On the cuspidality criterion for the Asai transfer to GL(4)

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Introduction

Let F be a number field and K a quadratic algebra over F, i.e., either $F \times F$ or a quadratic field extension of F. Denote by G the F-group defined by $\mathrm{GL}(2)/K$. Then, given any cuspidal automorphic representation π of $G(\mathbb{A}_F)$, one has (cf. [8], [9]) a transfer to an isobaric automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_F)$ corresponding to the L-homomorphism ${}^LG \to {}^L\mathrm{GL}(4)$. Usually, Π is called the Rankin-Selberg product when $K = F \times F$, and the Asai transfer when K is a quadratic extension. (See also [4]) In the former case, π is a pair (π_1, π_2) of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$, and Π is denoted $\pi_1 \boxtimes \pi_2$, while in the latter case, Π is denoted $\mathrm{As}_{K/F}(\pi)$.

The main purpose of this Note is the following. In the Appendix of [10], a criterion is given for deciding when Π is cuspidal, which is correct for non-dihedral forms π, but has to be modified for the dihedral ones. This error was encountered in the key Asai case by the first author in his work with Anandavardhanan [1]. Here we give two different proofs of the corrected cuspidality criterion. The first one is due to the second author, which slightly modifies and adapts his original arguments in [8], [10], while the second one, due to the first author, is different and may generalize to other situations. There is hardly any difference in the Rankin-Selberg case, so we give a unified single proof in that case. We hope that it is appropriate to present the proofs here as an appendix because Krishnamurthy also needs the corrected criterion for use in his work presented in [5].

The criterion has a natural analogue when F is a local field, where by a cuspidal representation we will mean (as in [7]) a discrete series representation. In order to treat the local and global cases simultaneously, let us write C_F for F^*, resp. the idele class group $\mathbb{A}_F^*/F^*$, in the former, resp. latter, case. In each case class field theory furnishes a natural isomorphism of C_F with the abelianization of the Weil group W_F, and we will, by abuse of notation, use the same letter to denote the corresponding characters of C_F and W_F. Let $\mathcal{A}_n(F)$ denote the set of irreducible isobaric automorphic, resp. admissible, representations of $\mathrm{GL}_n(\mathbb{A}_F)$, resp. $\mathrm{GL}_n(F)$, when F is global, resp. local. Then one knows (cf. [7], [6]) that for every π in $\mathcal{A}_n(F)$, there is a unique partition $n_1 + \cdots + n_r$, and cuspidal representations π_j of $\mathrm{GL}(n_j)$, $1 \leq j \leq r$, such that there is, in the sense of Langlands (cf. [7], [6]), an isobaric sum decomposition $\pi = \boxplus_{j=1}^r \pi_j$, which means in particular that the L-function (resp. ε-factor) of π is the product of the corresponding ones of the π_j. We will say that π is of (isobaric) type $(n_1, \ldots, n_r)$, and call each π_j an isobaric summand of π. We will normalize the order so that $n_i \geq n_j$ if $i \geq j$.

For convenience, we write down the full cuspidality criterion in the Rankin-Selberg and Asai situations, though a correction is needed only in the dihedral case. Recall that a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is dihedral iff $\pi \simeq \pi \otimes \delta$ for a quadratic character δ of C_F. In this case, if E denotes the quadratic extension of F over which δ becomes trivial, there is a character χ of C_E

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such that π is $I_E^F(\chi)$, the representation (automorphically) induced by χ, implying in particular that $L(s,\pi) = L(s,\chi)$. When we began writing this Appendix, we had the criterion that in the Asai case, $\mathrm{As}_{K/F}(\pi)$ is cuspidal iff we have either (i) π, π^θ are twist equivalent, or (ii) π is induced from a biquadratic extension M of F containing K. In the course of writing down the proof we realized that when π is dihedral, (i) actually implies (ii), and the statement below reflects that. We thank Krishnamurthy for a helpful conversation regarding Lemma D in section 2.

We take this opportunity to give, in addition to the cuspidality criterion, the precise occurrence of the various possible isobaric types $(n_1, \dots, n_r)$ of Π.

Theorem A *Let F be a number field or a local field, and π, π' cuspidal representations in $\mathcal{A}_2(F)$. Denote by Π the Rankin-Selberg product $\pi \boxtimes \pi'$, which is in $\mathcal{A}_4(F)$. Then we have the following:*

(a) If π or π' is non-dihedral, then Π is non-cuspidal iff π, π' are twist-equivalent, i.e., $\pi' \simeq \pi \otimes \nu$, for an idele class character ν of F;

(b) If π, π' are both dihedral, then Π is non-cuspidal iff they are both induced from a common quadratic extension.

(c) Π is of type $(3,1)$ iff π, π' are twist equivalent and non-dihedral, while it is of type $(2,1,1)$, resp. $(1,1,1,1)$, iff π, π' are twist equivalent and dihedral, both induced from a unique, resp. non-unique, quadratic extension. Π is of type $(2,2)$ iff π, π' are not twist equivalent, but are both dihedral, induced from a common quadratic extension.

Now we turn to the statement of the result in the Asai situation.

Theorem B *Let F be a number field or a local field, K/F a quadratic extension with non-trivial automorphism θ, and π a cuspidal representation in $\mathcal{A}_2(K)$. Denote by Π the Asai transfer $As_{K/F}(\pi)$, which is in $\mathcal{A}_4(F)$. Then we have the following:*

(a) If π is non-dihedral, then Π is non-cuspidal iff π and $\pi \circ \theta$ are twist-equivalent;

(b) If π is dihedral, then Π is non-cuspidal iff π is induced from a quadratic extension M of K which is biquadratic over F.

(c) Π admits an isobaric summand χ in $\mathcal{A}_1(F)$ iff $\pi^\theta \simeq \pi^\vee \otimes \chi$, for a θ-invariant character χ; in this situation, Π is of type $(3,1)$ iff it is non-dihedral, while it is of type $(2,1,1)$, resp. $(1,1,1,1)$, iff it is dihedral, induced from a unique, resp. non-unique, quartic Galois extension M of F containing K.

(d) Π is of type $(2,2)$ iff π is dihedral, induced from a biquadratic extension of F, and there is no θ-invariant character χ occurring as an isobaric summand of $\pi \boxtimes \pi^\theta$.

The first assertion of part (c) can also be deduced from [3].

1 The Rankin-Selberg case: Proof of Theorem A

When π, π' are both non-dihedral, the proof of (a) is given in [10] in the global case, which also works in the local non-archimedean case when π, π' are both supercuspidal. Suppose π is special, with F local non-archimedean, with parameter $\sigma = \mathrm{St}(\nu) : W_F \times \mathrm{SL}(2,\mathbb{C}) \to GL(2,\mathbb{C})$, $(w,g) \mapsto \nu(w)g$, where $\nu \in \mathcal{A}_1(F)$. If π' is supercuspidal, which need not be dihedral, with parameter $\sigma' : W_F \to \mathrm{GL}_2(\mathbb{C})$, then by [8], Π is the generalized special representation, hence square-integrable, of $\mathrm{GL}_4(F)$

with parameter $\sigma \otimes \sigma'$, which is irreducible. If π, π' are both special, say of the form $\mathrm{St}(\nu)$, $\mathrm{St}(\nu')$ for characters ν, ν' of C_F, then $\pi' \simeq \pi \otimes \nu'\nu^{-1}$, and Π is non-cuspidal of type $(3,1)$. Note that the case $F = \mathbb{C}$ does not occur here, as there is no cuspidal element of $\mathcal{A}_2(F)$, and if $F = \mathbb{R}$, π, π' are both dihedral. So we may disregard the archimedean case for part (a).

When π' is dihedral, say of the form $I_K^F(\chi)$, we have

$$\Pi \simeq I_K^F(\pi_K \otimes \chi),$$

where π_K is the base change of π to $\mathrm{GL}(2)/K$, which is cuspidal since π is non-dihedral. By [2], the (automorphically) induced representation Π is non-cuspidal iff $\pi_K \otimes \chi$ is invariant under the non-trivial automorphism θ of K/F, i.e., iff

$$\pi_K \simeq \pi_K \otimes \lambda, \text{ with } \lambda = \chi^\theta/\chi, \tag{1.1}$$

where χ^θ denotes $\chi \circ \theta$, resp. $\chi^{[\theta]} : w \mapsto \chi(\tilde\theta w \tilde\theta^{-1})$, if viewed as a character of C_K, resp. W_K; here $\tilde\theta$ denotes any element of W_F which lifts $\theta \in W_{K/F}$. Taking central characters, we see that $\lambda^2 = 1$ when (1.1) holds, and moreover, $\lambda^\theta = \lambda^{-1}$. Hence λ is θ-invariant and we may write λ as the base change $\nu_K = \nu \circ N_{K/F}$, for a character ν of C_F. Then π and $\pi \otimes \nu$ have the same base change to $GL(2)/K$, and since π_K is cuspidal, π is isomorphic to either $\pi \otimes \nu$ or to $\pi \otimes (\nu\delta)$, where δ is the quadratic character of C_F attached to K/F. In either case, noting that $\delta \neq \nu$ as $\nu_K = \lambda \neq 1$, we see that π admits a non-trivial quadratic twist, contradicting its non-dihedral nature. So Π must be cuspidal if exactly one of $\{\pi, \pi'\}$ is dihedral. This finishes the proof of part (a).

Let us now turn to (b), and assume that π, π' are both dihedral. If they are induced from a common quadratic extension K, i.e., respectively of the form $I_K^F(\chi)$ and $I_K^F(\chi')$ respectively, it follows by Mackey that we have the isobaric sum decomposition

$$\Pi = I_K^F(\chi) \boxtimes I_K^F(\chi') \simeq I_K^F(\chi\chi') \boxplus I_K^F(\chi^\theta\chi'). \tag{1.3}$$

Hence Π is not cuspidal in this case.

We need to prove the converse, which leads us to the situation when $\pi = I_E^F(\nu)$ and $\pi' = I_K^F(\mu)$, for characters ν, μ of distinct quadratic extensions E, K of F. The assertion in this case is that if $\Pi := \pi \boxtimes \pi'$ is not cuspidal, then π, π' are both induced from a *common* quadratic extension M of F. We have

$$\Pi \simeq I_K^F(\pi_K \otimes \mu), \tag{1.4}$$

where π_K denotes the base change of π to K. If π_K is not cuspidal, then π, π' are both induced from K, and we are done. So we may assume that π_K is cuspidal. Similarly, we may assume that π'_E is cuspidal.

Let θ be the non-trivial automorphism of K/F. When Π is not cuspidal, $\pi_K \otimes \mu$ must be θ-invariant, yielding

$$\pi_K \simeq \pi_K \otimes \lambda, \text{ with } \lambda = \mu^\theta/\mu. \tag{1.5}$$

The character λ must be quadratic as seen by comparing central characters. (If λ were trivial, π' would not be cuspidal.) Now, since $\theta^2 = 1$,

$$\lambda^\theta = \mu/\mu^\theta = \lambda^{-1} = \lambda.$$

Hence λ is θ-invariant, hence is the pull-back by norm of a character ξ of F. Thus

$$\pi_K \simeq (\pi \otimes \xi)_K.$$

Then

$$\pi \simeq \pi \otimes \xi \ \text{ or } \ \pi \simeq \pi \otimes \xi\delta, \tag{1.6}$$

where δ the quadratic character of F attached to K/F. Since $\delta \neq 1$, ξ and $\xi\delta$ are distinct. Moreover, since $\xi_K = \lambda \neq 1$, $\xi \neq 1, \delta$.

Let M, M' be the quadratic extensions of F corresponding to $\xi, \xi\delta$ respectively. Then π is induced from either M or M'.

Next we work with π'. The central character ω' of π' is δ times the restriction μ_0 of μ to F. Also,

$$\mathrm{sym}^2(\pi') = I_K^F(\mu^2) \boxplus \mu_0. \tag{1.7}$$

Since $\lambda = \xi \circ N_{K/F}$, $\lambda^2 = 1 = (\mu/\mu^\theta)^2$, so μ^2 is θ-invariant. Thus

$$Ad(\pi') = \mathrm{sym}^2(\pi') \otimes \omega'^{-1} \ = \xi \boxplus \xi\delta \boxplus \delta. \tag{1.8}$$

Let us, by abuse of notation, use π' to also denote the associated 2-dimensional dihedral representation of the Weil group W_F. Then

$$End(\pi') \simeq Ad(\pi') \oplus 1,$$

which, when restricted to either of three different quadratic extensions, namely K, M and M', contains the trivial representation twice. This implies that the restriction of π' to any of these three quadratic extensions is reducible; hence π' is induced from every one of these extensions. On the other hand, since M/K (resp. M'/K) is cut out by ξ (resp. $\xi\delta$), (1.6) implies that π is induced from either M or M'. Thus π and π' are both induced by a common quadratic extension, completing the proof of part (b).

Now on to part (c). If $\Pi = \pi \boxtimes \pi'$ is non-cuspidal, then we have seen that π, π' are both non-dihedral or both dihedral. The possible isobaric types are $(3,1), (2,2), (2,1,1)$, and $(1,1,1,1)$. Clearly, a GL(1) factor is a summand iff π, π' are twist equivalent. Moreover, there is more than one character of C_F appearing in the isobaric sum decomposition iff π, and hence π', admits a self-twist by a non-trivial character, i.e., they are dihedral. When π, π' are both non-dihedral, we know (by part (a)), that Π is non-cuspidal iff $\pi' \simeq \pi \otimes \nu$, for some $\nu \in \mathcal{A}_1(F)$, which gives

$$\Pi \simeq (\mathrm{sym}^2(\pi) \otimes \nu) \boxplus \omega\nu, \tag{1.9}$$

where ω is the central character of π and $\mathrm{sym}^2(\pi)$ is the symmetric square representation in $\mathcal{A}_3(F)$ attached to π, which is cuspidal by Gelbart-Jacquet since π is non-dihedral. Hence (1.9) is an isobaric sum decomposition of type $(3,1)$. Consequently, the isobaric type $(2,2)$ occurs iff π, π' are dihedral and not twist equivalent. □

2 The Asai case: The first proof of Theorem B

Let $(K/F, \theta, \pi)$ be as in Theorem B, with Π denoting the Asai transfer $\mathrm{As}_{K/F}(\pi)$ in $\mathcal{A}_4(F)$. Then the base change Π_K to $\mathrm{GL}(4)/K$ is (isomorphic to) the Rankin-Selberg product $\pi \boxtimes (\pi \circ \theta)$. When F is global, or when F is local with π non-dihedral and non-special, a proof of the assertion of part (a) of Theorem B is in [10]. If F is non-archimedean and π is special attached to $\nu \in \mathcal{A}_1(K)$, then $\pi \circ \theta$ is also special, attached to $\nu\theta$, hence twist equivalent to π, and $\Pi_K = \pi \boxtimes (\pi \circ \theta)$ is non-cuspidal of type $(3,1)$.

Let us now turn to the proof of part (b), so that π is a dihedral representation $I_E^K(\psi)$, where ψ is a character of C_E. Again we will, by abuse of notation, identify π with the corresponding 2-dimensional induced representation of W_K, which is irreducible as π is cuspidal. So we may also think of $\mathrm{As}_{K/F}(\pi)$ as a 4-dimensional representation of W_F. Note also that in this (dihedral) case, if $\tilde{\theta}$ is an element of $G_F = \mathrm{Gal}(\overline{F}/F)$ lifting θ, $\pi \circ \theta$ is also dihedral of the form $I_{E^\theta}^K(\psi^\theta)$, where $E^\theta = \tilde{\theta}(E)$, and $\psi^\theta(x) = \psi(\tilde{\theta}x\tilde{\theta}^{-1})$, for all x in W_{E^θ}.

Lemma C *Let $(\pi, K/F, \theta)$ be as above, with π dihedral. Then $\Pi_K \simeq \pi \boxtimes (\pi \circ \theta)$ is non-cuspidal iff π and $\pi \circ \theta$ are induced from a common quadratic extension M/K which is Galois over F.*

Proof. By Theorem A we know that the non-cuspidality of Π_K is equivalent to both π and $\pi \circ \theta$ being induced from a common quadratic extension M/K. If M is the unique quadratic extension of K inducing π, then it is necessarily Galois over F as $\pi \circ \theta$ is induced from a Galois conjugate of M. If, on the other hand, π is induced from more than one, hence three, quadratic extensions E_j, $j \in \{1, 2, 3\}$, of K cut out by quadratic characters δ_j of W_K, necessarily with $E_1E_2 = E_1E_3 = E_2E_3$, then $\pi \circ \theta$ is induced from each E_j^θ, $1 \leq j \leq 3$. We must have $M = E_i = E_j^\theta$, for some $i, j \leq 3$. If $i = j$, M itself is Galois over F, and we are done. So let $i \neq j$. Then $E_i^\theta = E_j$, and the remaining third quadratic extension E_k, say, must be preserved by θ and hence Galois over F. It is also a common inducing field for π and $\pi \circ \theta$. $\square$

Now let M be a quartic Galois extension of F containing K, with $\pi = I_M^K(\lambda)$. Then $\pi^\theta = I_M^K(\lambda^{\tilde{\theta}})$, where $\tilde{\theta}$ is a lift of θ to $\mathrm{Gal}(M/F)$. Denote by τ the non-trivial element of $\mathrm{Gal}(M/K)$. Then, as we have seen in section 1, one has by Mackey,

$$\Pi_K \simeq \pi \boxtimes (\pi \circ \theta) \simeq V \boxplus V', \text{ where } V = I_M^K(\lambda\lambda^{\tilde{\theta}}),\ V' = I_M^K(\lambda^\tau\lambda^{\tilde{\theta}}). \tag{2.1}$$

Consequently, the base change of Π_M of Π to M is an isobaric sum of the members of the following set:

$$S := \{\lambda\lambda^{\tilde{\theta}}, \lambda^\tau\lambda^{\tilde{\theta}\tau}, \lambda^\tau\lambda^{\tilde{\theta}}, \lambda\lambda^{\tilde{\theta}\tau}\}. \tag{2.2}$$

When M/K is bi-quadratic, $\tilde{\theta}$ has order 2, and preserves $V_M \simeq \lambda\lambda^{\tilde{\theta}} \boxplus \lambda^\tau\lambda^{\tilde{\theta}\tau}$. It follows that $V^\theta \simeq V$, and Π is non-cuspidal (and not of type $(3,1)$). Part (b) of Theorem B will be proved if we establish the following

Lemma D *Suppose M/F is cyclic. Then Π is non-cuspidal iff π^θ is twist equivalent to π, and in this case, π is also induced from a biquadratic extension M' of F containing K.*

Proof. Note that the set S above has 2 or 4 elements, and that Π is cuspidal when $|S| = 4$ *and* $\mathrm{Gal}(M/F)$ has a unique orbit in S. Since M/F is cyclic here, $\tilde{\theta}$ is of order 4, with $\tau = \tilde{\theta}^2$. S has two elements iff $\lambda\lambda^{\tilde{\theta}}$ is τ-invariant, i.e., $\lambda\lambda^{\tilde{\theta}} = \chi_M$, for some character χ of C_K. Hence $I_M^K(\lambda^{\tilde{\theta}}) \simeq I_M^K(\lambda^{-1}\chi_M)$, implying that when Π is non-cuspidal,

$$\pi^\theta \simeq \pi^\vee \otimes \chi. \tag{2.3}$$

Conversely, suppose (2.3) holds. Then Π_K contains a character, hence Π must contain a character or a two dimensional subrepresentation of W_F, completing this part of Lemma D.

It remains to show that when (2.3) holds, π is induced from a biquadratic extension as well. For any character ξ of C_K, let ξ_0 denote its restriction to C_F. Put $\chi' = \omega^{-1}\chi$, so that $\pi^\theta \simeq \pi \otimes \chi'$. Taking central characters, we get $\omega^\theta = \omega\chi'^2$ implying that ${\chi'_0}^2 = 1$, and $(\chi'^\theta/\chi')^2 = 1$. If χ'_0 is quadratic and unequal to δ, the quadratic character of C_F attached to K, then π will be induced from the biquadratic extension EK, where E is the extension of F cut out by χ'_0. If $\chi'_0 = 1$, then $\chi' = \alpha/\alpha^\theta$ for some character α, implying $\pi \otimes \alpha$ to be a base change of some $\eta \in \mathcal{A}_2(F)$. Let ν be the quartic character of C_F corresponding to M, and $\epsilon = \nu_K$ the quadratic character of C_K corresponding to the quadratic extension M of K. Since $\pi \simeq \pi \otimes \epsilon$, we have $\eta_K \simeq \eta_K \otimes \epsilon$, and so $\eta_K \simeq \eta_K \otimes \nu_K$. Since η is cuspidal, $\eta \otimes \nu$ is isomorphic to η or $\eta \otimes \delta$. Either way, by taking central characters, get $\nu^2 = 1$, which gives a contradiction. So we may assume that $\chi'_0 = \delta$. Then, $As(\pi) \simeq As(\pi^\theta) \simeq As(\pi) \otimes \chi'_0$, and so $As(\pi)$ admits a self-twist by δ. Then, if any character ν of C_F occurs in the isobaric sum decomposition of $As(\pi)$, $\nu\delta$ will also occur in $As(\pi)$. Hence over K, $\pi \boxtimes \pi^\theta$ will contain ν_K with multiplicity 2, which contradicts the cuspidality of π. So θ must move χ, and $\xi := \chi'^\theta/\chi'$ is a quadratic character of C_K fixed by θ such that $\xi_0 = 1$. Such a ξ necessarily cuts out a biquadratic extension M' of F containing K, since otherwise M' would be cyclic and $\xi = \beta_K$ for a quartic character β of C_F, implying that $\xi_0 = \beta^2 \neq 1$. Since ξ appears in $\pi \boxtimes \pi^\vee$, π is induced from (the biquadratic extension) M'. □

Now on to the proof of part (c). We may assume that $\Pi = \mathrm{As}_{K/F}(\pi)$ is not cuspidal. Recall that $\pi^\vee \simeq \pi \otimes \omega^{-1}$, and $\mathrm{Ad}(\pi) \simeq \mathrm{sym}^2(\pi) \otimes \omega^{-1}$, where ω is the central character of π. If π is not dihedral, then $\Pi_K = \pi \boxtimes (\pi \circ \theta)$ is non-cuspidal iff (2.3) holds for some $\chi \in \mathcal{A}_1(K)$, in which case

$$\Pi_K \simeq (\mathrm{Ad}(\pi) \otimes \chi) \boxplus \chi, \tag{2.4}$$

which is isobaric of type $(3,1)$ (since $\mathrm{sym}^2(\pi)$ is cuspidal for non-dihedral π). Since $\mathrm{Gal}(K/F)$ must preserve the two summands in this case, $\mathrm{As}_{K/F}(\pi)$ is also of type $(3,1)$, and χ is forced to be θ-invariant.

It is left to focus on the case when π is dihedral with Π non-cuspidal. Then by (b), there is a biquadratic extension M of F containing K with non-trivial automorphism τ of M/K, such that $\pi = I_M^K(\lambda)$ and (2.1) holds, with $\tilde{\theta}^2 = 1$ and $\tilde{\theta}\tau = \tau\tilde{\theta}$. Note that $\omega = \lambda_0\epsilon_K$, where λ_0 is the restriction of λ to C_K and ϵ is a quadratic character of C_F such that M is cut out over K by $\epsilon_K = \epsilon \circ N_{K/F}$. From the proof of Theorem A we know that Π_K is of type $(2,2)$ iff π is not twist equivalent to π^θ, and in this case Π, being non-cuspidal, is also of type (2,2). So we may assume that (2.3) holds as well, which implies, since $(\lambda_0)_M = \lambda\lambda^\tau$, that $\lambda^{\tilde{\theta}} \in \{(\lambda^\tau)^{-1}\chi_M, \lambda^{-1}\chi_M\}$. So (2.1) yields, since $I_M^K(\chi_M) \simeq \chi \boxplus \chi\epsilon_K$,

$$\Pi_K \simeq W \boxplus W', \quad \text{where} \quad W = I_M^K((\lambda/\lambda^\tau)\chi_M), \ W' = \chi \boxplus \chi\epsilon_K. \tag{2.5}$$

Since ϵ_K is a base change from F, θ fixes it, and consequently, W' descends to a cuspidal element of $\mathcal{A}_2(F)$ iff χ is not θ-invariant. By Theorem A, Π_K is of type $(2,1,1)$ or $(1,1,1,1)$ depending on whether or not π and π^θ are induced only from M, or also from another quadratic extension M'/K. In the former case, λ/λ^τ is not τ-invariant and so W is cuspidal, which implies that θ preserves W and W', hence Π is of isobaric type $(2,1,1)$, resp. $(2,2)$, if χ is, resp. is not, θ-invariant. So we may suppose we are in the latter case, when π is induced from three quadratic extensions M, M', M'' say, with M' being associated to the restriction ξ of the quadratic τ-invariant character λ/λ^τ to C_K, and with M'' associated to $\xi\epsilon_K$. We obtain

$$\Pi_K \simeq (\mathrm{Ad}(\pi) \otimes \chi) \boxplus \chi \simeq \xi\chi \boxplus \xi\chi\epsilon_K \boxplus \chi \boxplus \chi\epsilon_K. \tag{2.6}$$

Since π admits self-twists under ϵ_K, ξ, and $\xi\epsilon_K$, we have $\pi^\theta \simeq \pi^\vee \otimes \psi$, for any ψ in the set Σ of the four characters occurring on the right of (2.6). Suppose M'/F is Galois. In this case ξ is fixed by θ, and Π is of isobaric type $(1,1,1,1)$, resp. $(2,2)$, if χ is, resp. is not, θ-invariant. It remains to consider when M'/F is non-Galois, so that θ sends ξ to $\xi\epsilon_K$. Then Π admits an isobaric summand in $\mathcal{A}_1(F)$ iff ψ is θ-invariant for some $\psi \in \Sigma$, in which case ψ and $\psi\epsilon_K$ are the only characters in

Σ fixed by θ. Hence Π is of type $(2,1,1)$ or $(2,2)$ when M'/F is non-Galois, depending on whether or not there is some $\psi \in \Sigma$ fixed by θ.

Finally part (d) follows in the non-dihedral case from (c) since we showed in the course of the proof of (a) that Π is of type $(3,1)$ when it is non-cuspidal. When π is dihedral, (d) is immediate from (c). □

3 The Asai case: The second proof of Theorem B

We will focus here on the proof of only the key part of (b), namely that given $(\pi, K/F, \theta)$ as in Theorem B, $\Pi = \mathrm{As}_{K/F}(\pi)$ is non-cuspidal iff we have *either* (i) π and π^θ are twist equivalent, *or* (ii) π is a dihedral representation induced from a biquadratic extension M of F containing K. Since it is well known for non-dihedral π (see [9], [10]) that the non-cuspidality is in fact equivalent to (i), we may assume that π is dihedral. We will work totally on the Weil group side and treat π as a representation of W_K, with cuspidality corresponding to irreducibility; we will also say that π has CM by K. Then $\Pi \in \mathcal{A}_4(F)$ identifies with the tensor induction $\mathrm{As}(\pi)$ of π to a 4-dimensional representation of W_F, with the restriction Π_K to W_K being $\pi \otimes \pi^\theta$. Of course $\mathrm{As}(\tau)$ makes sense for any representation τ of W_K, not necessarily two-dimensional, in particular for $\tau = \pi \otimes \pi^\theta$. If M/k is any quadratic extension, let $\delta_{M/K}$ be the corresponding order 2 character of W_K.

In this section we will repeatedly use the following well-known lemma from class field theory. Recall that given a subgroup H of finite index in a group G, there is the notion of the transfer map from $G/[G,G]$ to $H/[H,H]$ which allows one to transfer characters of H to characters of G. If K is a finite extension of a local field (resp. global field), then the transfer map from the characters of W_K to the characters of W_F corresponds to the restriction of the associated characters of $K^\times$ (resp. $\mathbb{A}_K^*/K^\times$) to those of $F^\times$ (resp. $\mathbb{A}_F^*/F^*$).

Lemma E *Let M be a quadratic extension of K which is in turn a quadratic extension of F, which is either a local or a global field. Then M is Galois over F if and only if $\delta_{M/K}$ is invariant under the Galois group of K over F. If $\delta_{M/K}$ is invariant under the Galois group of K over F, then its transfer to W_F is trivial if and only if M is bi-quadratic over F. If the Galois group of M over F is $\mathbb{Z}/4$, then the transfer of $\delta_{M/K}$ to W_F is $\delta_{K/F}$. If M is not Galois over F, then the transfer of $\delta_{M/K}$ to W_F is nontrivial, and not $\delta_{K/F}$.*

Recall that a representation τ of W_F is irreducible iff $\tau \otimes \tau^\vee$ contains the trivial representation exactly once. We apply this to $\tau = \mathrm{As}(\pi)$. Since

$$\mathrm{As}(\pi) \otimes \mathrm{As}(\pi)^\vee = \mathrm{As}(\pi \otimes \pi^\vee),$$

we need to analyze $\mathrm{As}(\pi \otimes \pi^\vee)$.

Assume that π is CM by three distinct quadratic extensions of K, with quadratic characters $\alpha, \beta, \alpha\beta$. In this case,

$$\pi \otimes \pi^\vee = 1 \oplus \alpha \oplus \beta \oplus \alpha\beta,$$

and therefore,

$$\mathrm{As}(\pi \otimes \pi^\vee) = \mathrm{As}(1 \oplus \alpha \oplus \beta \oplus \alpha\beta).$$

It is easy to see that for any two representations V_1 and V_2 of W_K,

$$\mathrm{As}(V_1 \oplus V_2) = \mathrm{As}(V_1) \oplus \mathrm{As}(V_2) \oplus \mathrm{Ind}_K^F(V_1 \otimes V_2^\theta).$$

Using this, and recalling that the Asai lift of a character of W_K is just its transfer to W_F, we can calculate $\mathrm{As}(\pi \otimes \pi^\vee) = \mathrm{As}(1 \oplus \alpha \oplus \beta \oplus \alpha\beta)$, and find that $\mathrm{As}(\pi \otimes \pi^\vee)$ contains more than 1 copy of the trivial representation if and only if either of the following happens:

1. The transfer of one of the characters $\alpha, \beta, \alpha\beta$ of W_K to W_F is trivial.

2. For two distinct characters γ, η among, $\alpha, \beta, \alpha\beta$,

$$\mathrm{Ind}_K^F(\gamma\eta^\theta),$$

   contains the trivial representation.

In case (1), π has a self-twist by a character, say α, such that the corresponding quadratic extension M_α of K is bi-quadratic over F by Lemma 1.

In case (2), we get that $\gamma\eta^\theta$ is the trivial character. Therefore if none of the characters $\alpha, \beta, \alpha\beta$, are invariant under $\mathrm{Gal}(K/F)$, then there are two characters among this which are Galois conjugate, say $\beta = \alpha^\theta$. In this case $\alpha\beta = \alpha\alpha^\theta$, so we have a non-trivial twist of π which is Galois invariant. Since this twisting character is $\alpha\alpha^\theta$ with α quadratic, it must be trivial on $F^\times$, and hence gives rise to a bi-quadratic extension of F by Lemma 1.

If π has CM by a unique quadratic extension L of K, then if $\mathrm{As}(\pi)$ is non-cuspidal, we see by Theorem A that π and π^θ must both be induced from L, forcing it to be Galois over F. To end the proof of (part (b) of) Theorem B, it remains to prove that in this case, $\mathrm{Gal}(L/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$, or rather that the Galois group cannot be $\mathbb{Z}/4$, since it is quite straightforward to see that if $\mathrm{Gal}(L/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$, then indeed $\mathrm{As}(\pi)$ is not cuspidal.

Assume in the rest of the proof that L is a quadratic extension of K which is Galois over F with $\mathrm{Gal}(L/F) = \mathbb{Z}/4 = \langle\sigma\rangle$ (so that $\theta = \sigma^2$), and that $\pi = \mathrm{Ind}_L^K(\chi)$ arises from no other quadratic extension of K, with the further property that $\pi^\sigma \neq \pi^\vee \otimes \mu$ for any character μ of W_K. The first condition translates into the condition that the character χ/χ^{σ^2} is not of order 2, and the second implies that the character $\chi^{\sigma^2}\chi^{\sigma^3}$ is not invariant under σ^2.

Since $\pi \otimes \pi^\sigma = \mathrm{Ind}_L^K(\chi\chi^\sigma) \oplus \mathrm{Ind}_L^K(\chi^{\sigma^2}\chi^\sigma)$, we find that it is a sum of two distinct irreducible representations permuted by σ, therefore $\mathrm{As}(\pi)$ must be irreducible.

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