_\ell(\pi)) = L(s, \pi)$, i.e., $\forall \mathcal{P} \nmid \mathcal{N}$,

$$\mathrm{tr}\,(Fr_{\mathcal{P}}|W_\ell(\pi)) \;=\; a_{\mathcal{P}}(\pi).$$

By Brylinski-Labesse, as refined by Blasius,

$$\mathrm{tr}\,(Fr_{\mathcal{P}}|V_\ell(\pi)) \;=\; L(s, \pi_{\mathcal{P}}; r), \;\; \forall \mathcal{P} \nmid \mathcal{N}.$$

Since $V_\ell(\pi)$ and $W_\ell(\pi)$ are semisimple, we get the following isomorphism by Tchebotarev:

$$V_\ell(\pi)|_{\mathcal{G}_K} \simeq \otimes_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} W_\ell(\pi)^{[\tau]}$$

To be precise, $V_\ell(\pi)$ is the *tensor induction* ([CuR]) of $W_\ell(\pi)$ from K to $\mathbb{Q}$:

$$V_\ell(\pi) \simeq {}^{\otimes}\mathrm{Ind}_{\mathcal{G}_K}^{\mathcal{G}_{\mathbb{Q}}}(W_\ell(\pi))$$

This identity makes it possible to compute the Tate classes.

We need the following result:

Theorem C ([Ra3,4]) *The L-function of $V_\ell(\pi)$ admits a meromorphic continuation to the whole s-plane with a functional equation of the form*

$$L(s, V_\ell(\pi)^\vee) = \varepsilon(s, V_\ell(\pi))L(5-s, V_\ell(\pi)),$$

where $\varepsilon(s, V_\ell(\pi))$ is an invertible function on $\mathbb{C}$ and the superscript $\vee$ indicates the dual.

From this one also gets the analogous statement about the L-function of X, which is a product of these $L(s, V_\ell(\pi))$ with an abelian L-function.

8 Strategy for algebraicity

When $r_\ell(\pi) \neq 0$, we show first that $L(s, V_\ell(\pi))$ has a pole at $s = 3$. But then we also show, using a specific integral representation, that for a suitable quadratic field $F \subset K$, the function $L_{1,F}(s) := L(s, {}^{\otimes}\mathrm{Ind}_{\mathcal{G}_K}^{\mathcal{G}_F}(W_\ell(\pi)) \otimes \nu)$ has a *simple* pole. What we do then is to construct an algebraic cycle ${}^{\nu}Z_F$, and prove $\int_{{}^{\nu}Z_F} \omega \neq 0$ for a $(2,2)$-form ω by realizing it as $\mathrm{res}_{s=2} L_{1,F}(s)$. Using the previous section, we first prove

Proposition *For any Dirichlet character χ,*

$$r_\ell(\pi, \chi) = r_{\mathrm{an}}(\pi, \chi) \leq 2$$

$r_\ell(\pi, \chi) = 1$ iff a twist $\pi \otimes \nu$ is fixed by an involution $\tau \in$Gal$(K/\mathbb{Q})$, while $r_\ell(\pi, \chi) = 2$ iff K is biquadratic and $\pi \otimes \nu$ is Gal$(K/\mathbb{Q})$-invariant.

Now the problem is to construct algebraic cycles in the $(\pi\otimes\nu)$-eigenspace, which are of infinite order modulo homological equivalence, to account for these poles when $\pi \otimes \nu$ is fixed by one or more (non-trivial) elements of $\mathrm{Gal}(K/\mathbb{Q})$.

9 Cycles

Let F be a quadratic subfield of K, with corresponding embedding $\mathfrak{h}^2 \hookrightarrow \mathfrak{h}^4$. Recall that if $g \in \mathrm{GL}_2^+(K)$, the image of the translate $g\mathfrak{h}^2$ under the projection $\mathfrak{h}^4 \to \Gamma\backslash\mathfrak{h}^4$ defines a surface $\Delta_g\backslash\mathfrak{h}^2$ with closure $Z_{F,g}$ in X. This is an example of a Hirzebruch-Zagier cycle; see [K] for a definition of such cycles for orthogonal Shimura varieties.

For any abelian character μ, there is a *μ-twisted Hirzebruch-Zagier cycle* $Z^{\mu}_{F,g}$ of codimension 2 in X. This is defined (cf. [Ra4]) by composing the above construction with a *twisting correspondence* defined by μ, which sends, for every π, the π-isotypic subspace onto the $(\pi \otimes \mu)$-isotypic subspace of the cohomology.

Suppose $r_\ell(\pi, \chi)(= r_{\rm an}(\pi, \chi))$ is > 0. Then there is a quadratic subfield F and a cusp form π_1 on $\mathrm{GL}(2)/F$ such that $\pi \otimes \nu \simeq (\pi_1)_K$. As one would hope, a twisted Hirzebruch-Zagier cycle $Z^{\mu}_{F,g}$ provides the requisite algebraic cycle to get the Tate conjecture.

There is a real **subtle point** here which separates it from the work of Harder, Langlands and Rapoport ([HLR]) on the divisors on Hilbert modular surfaces:

The period of a $(2,2)$-form on X (defined by π) over $Z^{\mu}_{F,g}$ is non-zero, but it is the residue of a **different L-function***, namely $L_{1,F}(s)$, which* **does not divide** $L(s, V_\ell(\pi))$!

The residues of the two L-functions are presumably related in a non-trivial way, but this is not known. It is an intriguing problem to try to understand this better.

10 Hodge classes

By hypothesis, we need only consider those π which are of level $\mathcal{N} \neq \mathcal{O}_K$. We may then fix a prime divisor $\mathcal{P}$ of $\mathcal{N}$ and consider a quaternion algebra B/K which is ramified only at three infinite places and at $\mathcal{P}$. By the Eichler-Shimizu-Jacquet-Langlands correspondence, there exists a corresponding cusp form π' on B^* giving rise to a submotive $V(\pi')$ of $H^4(R_{K/\mathbb{Q}}(C))$ for a Shimura curve $C = \Delta\backslash\mathfrak{h}$ defined over K. As $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-modules,

$$V_\ell(\pi') \simeq V_\ell(\pi),$$

implying that

$$(i) \qquad r_{\ell,E}(\pi') = r_{\ell,E}(\pi),$$

for any number field $E \supset K$.

On the Hodge side we need the following result, proved jointly with V.K. Murty, which is really a statement about periods:

Theorem ([Mu-Ra2]) *As $\mathbb{Q}$-Hodge structures,*

$$V_B(\pi') \simeq V_B(\pi).$$

The proof compares the coefficients of the Shimura liftings of π and π' to forms of weight 3/2, which live on the two-sheeted covering group of $\mathrm{GL}(2)/K$.

By Deligne ([DMOS]), every Hodge class on an abelian variety gives rise to a Tate class. One gets from this the equality over a sufficiently large field $E \supset K$:

$$(ii) \qquad r_{\ell,E}(\pi') = r_{\mathrm{Hg}}(\pi').$$

On the other hand, by the Theorem with Murty, one also gets

$$(iii) \qquad r_{\mathrm{Hg}}(\pi') = r_{\mathrm{Hg}}(\pi).$$

Combining (i), (ii) and (iii), we see that the Hodge conjecture for $V_B(\pi)$ follows from the Tate conjecture for $V_\ell(\pi)$ over E.

11 Where the cycles come from

The method of proof furnishes the following:

Theorem B′ *Let F be a quadratic subfield of K. Then a twisted Hirzebruch-Zagier cycle of codimension 2 on X associated to F contributes to $V(\pi)$ iff a twist of π is a base change from F. When K is biquadratic and a twist of π is base changed from $\mathbb{Q}$, i.e., attached to an elliptic cusp form h, we get*

$$r_{\mathrm{alg},\mathbb{Q}}(\pi) = 2,$$

with the Hecke twisted Hilbert modular surfaces from two subfields F_1, F_2, say, give non-trivial independent algebraic classes of codimension 2.

As a consequence, the decomposable part of $CH^3(X_\pi, 1)$ has rank at least 2 when π is a base change from $\mathbb{Q}$ and K biquadratic. (Here X_π refers to the submotive of $[X]$ in degree 4 cut out by π.) But since $\mathbb{Z}^* = \{\pm 1\}$, such classes will not come from a regular, proper model $\mathcal{X}$ of X (when such a model exists).

Now let $N > 0$ be defined by $N\mathbb{Z} = \mathcal{N} \cap \mathbb{Z}$. Then when the modular curve $X_0(N)$ has positive genus, there exists at least one (elliptic) newform h

of weight 2 and level N, and the base change to K of the associated cuspidal automorphic representation on $\mathrm{GL}(2)/\mathbb{Q}$, assures the existence of a π as in Theorem B′. Hence Theorem B follows from Theorem B′.

12 What to expect

A straight-forward calculation shows that

$$-\mathrm{ord}_{s=2} L_\infty(s, V(\pi)) \,=\, 3.$$

In the *biquadratic* case, if a twist of π is a base change from $\mathbb{Q}$, we have

$$r_{\mathrm{an},\mathbb{Q}}(\pi^\vee) \,=\, 2.$$

Hence by the functional equation,

$$\mathrm{ord}_{s=2} L(s, V(\pi)) \,=\, 1.$$

In this case, Beilinson predicts (in [Be]) the existence of a non-trivial class β in $CH^3(X_\pi, 1)_\mathbb{Q}$ which comes from a proper model $\mathcal{X}$ over $\mathbb{Z}$, to account for this simple zero of the L-function.

In the *cyclic* case, $r_{\mathrm{an},\mathbb{Q}}(\pi^\vee) = 1$, and so the we should have two independent classes in the higher Chow group. The general philosophy is that it is much harder to produce classes in a motivic cohomology group (or a Selmer group) which is conjecturally of rank bigger than one, and this is why we are not at present concentrating on this (cyclic) case.

13 Elements in $\mathbf{CH^3(X, 1)_\mathbb{Q}}$

Let $K/\mathbb{Q}$ be biquadratic with quadratic subfields F_1, F_2, F_3. For $g_1, g_2, g_3 \in \mathrm{GL}_2^+(K)$, consider the surfaces $Z_i = Z_{F_i, g_i}, 1 \leq i \leq 3$, which are Hecke translates of the three Hilbert modular surfaces in X associated to $\{F_i | 1 \leq i \leq 3\}$. Put

$$C_{i,j} \,=\, Z_i \cap Z_j, \quad \text{for} \quad 1 \leq i \neq j \leq 3.$$

Theorem C *One can find g_1, g_2, g_3 in $\mathrm{GL}_2^+(K)$, and an integer $m > 0$, such that, up to modifying the construction by decomposable classes supported on* *the boundary and on $X_{\rm res}$, we have for each $i \leq 3$ and a permutation (i,j,k) of $(1,2,3)$,*

$$m(C_{i,j} - C_{i,k}) \;=\; \mathrm{div}(f_i)$$

for some functions f_i on Z_i.

Here is the **basic idea**: Each Z_i is a Hecke translate of a Hilbert modular surface S_i, which is simply connected. So $Pic(Z_i) = NS(Z_i)$. We show that the homology class $[g_i^{-1}(C_{i,j} - C_{i,k})]$ is trivial in $H^2(S_i, \mathbb{Q}(1))$, up to modifying by the *trivial classes* coming from the boundary and the residual part. Thanks to the explicit description of the algebraic cycles (modulo homological equivalence) on Hilbert modular surfaces (see [HLR], [MuRa1]), one knows that in the situation we are in, the divisor classes are spanned by Hecke translates of modular curves. From this it is not too difficult to show (for suitable $\{g_i\}$) the homological triviality (modulo trivial cycles) of $[g_i^{-1}(C_{i,j} - C_{i,k})]$ *when* the Hilbert modular surface S_i has geometric genus 1; in fact it is enough to know that there is a unique base changed newform π_i of weight 2 over F_i for this. In general one has to deal with several newforms, and one uses a delicate refinement of an argument of Zagier ([Z], page 243). The subtlety comes from the fact that one needs to deal with three quadratic fields at the same time.

A simple example is when $g_1^{-1}g_2, g_1^{-1}g_3$ are diagonal matrices in $\mathrm{GL}_2^+(K)$, not in $\mathrm{GL}_2^+(\mathbb{Q})$, fixed by $\mathrm{Gal}(K/F_1)$, $\mathrm{Gal}(K/F_3)$ respectively. For each $i \leq 3$, C_{ij}, C_{ik} are Hecke translates of modular curves on Z_i.

Thanks to Theorem C, the formal sum

$$\sum_{i=1}^{3} (Z_i, f_i)$$

satisfies $\sum_i \mathrm{div}(f_i) = 0$ as a codimension 2 cycle on X, and hence defines a class in $CH^3(X,1) \otimes \mathbb{Q}$.

Problem: *Compute, for $\omega \in H^{(2,2)}(X(C))$,*

$$\sum_i \int_{Z_i} \log|f_i|\,\omega$$

We can understand this period integral a bit better in the analogous situation where X is the four-fold product of a modular curve $X_0(N)$ for prime level N, the simplification arising from the fact that one can reduce to considering f of the form $\sum_{j=1}^{r} f_{1,j} \otimes f_{2,j} \otimes f_{3,j} \otimes f_{4,j}$, with each $f_{i,j}$ a translate of a modular unit on $X_0(N)$. Here the Hilbert modular surfaces are replaced by the twisted images of $X_0(N)^2 \to X_0(N)^4$, which are not simply connected, but anyhow, their Pic^0 is generated by elementary divisors of degree 0, and this suffices. Work is in progress to prove for $N = 11$, by a combination of theoretical and numerical arguments, that the integral is non-zero for a suitable choice; this will show that the class is of infinite order. We hope to investigate if such a class can give any information on the Bloch-Kato Selmer group at $s = 0$ of the Sym^4 motive, twisted by $\mathbb{Q}(2)$, of $X_0(N)$.

14 The modular complex

Let X be a smooth, toroidal compactification of a Shimura variety of dimension n over its natural field of definition k. Consider the class of closed irreducible subvarieties of X, called *modular*, generated by components of the twisted Hecke translates of Shimura subvarieties, the boundary components, the components of their intersections, and so on. For every $p \geq 0$, let X^p_{mod} denote the set of such subvarieties of codimension p. The **modular points** in this setting will be the CM points and the points arising from successive intersections of components at the boundary. To give a concrete example, consider the case of a Hilbert modular surface X which is obtained by blowing up each cusp into a cycle of rational curves. When $\Gamma_0(\mathcal{N})$ is torsion-free, the *modular points* on X will be the CM points and the points where the rational curves over cusps intersect.

We can now consider the **modular analogue** of the Gersten complex, namely

$$\cdots \to \coprod_{W \in X^{p+2}_{\mathrm{mod}}} K_2(k(W)) \to \coprod_{Z \in X^{p+1}_{\mathrm{mod}}} k(Z)^* \to \coprod_{Y \in X^p_{\mathrm{mod}}} \mathbb{Z}.Y$$

We may look at the homology of this complex, and denote the resulting groups - the first two from the right – by $B^p(X)$ and $B^{p+1}(X, 1)$. There are natural maps

$$B^p(X) \to CH^p(X), \quad \text{and} \quad B^{p+1}(X, 1) \to CH^{p+1}(X, 1)$$

Denote the respective images by $CH^p_{\mathrm{mod}}(X)$ and $CH^{p+1}_{\mathrm{mod}}(X, 1)$. The nice thing about these groups is that since the modular subvarieties are all defined over number fields, the building blocks do not change from $\overline{\mathbb{Q}}$ to $\mathbb{C}$. It will be very interesting (exciting?) to try to verify, in some concrete cases of dimension ≥ 2, whether $CH^p_{\mathrm{mod}}(X)$ and $CH^{p+1}_{\mathrm{mod}}(X,1)$ are finitely generated over k.

Note that the classes we consider in this article are *modular* in this sense. When $n = 2m$ and when $L(s,M)$ has, for a simple submotive M of $H^n(X)$, a *simple zero* at $s = m$, resp. a *simple pole* at $s = m+1$, it is tempting to ask, in view of the known examples, if there is an element of $CH^{m+1}_{\mathrm{mod}}(X,1)$, resp. $CH^m_{\mathrm{mod}}(X)$, which *explains* it. When $n = 2m-1$ and $L(s,M)$ has a simple zero at $s = m$, one could again ask if it is explained by a *modular* element of $CH^m(X)^0$, the homologically trivial part. For modular curves $X_0(N)/\mathbb{Q}$, one has striking evidence for this in the work of Gross and Zagier. For Hilbert modular surfaces it is again true ([HZ], [HLR]). The situation is *not* the same when the order of pole or zero is 2 or more, especially over non-abelian extensions of k ([MuRa1]).

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Dinakar Ramakrishnan
Department of Mathematics
California Institute of Technology, Pasadena, CA 91125.
dinakar@its.caltech.edu