Contemporary Mathematics

Recovering Cusp Forms on $GL(2)$ from Symmetric Cubes

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To Ram Murty, with admiration!
Janmadina Shubhecchah!

Introduction

Let F be a number field with adele ring $\mathbb{A}_F$, and let π, π' be cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$, say with the same central character. If the symmetric squares of π and π' are equivalent, we know that π' will need to be an abelian, in fact quadratic, twist of π, which amounts to a multiplicity one statement for $\mathrm{SL}(2)$ ([**11**]). It is of interest to ask if the situation is the same for the symmetric cube transfer (from $\mathrm{GL}(2)$ to $\mathrm{GL}(4)$) constructed by Kim and Shahidi (cf. [**8**]). In an earlier paper [**12**], dedicated to Freydoon Shahidi, we showed that the answer is in the negative: If π is of icosahedral type in a suitable sense (which is meaningful even for π without an associated Galois representation), there is a cusp form π^* on $\mathrm{GL}(2)/F$, which we call the "conjugate" of π, having the same symmetric cube, but which is not an abelian twist of π. (We also showed there that such a π is algebraic when the central character ω is algebraic.) In this note we consider the converse direction and show that for π not of solvable polyhedral type, if $\mathrm{sym}^3(\pi) \simeq \mathrm{sym}^3(\pi')$ with π' not an abelian twist of π, then a certain degree 36 L-function has a pole at $s = 1$. If one knew the automorphy of the symmetric fifth power of π, which we do not assume in our principal result below, then such a pole would imply that π is icosahedral with π' twist equivalent to the conjugate π^*. The situation is simpler if one could associate a Galois representation to π.

Given a cusp form π on $\mathrm{GL}(2)/F$, one can define, for every $m \geq 1$, an admissible representation $\mathrm{sym}^m(\pi)$ of $\mathrm{GL}(m+1, \mathbb{A}_F)$, and the principle of functoriality predicts that it is automorphic, which is known (without any hypothesis on π) for $m \leq 4$ ([**3**] for $m = 2$, [**8**] for $m = 3$, [**6**] for $m = 4$). We will say that π is *solvable polyhedral* if $\mathrm{sym}^m(\pi)$ is Eisensteinian for some $m \leq 4$.

Theorem A. *Let F be a number field, and π, π' be cuspidal automorphic representations of* $\mathrm{GL}(2, \mathbb{A}_F)$, *such that π is not solvable polyhedral, with central characters ω, ω' respectively. Suppose we have* $\mathrm{sym}^3(\pi) \simeq \mathrm{sym}^3(\pi')$. *Then π' is also not solvable polyhedral. Also, up to replacing π' by a cubic twist, $\omega' = \omega$,* $\mathrm{sym}^4(\pi') \simeq \mathrm{sym}^4(\pi)$, *and exactly one of the following happens*:

(a) $\pi' \simeq \pi$;

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(b) *The functorial product* $\Pi := \pi \boxtimes \mathrm{sym}^2(\pi')$ *is cuspidal on* $\mathrm{GL}(6)/F$, *and the formal Rankin–Selberg L-function* $L^S(s, \mathrm{sym}^5(\pi) \times (\Pi \otimes \omega)^\vee)$ *is meromorphic and has a pole at* $s = 1$, *where* S *is any finite set of places containing the Archimedean and ramified primes for* π, π'.

Furthermore, in case (b), *we also have*

$$-\operatorname{ord}_{s=1} L^S(s, \mathrm{sym}^5(\pi) \times (\Pi' \otimes \omega)^\vee) \geq 1,$$

where $\Pi' := \mathrm{sym}^2(\pi) \boxtimes \pi'$.

When we are in case (b) of Theorem A, we will say that π is of *icosahedral type*. In [**12**], we used a closely related condition of being s-icosahedral, which is equivalent to the one in part (b) of Theorem A above *if* $\mathrm{sym}^5(\pi)$ is automorphic, as one can see by using the corollary below.

The functorial product $\boxtimes$ used in Theorem A above, also called the Rankin–Selberg product (of automorphic forms), from $\mathrm{GL}(k) \times \mathrm{GL}(r)$ to $\mathrm{GL}(kr)$ is known to exist for $(k, r) = (2, 2)$ ([**11**]) and for $(k, r) = (2, 3)$ by Kim–Shahidi ([**8**]).

When π is of solvable polyhedral type, there is an associated 2-dimensional $\mathbb{C}$-representation ρ of the global Weil group W_F with solvable image, and the fibres of the symmetric cube may be evaluated directly without recourse to automorphic L-functions.

Corollary B. *Suppose* π *is of icosahedral type* (*as in part* (b) *of Theorem* A), *with* π, π' *having the same symmetric cubes and central character*, *but not twist equivalent. If* $\mathrm{sym}^5(\pi)$ *is in addition automorphic*, *then it is equivalent to* $\Pi = \pi \boxtimes \mathrm{sym}^2(\pi')$, *and also to* $\Pi' := \pi' \boxtimes \mathrm{sym}^2(\pi)$. *In particular*, *it is cuspidal. In this case*, $\mathrm{sym}^m(\pi)$ *is also automorphic for* $m = 6, 7$, *and we have isobaric sum decompositions*

$$\mathrm{sym}^6(\pi) \simeq \left((\pi \boxtimes \pi') \otimes \omega^2\right) \boxplus \left(\mathrm{sym}^2(\pi') \otimes \omega^2\right).$$

and

$$\mathrm{sym}^7(\pi) \simeq \left(\mathrm{sym}^2(\pi) \boxtimes \pi' \otimes \omega^2\right) \boxplus \left(\pi' \otimes \omega^3\right).$$

For holomorphic Hilbert modular newforms of weight ≥ 2 generating π, there has been a lot of progress recently on the automorphy of the symmetric fifth power of π (and more), due to the striking (independent) works of Dieulefait ([**2**]) and of Clozel and Thorne ([**1**]). But in this situation one can directly describe the fibres of the symmetric cube (and higher powers) by using the openness of the image of ρ à la Serre and Ribet. See [**10**] for an elegant general result for ℓ-adic representations. For π defined by a form of weight one, which has an attached Galois representation ρ of Artin type (cf. [**15**, Thm. 2.4.1]), one has a lot of information on the symmetric powers (see [**14**]).

In [**12**] it was proved that when π is of icosahedral type, the finite part π_f is $\mathbb{Q}[\sqrt{5}]$-rational when $\omega = 1$, but not rational over $\mathbb{Q}$, and that π' is its Galois conjugate π^* by the non-trivial automorphism of $\mathbb{Q}[\sqrt{5}]$. The fibre of the symmetric cube transfer has two elements in this case (up to character twists). For general ω, an icosahedral π is only rational over the field generated over $\mathbb{Q}[\sqrt{5}]$ by the values of ω. This paper may be viewed as a completion of [**12**], but can be read independently.

Here is some general philosophy, which is not needed for this Note, but underlies the motivation. Langlands conjectures that given any cusp form π on $\mathrm{GL}(n)/F$, there is an associated irreducible, reductive subgroup $G(\pi)$ of $\mathrm{GL}(n,\mathbb{C})$, generalizing the construction of the Zariski closure of the image of a Galois representation ρ, if one may be associated to π. Such a $G(\pi)$ is expected to exist for any π, even one which is not algebraic, and for any finite-dimensional representation r of $\mathrm{GL}(n,\mathbb{C})$, the way the restriction of r to $G(\pi)$ decomposes should explicate the analytic behavior of $L(s,\pi;r)$. When $n=2$, there are not too many choices for $G(\pi)$, and if it does not contain $\mathrm{SL}(2,\mathbb{C})$, then its image in $\mathrm{PGL}(2,\mathbb{C})$ will be finite or its connected component must be a torus; the former should correspond to when π is dihedral, tetrahedral, octahedral or icosahedral, while the latter to when π is dihedral, though of infinite image. The solvable polyhedral case is when one is in the dihedral, tetrahedral or octahedral situation, and here one can give an automorphic definition by the works of Kim and Shahidi ([**7**]). For our study of the fibres of the symmetric cube transfer, the icosahedral case is the one of interest. For a π which is not polyhedral, $G(\pi)$ should contain $\mathrm{SL}(2,\mathbb{C})$ and so all the symmetric power L-functions of π should be entire. It may be remarked that the article [**9**] of Ram Murty with C.S Rajan investigates, under a hypothesis, certain analytic consequences of that case *of general type*.

This article is dedicated to Ram Murty, a friend whose works I have long read with interest. We thank the referee for some comments which improved the presentation of the article.

1. Preliminaries

Let F be a number field. If $\pi_1,\dots,\pi_k$ are isobaric automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_F),\dots,\mathrm{GL}_{n_k}(\mathbb{A}_F)$ respectively, and if $r^1,\dots,r^k$ are polynomial representations of $\mathrm{GL}_{n_1}(\mathbb{C}),\dots,\mathrm{GL}_{n_k}(\mathbb{C})$, then for any (idele class) character μ of F, we have the associated Langlands L-function

$$L(s,\pi_1,\dots,\pi_k;r^1\otimes\cdots\otimes r^k\otimes\mu) \tag{1.1}$$

of degree $d=\sum_{j=1}^k \dim(r^j)$, equipped with an Euler product over the places v of F, convergent in a right half plane. Let S denote the finite set of places of F made up of the union of the Archimedean places and the finite places where some π_j or μ is ramified. Then for every finite place v outside S of norm q_v and uniformizer ϖ_v, there are conjugacy classes $A_v(\pi_{1,v}),\dots,A_v(\pi_{k,v})$ in $\mathrm{GL}_{n_1}(\mathbb{C}),\dots,\mathrm{GL}_{n_k}(\mathbb{C})$ respectively, such that the v-factor of (1.1), denoted by $L_v(s,\pi_1,\dots,\pi_k;r^1\otimes\cdots\otimes r^k\otimes\mu)$, is defined to be

$$\det\Big(I_d-q_v^{-s}\mu(\varpi_v)r^1\big(A_v(\pi_1)\big)\otimes\cdots\otimes r^k\big(A_v(\pi_k)\big)\Big)^{-1}. \tag{1.2}$$

Even at a ramified (resp. Archimedean) place v, one can use the local Langlands correspondence, established for $\mathrm{GL}(n)$ by Harris–Taylor and Henniart, to define the corresponding local factor, but we will not need to use it.

We will denote by r_m the standard (m-dimensional) representation of $\mathrm{GL}(m,\mathbb{C})$, and by sym^j, resp. Λ^j, the symmetric, resp. alternating, jth power of r_m. For any Euler product, $L(s)=\prod_v L_v(s)$, and a set T of places of F, we will denote by $L^T(s)$ the incomplete Euler product $\prod_{v\notin T} L_v(s)$.

For $k=2$ and $r^j=r_{n_j}$, the L-function (1.2) is called the Rankin–Selberg L-function of the pair (π_1,π_2) ([**4**,**13**]), which admit meromorphic continuation and a standard functional equation, such that when both π_j are cuspidal, there is a pole at $s=1$ iff $n_1=n_2$ and moreover, π_2 is the contragredient $\pi_1^\vee$ (of π_1). It is expected (by Langlands' functoriality principle) that there exists an isobaric automorphic form $\pi_1 \boxtimes \pi_2$, called the Rankin–Selberg product or the automorphic tensor product, on $\mathrm{GL}(n_1 n_2)/F$, whose standard L-function agrees with $L(s, \pi_1 \times \pi_2)$, which is a shorthand for $L(s, \pi_1, \pi_2; r_{n_1} \otimes r_{n_2})$. This is known to be true for $n_1 = n_2 = 2$ ([**11**]) and for $(n_1, n_2) = (2, 3)$ by Kim-Shahidi ([**8**]).

For any cuspidal (and hence isobaric) automorphic form π on $\mathrm{GL}(2)/F$, a fundamental result we will use is the existence, for $j \leq 4$, of the symmetric j-th power transfer to $\mathrm{GL}(j+1)/F$, which is classical (due to Gelbart–Jacquet) for $j = 2$ (cf. [**3**]) and for $j = 3, 4$ due to Kim–Shahidi ([**8**]) and Kim ([**6**]). For $j = 2$, one knows (by [**3**]) that (for cuspidal π), $\mathrm{sym}^2(\pi)$ is cuspidal iff π is not *dihedral*, i.e., monomial attached to a character of a quadratic extension K/F. A beautiful result of Kim and Shahidi ([**7**]) asserts that for $j = 3$, resp. $j = 4$, $\mathrm{sym}^j(\pi)$ is cuspidal iff π is not *tetrahedral*, resp. *octahedral*, which means $\mathrm{sym}^2(\pi)$ is not monomial attached to a character of a cyclic cubic, resp. non-normal cubic, extension E/F. We will say that π is *solvable polyhedral* iff it is dihedral, tetrahedral or octahedral.

We will call a cusp form π *s-icosahedral* iff there is another cusp form π^* such that

$$L^S(s, \pi; \mathrm{sym}^5) = L^S(s, \mathrm{sym}^2(\pi^*) \boxtimes \pi \otimes \omega).$$

We will at times write, by abuse of notation $L\big(s, \mathrm{sym}^m(\pi)\big)$ for any m, though $\mathrm{sym}^m(\pi)$ is only known to be an admissible representation of $\mathrm{GL}(m+1, \mathbb{A}_F)$ for general m.

We will also have occasion to make use of Henry Kim's exterior square functoriality $\eta \mapsto \Lambda^2(\eta)$ (cf. [**6**]) from $\mathrm{GL}(4)/F$ to $\mathrm{GL}(6)/F$, such that the standard L-function of $\Lambda^2(\eta)$ agrees with $L(s, \eta; \Lambda^2)$.

2. Two Lemmas

Let π, π' be cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$ which are not abelian twists of each other, and *not solvable polyhedral*, with respective central characters ω, ω', such that $\mathrm{sym}^3(\pi)$ and $\mathrm{sym}^3(\pi')$ are equivalent. Let S denote a finite set of places of F containing the Archimedean and ramified places (for π, π').

Lemma 2.1. *Suppose π, π' are as above, and are not twist equivalent. Then*

(a) *The automorphic representations*

$$\pi \boxtimes \pi', \quad \Pi := \pi \boxtimes \mathrm{sym}^2(\pi'), \quad \Pi' := \pi' \boxtimes \mathrm{sym}^2(\pi)$$

are all cuspidal. Moreover, $\pi \boxtimes \pi'$ does not admit any non-trivial self-twist.

(b) *When $\omega = \omega'$, $\Pi \simeq \Pi'$.*

Proof.

(a) Just the fact that π, π' are not dihedral implies that $\pi \boxtimes \pi'$ is cuspidal unless π' is an abelian twist of π (cf. [**11**]), which we have assumed to be not the case. So we have cuspidality in this case. Suppose $\pi \boxtimes \pi'$ is equivalent to $\pi \boxtimes \pi' \otimes \chi$ for a character χ. Since we know the fibres of $(\pi, \pi') \mapsto \pi \boxtimes \pi'$ (from [**11**]), we see that there must be characters χ_1, χ_2 such that $\chi = \chi_1 \chi_2$, $\pi \simeq \pi \otimes \chi_1$ and $\pi' \simeq \pi' \otimes \chi_2$. Since π, π' do not admit any self-twist, for otherwise they will be dihedral, we are forced to have $\chi_1 = \chi_2 = 1$, thus $\chi = 1$, and the assertion of part (a) is proved for $\pi \boxtimes \pi'$.

Next we will show the assertions for Π and note that by symmetry they also hold for Π'. Consider the Rankin–Selberg L-function

$$L^S(s, \Pi \times \Pi^\vee),$$

which can be rewritten as

$$L^S(s, (\pi \boxtimes \pi^\vee) \times \big(\mathrm{sym}^2(\pi') \boxtimes \mathrm{sym}^2(\pi')^\vee)\big),$$

with isobaric decompositions

$$\pi \boxtimes \pi^\vee = \big(\mathrm{sym}^2(\pi) \otimes \omega^{-1}\big) \boxplus \underline{1}$$

and

$$\mathrm{sym}^2(\pi') \boxtimes \mathrm{sym}^2(\pi')^\vee \simeq \Big(\mathrm{sym}^4(\pi') \otimes {\omega'}^{-2}\Big) \boxplus \Big(\mathrm{sym}^2(\pi') \otimes {\omega'}^{-1}\Big) \boxplus \underline{1},$$

which may be taken as its definition. (Here $\underline{1}$ denotes the trivial automorphic representation of $\mathrm{GL}_1(\mathbb{A}_F)$.) Thus we have the factorization

$$L^S(s, \Pi \times \Pi^\vee) = L_1^S(s) L_2^S(s),$$

where $L_1^S(s)$ is defined to be

$$L^S(s, \mathrm{sym}^2(\pi) \times \mathrm{sym}^4(\pi') \otimes (\omega{\omega'}^2)^{-1}) L^S(s, \mathrm{sym}^2(\pi) \times \mathrm{sym}^2(\pi') \otimes (\omega\omega')^{-1}) \\ \times L^S(s, \mathrm{sym}^2(\pi) \otimes \omega^{-1}),$$

and

$$L_2^S(s) \coloneqq L^S(s, \mathrm{sym}^4(\pi') \otimes {\omega'}^{-2}) L^S(s, \mathrm{sym}^2(\pi') \otimes {\omega'}^{-1}) \zeta_F^S(s).$$

Note that by Jacquet–Shalika ([**5**]), Π is cuspidal iff the (incomplete) Rankin–Selberg L-function $L^S(s, \Pi \times \Pi^\vee)$ has a simple pole at $s = 1$. So we have to show that $L_1^S(s) L_2^S(s)$ has a pole of order one. Since $\mathrm{sym}^m(\eta)$ is cuspidal for $\eta = \pi, \pi'$, we see that $L_2^S(s)$ has a simple pole at $s = 1$, and that the only possible pole of $L_1^S(s)$ could come from the factor $L^S(s, \mathrm{sym}^2(\pi) \times \mathrm{sym}^2(\pi') \otimes (\omega\omega')^{-1})$, which is usually written as $L^S\big(s, \mathrm{Ad}(\pi) \times \mathrm{Ad}(\pi')\big)$, where $\mathrm{Ad}(\pi)$ is the self-dual adjoint $\mathrm{sym}^2(\pi) \otimes \omega^{-1}$. This factor can have a pole iff $\mathrm{Ad}(\pi)$ and $\mathrm{Ad}(\pi')$ are equivalent, which by [**11**] can happen iff π is an abelian twist of π', which we have assumed to be not the case. We are now done with proving part (a).

(b) Suppose $\omega = \omega'$. To prove that $\Pi \simeq \Pi'$, we only have to show the existence of a pole at $s = 1$ of

$$L^S(s, \Pi \times {\Pi'}^\vee) = L^S(s, (\pi \boxtimes \mathrm{sym}^2(\pi)^\vee) \times \big(\pi' \boxtimes \mathrm{sym}^2(\pi')^\vee\big)^\vee).$$

We have

$$\pi \boxtimes \mathrm{sym}^2(\pi)^\vee \simeq \mathrm{sym}^3(\pi) \otimes \omega^{-2} \boxplus \pi \otimes \omega^{-1},$$

and similarly for the corresponding expression involving π'. Thus $L^S(s, \Pi \times {\Pi'}^\vee)$ factors as

$$L^S(s, \mathrm{sym}^3(\pi) \times \mathrm{sym}^3(\pi')^\vee) L^S(s, \mathrm{sym}^3(\pi) \times {\pi'}^\vee \otimes \omega^{-1}) L^S(s, \pi \times \mathrm{sym}^3(\pi')^\vee \otimes {\omega'}^{-1})$$

times the entire function (since π, π' are not twist equivalent):

$$L^S(s, \pi \times {\pi'}^\vee).$$

Note that $L^S(s, \mathrm{sym}^3(\pi) \times \mathrm{sym}^3(\pi')^\vee)$ has a simple pole at $s = 1$ since the symmetric cubes of π, π' are cuspidal and equivalent. The remaining two L-functions dividing $L^S(s, \Pi \times {\Pi'}^\vee)$ are entire and non-zero at $s = 1$. □

LEMMA 2.2. *Let π, π' be as in Lemma* 2.1. *Then, up to replacing π by a cubic twist, we have*

(a) $\omega = \omega'$.
(b) $\mathrm{sym}^4(\pi) \simeq \mathrm{sym}^4(\pi')$.

PROOF. At any $v \notin S$ with uniformizer ϖ_v, let the corresponding conjugacy classes $A_v(\pi), A_v(\pi')$ (of π, π') be represented by $\mathrm{diag}(\alpha_v, \beta_v), \mathrm{diag}(\alpha'_v, \beta'_v)$ respectively, so that

$$\omega_v(\varpi_v) = \alpha_v\beta_v,\ \omega'_v(\varpi_v) = \alpha'_v\beta'_v.$$

A direct calculation shows

$$\Lambda^2(\mathrm{sym}^3(\pi_v)) \simeq \big(\mathrm{sym}^4(\pi_v) \otimes \omega_v\big) \oplus \omega_v^3. \tag{2.1}$$

This is the v-factor of the exterior square of the cusp form $\mathrm{sym}^3(\pi)$ by Kim [**6**]. By the strong multiplicity one theorem for isobaric automorphic representations ([**5**], we obtain a global equivalence

$$\Lambda^2\big(\mathrm{sym}^3(\pi)\big) \simeq \big(\mathrm{sym}^4(\pi) \otimes \omega\big) \boxplus \omega^3. \tag{2.2a}$$

Similarly, as the symmetric cubes of π and π' are equivalent,

$$\Lambda^2\big(\mathrm{sym}^3(\pi)\big) \simeq \big(\mathrm{sym}^4(\pi') \otimes \omega'\big) \boxplus {\omega'}^3. \tag{2.2b}$$

Now since π, π' are not solvable polyhedral, $\mathrm{sym}^4(\pi)$ and $\mathrm{sym}^4(\pi')$ are cuspidal. Hence (2.2b) implies that

$$-\operatorname{ord}_{s=1} L^S\big(s, \Lambda^2\big(\mathrm{sym}^3(\pi)\big) \otimes {\omega'}^{-3}\big) = 1.$$

But this L-function equals, by applying (2.2a):

$$L^S(s, \mathrm{sym}^4(\pi) \otimes \omega{\omega'}^{-3})L^S(s, (\omega/\omega')^3),$$

which will have no pole at $s = 1$ if ω^3 is distinct from ${\omega'}^3$, giving a contradiction. Thus we must have $\omega^3 = {\omega'}^3$, yielding part (a).

Comparing (2.2a) and (2.2b), we also get part (b). $\square$

3. Proof of Theorem A

Let π, π' be as in Theorem A. Since π is not solvable polyhedral, $\mathrm{sym}^3(\pi)$ is cuspidal and does not admit a quadratic self-twist. As π' and π have the same symmetric cubes, the same statements hold for $\mathrm{sym}^3(\pi')$, implying by [**7**] that π' is also not solvable polyhedral.

There is nothing to prove if π and π' are abelian twists of each other, so we may assume that they are not. Then by part (a) of Lemma 2.2, we may assume that π and π' have the same central character ω.

Consider the functorial product $\pi \boxtimes \pi'$ which, by Lemma 2.1, is cuspidal and not equivalent to any non-trivial abelian twist of itself. We have, for any character μ of F,

$$L^S(s, \mathrm{sym}^4(\pi) \times (\pi \boxtimes \pi') \otimes \mu) = L^S(s, \mathrm{sym}^5(\pi) \times \pi' \otimes \mu)L^S(s, \mathrm{sym}^3(\pi) \times \pi' \otimes \mu\omega), \tag{3.1}$$

which also equals (by replacing $\mathrm{sym}^4(\pi)$ by $\mathrm{sym}^4(\pi')$ in the left hand side L-function of (3.1))

$$L^S(s, \mathrm{sym}^5(\pi') \times \pi \otimes \mu)L^S(s, \mathrm{sym}^3(\pi') \times \pi \otimes \mu\omega). \tag{3.2a}$$

We may rewrite (3.2a) by replacing $\mathrm{sym}^3(\pi')$ therein by $\mathrm{sym}^3(\pi)$ and decomposing $\mathrm{sym}^3(\pi)\times\pi$, to obtain

$$\text{(3.2b)}\qquad L^S(s,\mathrm{sym}^5(\pi')\times\pi\otimes\mu)L^S(s,\mathrm{sym}^4(\pi)\otimes\mu\omega)L^S(s,\mathrm{sym}^2(\pi)\otimes\mu\omega^2).$$

Appropriately twisting (3.1) and (3.2b) by $\mathrm{sym}^2(\pi)^\vee\otimes(\mu\omega^2)^{-1}$ $(=\mathrm{sym}^2(\pi)\otimes(\mu\omega^4)^{-1})$, we are led to identify

$$\text{(3.3a)}\quad L^S(s,\mathrm{sym}^5(\pi)\times\mathrm{sym}^2(\pi)\times\pi'\otimes\omega^{-4})L^S(s,\mathrm{sym}^3(\pi)\times\mathrm{sym}^2(\pi)\times\pi'\otimes\omega^{-3})$$

with

$$\text{(3.3b)}\quad L^S(s,\mathrm{sym}^5(\pi')\times\mathrm{sym}^2(\pi)\times\pi\otimes\omega^{-4})L^S(s,\mathrm{sym}^4(\pi)\times\mathrm{sym}^2(\pi)\otimes\omega^{-4})\\ \times L^S(s,\mathrm{sym}^2(\pi)\times\mathrm{sym}^2(\pi)^\vee).$$

We have, for any character ν of F,

$$\text{(3.4a)}\quad L^S(s,\mathrm{sym}^5(\pi')\times\mathrm{sym}^2(\pi)\times\pi\otimes\nu)\\ = L^S(s,\mathrm{sym}^5(\pi')\times\mathrm{sym}^3(\pi')\otimes\nu)L^S(s,\mathrm{sym}^5(\pi')\times\pi\otimes\nu\omega),$$

which equals the product of $L^S(s,\mathrm{sym}^5(\pi')\times\pi\otimes\nu\omega)$ with

$$\text{(3.4b)}\quad \Lambda(s):=L^S(s,\mathrm{sym}^8(\pi')\otimes\nu)L^S(s,\mathrm{sym}^6(\pi')\otimes\nu\omega)^2L^S(s,\mathrm{sym}^4(\pi')\otimes\nu\omega^2).$$

Note that

$$\text{(3.5)}\qquad \Lambda(s)L^S(s,\pi'\times\pi'\otimes\nu\omega^3)==L^S(s,\mathrm{sym}^4(\pi')\times\mathrm{sym}^4(\pi')\otimes\nu,$$

which shows that $\Lambda(s)$ is invertible at $s=1$, being the ratio of two meromorphic functions with simple poles at $s=1$. We now claim that $L^S(s,\mathrm{sym}^5(\pi')\times\mathrm{sym}^2(\pi)\times\pi\otimes\nu)$ is meromorphic with no pole or zero at $s=1$. In view of (3.4a) and (3.4b), it suffices to show that $L^S(s,\mathrm{sym}^5(\pi')\times\pi\otimes\nu\omega)$ is (meromorphic and) invertible at $s=1$. (It is essential to note that we can prove this without knowing the automorphy of $\mathrm{sym}^5(\pi')$, but exploiting the facts that (i) $\mathrm{sym}^3(\pi')$ and $\mathrm{sym}^3(\pi)$ are equivalent, with π',π not twist equivalent and not solvable polyhedral, and (ii) $\mathrm{sym}^j(\pi)$ and $\mathrm{sym}^j(\pi')$ are automorphic for $j\leq 4$.) Indeed, to this end let us first note the factorization

$$\text{(3.6)}\quad L^S(s,\mathrm{sym}^3(\pi')\times\mathrm{sym}^2(\pi')\boxtimes\pi\otimes\otimes\nu\omega)\\ = L^S(s,\mathrm{sym}^5(\pi')\times\pi\otimes\nu\omega)L^S(s,\mathrm{sym}^3(\pi')\times\pi\otimes\nu\omega^2)L^S(s,\pi'\times\pi\otimes\nu\omega^3),$$

whose left hand side, due to the equivalence of the symmetric cubes of π and π', can be reexpressed as

$$\text{(3.7)}\quad L^S(s,\mathrm{sym}^2(\pi')\times\mathrm{sym}^3(\pi)\boxtimes\pi\otimes\otimes\nu\omega)\\ = L^S(s,\mathrm{sym}^2(\pi')\times\mathrm{sym}^4(\pi)\otimes\nu\omega)L^S(s,\mathrm{sym}^2(\pi')\times\mathrm{sym}^2(\pi')\otimes\nu\omega^2)$$

The claim follows.

Consequently, the expression in(3.3b) has a simple pole at $s=1$, which results in a simple pole (at $s=1$) of (3.3a). On the other hand,

$$\text{(3.8a)}\quad L^S(s,\mathrm{sym}^3(\pi)\times\mathrm{sym}^2(\pi)\times\pi'\otimes\omega^{-4})=L^S(s,\mathrm{sym}^3(\pi')\times\mathrm{sym}^2(\pi)\times\pi'\otimes\omega^{-4}),$$

which factors as

$$\text{(3.8b)}\qquad L^S(s,\mathrm{sym}^4(\pi')\times\mathrm{sym}^2(\pi)\otimes\omega^{-4})L^S(s,\mathrm{sym}^2(\pi')\times\mathrm{sym}^2(\pi)\otimes\omega^{-3}).$$

The first L-function of (3.8b) has no pole since (by virtue of π' not being solvable polyhedral) $\mathrm{sym}^4(\pi')$ is cuspidal. And the second L-function of (3.8b) has a pole iff $\mathrm{Ad}(\pi) = \mathrm{sym}^2(\pi)\otimes\omega^{-1}$ is equivalent to $\mathrm{Ad}(\pi')$, which would imply, by [**11**], that π and π' are twist equivalent, which is not the case. Thus the only possibility, by looking at (3.3a), is to have

$$-\operatorname{ord}_{s=1} L^S(s, \mathrm{sym}^5(\pi)\times\Pi\otimes\omega^{-4}) = 1, \tag{3.9}$$

where, as before,

$$\Pi = \mathrm{sym}^2(\pi)\boxtimes\pi'.$$

Note that

$$(\Pi\otimes\omega)^\vee \simeq \Pi\otimes\omega^{-4},$$

which gives the first assertion of case (b) concerning the existence of a pole. The second assertion also follows since we know by Lemma 2.1 that Π and Π' are equivalent. All this has resulted under the assumption that we are not in case (a), i.e., that π,π' are not twist equivalent. Thus we get the dichotomy of Theorem A. □

4. Proof of Corollary B

Suppose in addition to our working hypotheses, we also know that $\mathrm{sym}^5(\pi)$ is automorphic. As before, by replacing π' by a cubic twist if necessary, we may assume that it has the same central character ω as π. Then in case (b) of Theorem A, the fact that

$$-\operatorname{ord}_{s=1} L^S(s, \mathrm{sym}^5(\pi)\times(\Pi\otimes\omega)^\vee) \geq 1$$

implies that $\Pi\otimes\omega$ occurs in the isobaric decomposition of $\mathrm{sym}^5(\pi)$. However, both Π and $\mathrm{sym}^5(\pi)$ are representations of $\mathrm{GL}(6,\mathbb{A}_F)$, which forces the equivalence

$$\mathrm{sym}^5(\pi)\simeq\Pi\otimes\omega = \pi\boxtimes\mathrm{sym}^2(\pi')\otimes\omega, \tag{4.1}$$

since Π is cuspidal. Thus by applying part (b) of Lemma 2.1, $\mathrm{sym}^5(\pi)$ must be cuspidal too, and moreover, using the second assertion of part (b) of Theorem A, $\mathrm{sym}^5(\pi)$ is also equivalent to $\Pi'\otimes\omega = \pi'\boxtimes\mathrm{sym}^2(\pi)\otimes\omega$.

Now we have the identifications $\mathrm{sym}^2(\pi)\boxtimes\pi = \mathrm{sym}^3(\pi)\boxplus\pi\otimes\omega$, $\mathrm{sym}^3(\pi)\simeq\mathrm{sym}^3(\pi')$, and

$$\pi'\boxtimes\mathrm{sym}^3(\pi') = \mathrm{sym}^4(\pi')\boxplus\mathrm{sym}^2(\pi')\otimes\omega, \tag{4.2}$$

which can be taken to be the definition of $\pi'\boxtimes\mathrm{sym}^3(\pi')$. Thus we are able to realize, in our (icosahedral) case, the functorial product of $\mathrm{sym}^5(\pi)$ and π by setting

$$\mathrm{sym}^5(\pi)\boxtimes\pi\simeq\mathrm{sym}^4(\pi')\otimes\omega\boxplus\mathrm{sym}^2(\pi')\otimes\omega^2\boxplus\pi'\boxtimes\pi\otimes\omega^2. \tag{4.3}$$

On the other hand, we have the factorization

$$L(s,\mathrm{sym}^5(\pi)\times\pi) = L\big(s,\mathrm{sym}^6(\pi)\big)L(s,\mathrm{sym}^4(\pi)\otimes\omega), \tag{4.4}$$

which is in fact correct at every place by the work of Shahidi ([**13**]). By Lemma 2.2, $\mathrm{sym}^4(\pi)$ and $\mathrm{sym}^4(\pi')$ are equivalent, implying (by a comparison of (4.3) and (4.4), that

$$L\big(s,\mathrm{sym}^6(\pi)\big) = L(s,\mathrm{sym}^2(\pi')\otimes\omega^2)L(s,\pi'\boxtimes\pi\otimes\omega^2). \tag{4.5}$$

Thus we may realize $\mathrm{sym}^6(\pi)$ as the isobaric automorphic representation

$$\big(\mathrm{sym}^2(\pi')\otimes\omega^2\big)\boxplus\big(\pi'\boxtimes\pi\otimes\omega^2\big)\,. \tag{4.6}$$

The assertion about $\mathrm{sym}^7(\pi)$ is established in a similar manner, and the proof is left as an exercise. $\square$

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