Decomposition and Parity of Galois Representations Attached to GL(4)

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Introduction

Let F be a number field, and π an isobaric ([La]), algebraic ([Cℓ1]) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. We will call π *quasi-regular* iff at every archimedean place v of F, the associated n-dimensional representation σ_v of the Weil group W_{F_v} (defined by the archimedean local correspondence) is *multiplicity free*. For example, when $n = 2$ and $F = \mathbb{Q}$, a cuspidal π is quasi-regular exactly when it is generated by a holomorphic newform f of weight ≥ 1. Recall that π is *regular* iff at each archimedean v the *restriction* of σ_v to $\mathbb{C}^*$ is multiplicity free; hence any regular π is quasi-regular, but not conversely.

When F is totally real, an algebraic automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is said to be *totally odd* iff it is odd at each archimedean place v, i.e., iff the difference in the multiplicities of 1 and -1 as eigenvalues of complex conjugation in σ_v is at most 1; in particular if n is even, these two eigenvalues occur with the same multiplicity. One readily sees that any quasi-regular π is totally odd, but not conversely. When $n = 2$, π is said to be *even* (or that it has even parity) at an archimedean place v (of F) iff if it is not odd at v.

Still with F totally real, let c be one of the $[F : \mathbb{Q}]$ complex conjugations in the absolute Galois group $\mathfrak{G}_F = \mathrm{Gal}(\overline{F}/F)$. Recall that an n-dimensional $\overline{\mathbb{Q}}_p$-representation ρ of $\mathfrak{G}_F$ is *odd relative to c* if the trace of $\rho(c)$ lies in $\{1, 0, -1\}$. It is *odd* if it is so relative to every c. When $n = 2$, ρ is said to be *even relative to c* if it is not odd relative to c, which is the same as the determinant of $\rho(c)$ being 1.

One says that an isobaric, algebraic π on $\mathrm{GL}(n)/F$ has a fixed *archimedean weight* iff there is an integer w such that at every archimedean place v of F, the restriction to $\mathbb{C}^*$ of $\sigma_v \otimes |\cdot|^{(n-1)/2}$ is a direct sum of characters

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$z \to z^{p_j} \overline{z}^{q_j}$ with $p_j + q_j = w$ ($\forall j$). Every *cuspidal* algebraic π has such a weight ([Cℓ1]). The first object here is to provide a proof of the following assertion, which was established for the regular (algebraic), cuspidal case in an earlier preprint [Ra1] which has remained unpublished:

Theorem A   *Let F be a totally real number field, $n \leq 4$, p a prime, π a quasi-regular algebraic, isobaric automorphic representation of $GL_n(\mathbb{A}_F)$ of a fixed archimedean weight, and ρ an associated n-dimensional, Hodge-Tate $\overline{\mathbb{Q}_p}$-representation of $\mathfrak{G}_F$ whose local L-factors agree with those of π (up to a shift) at almost all primes P of F. Then the semisimplification ρ^{ss} (of ρ) does not contain any irreducible 2-dimensional Galois representation which is even relative to some complex conjugation c.*

Since π is *algebraic*, it is expected by a conjecture of Clozel ([Cℓ1] that there is an associated Galois representation ρ (see also [BuzG]). When π is *regular and selfdual* and F totally real, the existence of ρ has been well known for some time for GL(n) ([Cℓ2], [PL], [CℓHLN]). Now it is also known for π *non-selfdual* and regular, by the important recent work of Harris, Lan, Taylor and Thorne ([HLTT]). When π is quasi-regular, the algebraicity is already in [BHR] (for F totally real), where one finds the equivalent notion of a *semi-regular, motivically odd* cusp form, and much progress has been made in the construction of ρ in the selfdual case in the thesis of Goldring ([Go1, Go2]), when the base change of π to a suitable CM extension K with $K^+ = F$ is known to descend to a holomorphic form on a suitable unitary group associated to K/F.

Here are some remarks on the proof of Theorem A. When ρ^{ss} is a direct sum $\eta_1 \oplus \eta_2$ with each η_j two-dimensional, one sees easily that the summands must have the same parity. But then we are left with the subtle task of ruling out both of them being *even*. We first appeal to the exterior square construction of Kim ([K]), which supplies (using the Langlands-Shahidi method) an isobaric automorphic form $\Pi = \Lambda^2(\pi)$ on GL(6)/F, and then, more importantly, we make use of the fact that one has some control, thanks to the works of Shahidi ([Sh1, Sh2]) and Ginzburg-Rallis ([GR]) concerning the analytic properties of the exterior cube L-function of Π, which has degree 20. Luckily, this L-function is also related to the square of a twist of the symmetric square L-function of π, and we exploit this. We also appeal to base change for GL(n) ([AC]). The proof given here is a strengthened form of the one in [Ra1], and we avoid making use of either regularity or cuspidality (of π). This article is completely self-contained, however, and one does not need to refer to [Ra1].

When the semisimplification of an p-adic representation τ decomposes as $\oplus_{j=1}^{r} tau_j$ with each τ_j irreducible of dimension n_j, we will say that τ

has decomposition type $(n_1, n_2, \ldots, n_r)$. Similarly, if π is an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of the form $\boxplus_{j=1}^r \pi_j$, with each π_j cuspidal of $\mathrm{GL}_{n_j}(\mathbb{A}_F)$, we will say that π has isobaric type $(n_1, n_2, \ldots, n_r)$.

One motivation for the assertion of Theorem A (in [Ra1]) was to help establish the *irreducibility of ρ for π cuspidal and regular, algebraic* on $\mathrm{GL}(4)/F$, at least *for large enough p*, generalizing the classical results of Ribet for $n = 2$, and Blasius-Rogawski ([BRo2]) for $n = 3$ *and π essentially self-dual:*

Theorem B *Let F be totally real, $n \leq 4$, and π an isobaric, algebraic automorphic representation of $GL_n(\mathbb{A}_F)$ which is regular at every archimedean place, with associated Hodge-Tate representation ρ of $\mathfrak{G}_F$. Assume that ρ is crystalline and $p - 1$ is greater than the twice the largest difference in the Hodge-Tate weights. Then the (isobaric) type of π determines the (reducibility) type of ρ^{ss}. In particular, if π is cuspidal, then ρ is irreducible.*

It should be remarked that recently, such an irreducibility result for essentially selfdual, cuspidal π has been established by F. Calegari and T. Gee ([CaG]) *with no condition on p* (and even for $n = 5$), and the route they take appeals to, and further extends, recent modularity results, as well as [BCh] on signs of selfdual representations and the results on even Galois representations in [Ca]; see also [DV] and all other related references in [CaG]. In the non-selfdual case, the authors of [CaG] give a condensed form of a simple argument of [Ra1], which we give in its entirety in section 3 below, both to correct an identity slightly and also to check that the isobaric type of π agrees with the decomposition type of ρ.

Our proof of Theorem B makes use of Theorem A in conjunction with a theorem of Richard Taylor establishing the potential automorphy (over a totally real extension) of a class of 2-dimensional p-adic representations, as well as some tricks involving L-functions. Our original approach was to require potential automorphy over a common totally real extension *simultaneously* for two 2-dimensional odd representations of the type considered in [Ta2, Ta3]. The proof given here is simpler and does not require this more stringent condition. We also make use of the recent work [HLTT] on non-selfdual representations. In general, the arguments here are a bit more involved than in [Ra1] so as to deal (for Theorem A) also with the quasi-regular, non-regular case. If $F = \mathbb{Q}$, in many cases one can also use [Ki] for the proof of the $(2, 2)$-case of Theorem B (instead of [Ta2, Ta3]), once the even summands are ruled out (as in Theorem A).

As a companion to Theorem B, we will also establish the following

Theorem B *Let F be a totally real number field, p a rational prime. and σ, σ' 2-dimensional semisimple, odd, crystalline p-adic representations of*

$\mathfrak{G}_F$ *of the same weight w, such that each has distinct Hodge-Tate weights in an interval of length at most $(p-1)/2$. Assume moreover that $\sigma \oplus \sigma'$ is associated to an isobaric automorphic representation π of $GL_4(\mathbb{A}_F)$. Then there are a finite set S of places of $\mathbb{Q}$ such that $L^S(1 + w/2, \sigma) \neq 0$. Moreover,*

$$-\mathrm{ord}_{s=1} L^S(s, \sigma^\vee \otimes \sigma') = \dim(\sigma^\vee \otimes \sigma')^{\mathfrak{G}_F}.$$

This assertion is as predicted by the Tate conjecture when σ, σ' occur in the p-adic étale cohomology of smooth projective varieties. It is essentially a consequence of Theorem B if we also assume that π is regular and algebraic (in which case the oddness of σ, σ' would in fact be a consequence, thanks to Theorem A). Furthermore, this result is stated (for $F = \mathbb{Q}$) in [Ra1], and referred to by Skinner and Urban in [SU] (Thm. 3.2.4), without even assuming that $\sigma \oplus \sigma'$ is automorphic, and this stronger version will be the subject of a sequel to this paper. However, Theorem C as stated above appears to be sufficient for the application in [SU], section 3.2.6. To elaborate, in [SU], $\sigma \oplus \sigma'$ is modular, corresponding to a cusp form Π on $GSp(4)/\mathbb{Q}$. By Arthur's recent work ([A]), one can transfer Π to an isobaric automorphic form π on $GL(4)/\mathbb{Q}$ such that the degree 4 L-functions of π and of Π coincide at all but a finite number of primes. (One can say more but it is not needed for this application.) Now Theorem C above applies with this π associated to (σ, σ'). It will be interesting to extend this result without making the regularity assumption on the Hodge-Tate weights, possibly by making use of recent works of Calegari and Geraghty.

1   Reductions

From here on, let π be as in Theorem A, with associated ρ. Before deriving some lemmas, let us make some simple observations. To begin, note that if π is an isobaric sum of idele class characters, then ρ^{ss} must be a direct sum of one-dimensional p-adic representations. Indeed, each idele class character ν appearing as an isobaric summand of π is algebraic and corresponds to an p-adic character ν by [Se], and the fact that ρ^{ss} is a direct sum of such

characters follows by Tchebotarev. So we may take $2 \leq n \leq 4$; of course the case $n = 1$ is classical.

Next suppose there is a cuspidal automorphic representation β of $\mathrm{GL}_2(\mathbb{A}_F)$ occurring as an isobaric summand of π. Then by our hypothesis (in Theorem A), β is a totally odd, algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, which corresponds to a holomorphic Hilbert modular newform with multi-weight $(k_1, k_2, \ldots, k_m)$, $m = [F : \mathbb{Q}]$, such that the $k_j \geq 1$ are all of the same parity. By Blasius-Rogawski ([BRo1]), Taylor ([Ta1]) in the regular case ($k_j \geq 2$), and by Jarvis ([Jar]) when one of the k_j is 1, we know the existence of an irreducible 2-dimensional p-adic representation τ of the absolute Galois group of F attached to β, i.e., with coincident local factors almost everywhere. Then τ is totally odd, and there is nothing else to say if $n = 2$; therefore we assume in the rest of this section that n is 3 or 4. If we write $\pi = \beta \boxplus \beta'$, with β' is an algebraic, isobaric automorphic representation of $\mathrm{GL}_{n-2}(\mathbb{A}_F)$, then β' also corresponds to an $(n-2)$-dimensional p-adic representation τ' which is totally odd if $n = 4$.

Suppose $n = 3$ and π is a cuspidal. Then $\pi' := \pi \boxplus 1$ is an isobaric automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$ satisfying the same hypotheses of Theorem A, with corresponding Galois representation $\rho' = \rho \oplus 1$. The proof of Theorem A for π' will imply the same for π.

In view of the above remarks, we may assume from here on that $n = 4$ and that either π is cuspidal or isobaric of type $(3, 1)$. Let $\Lambda^2(\pi)$ denote the isobaric automorphic representation of $\mathrm{GL}_6(\mathbb{A}_F)$ defined by Kim ([K]), which corresponds to $\Lambda^2(\rho)$.

Assume moreover that

$$\rho^{\mathrm{ss}} \simeq \tau_1 \oplus \tau_2, \quad \text{with } \dim(\tau_j) = 2, \; j = 1, 2. \tag{1.1}$$

Lemma 1.1 *The decomposition* (1.1) *of* ρ^{ss} *precludes the possibility of* π *being of type* $(3, 1)$*, i.e., we cannot have an isobaric sum decomposition of the form*

$$(*) \qquad\qquad \pi \simeq \eta \boxplus \nu,$$

with η *a cuspidal automorphic representation of* $\mathrm{GL}_3(\mathbb{A}_F)$ *and* ν *an idele class character. In fact, when* π *is of type* $(3, 1)$*,* ρ^{ss} *must be of the same type, i.e.,* ρ *must contain a sub or a quotient which is irreducible of dimension* 3*.*

Proof Let ω, resp. ω_η, be the central character of π, resp. ν, so that $\omega_\nu = \omega \nu^{-1}$. Note that $(*)$ implies

$$\Lambda^2(\pi) \simeq (\eta^\vee \otimes \omega_\pi \nu^{-1}) \boxplus (\eta \otimes \nu),$$

which can be seen by checking the unramified local factors and applying the strong multiplicity one theorem ([JS1]). It follows that

$$\Lambda^2(\pi) \boxplus \nu^2 \boxplus \omega\nu^{-1} \simeq (\pi^\vee \otimes \omega\nu^{-1}) \boxplus (\pi \otimes \nu). \qquad (1.2)$$

The algebraicity of π implies the same for η, ν and ω, and the latter pair defines (by [Se]) p-adic characters, again denoted by ν, ω, of the absolute Galois group of F. We get (by Tchebotarev)

$$\Lambda^2(\rho^{\mathrm{ss}}) \oplus \nu^2 \oplus \omega\nu^{-1} \simeq (\rho^{\mathrm{ss}})^\vee \otimes \omega\nu^{-1} \boxplus (\rho^{\mathrm{ss}} \otimes \nu). \qquad (1.3)$$

On the other hand, applying the exterior square operation to both sides of the decomposition (1.1) yields

$$\Lambda^2(\rho^{\mathrm{ss}}) \oplus \nu^2 \oplus \omega\nu^{-1} \simeq (\tau_1 \otimes \tau_2) \oplus \omega_1 \oplus \omega_2 \oplus \nu^2 \oplus \omega\nu^{-1}, \qquad (1.4)$$

where ω_j is the determinant of τ_j. Comparing (1.3) and (1.4), and using (1.1) and the isomorphism $\tau_j^\vee \simeq \tau_j \otimes \omega_j^{-1}$, we see that τ_1 and τ_2 must both be reducible. If we write

$$\tau_j \simeq \mu_j \oplus \mu_j', \ \dim(\mu_j) = \dim(\mu_j') = 1,$$

then the characters on the right are both Hodge-Tate, since ρ has that property (by assumption), hence are locally algebraic and correspond to idele class characters of F (cf. [Se]). This then forces the isobaric decomposition of the form

$$\pi \simeq \mu_1 \boxplus \mu_1' \boxplus \mu_2 \boxplus \mu_2',$$

which contradicts the cuspidality of η.

To finish the proof of the Lemma, we still need to check that ρ cannot be irreducible when $(*)$ holds. For this, note that (1.3) still holds because its proof did not use (1.1). But then, since the left hand side contains 1-dimensionals, ρ cannot be irreducible.

$\square$

Lemma 1.2 *Let π be cuspidal and satisfy the hypotheses of Theorem A, together with the decomposition (1.1) of the semisimplification of the associated ρ. Let K be a totally real number field which is solvable and Galois over F. If the base change π_K of π to $GL(4)/K$ is Eisensteinian, then ρ must be irreducible.*

Proof If π_K is Eisensteinian, the cuspidality of π implies that $\mathrm{Gal}(K/F)$ acts transitively on the set of cuspidal automorphic representations occurring in the isobaric sum decomposition of π_K (cf. [AC]). This forces a

decomposition $\pi_K \simeq \beta_1 \boxplus \beta_2$, where each β_j is an isobaric automorphic representation of $\mathrm{GL}(2, \mathbb{A}_K)$. It follows that there are intermediate fields L, E such that $F \subset E \subset L \subset K$ with $[L : E] = 2$, and a cuspidal automorphic representation β of $\mathrm{GL}(2, \mathbb{A}_L)$ such that π_E is cuspidal and is the automorphic induction $I_L^E(\beta)$. Now since π_E is the base change of π, at every archimedean place w of E, the eigenvalue set of complex conjugation in the associated 4-dimensional representation σ_w of $W_{E_w} \simeq W_{\mathbb{R}}$ must remain $\{1, -1, 1, -1\}$. As L is totally real, w splits in it, say into $\{\tilde{w}_1, \tilde{w}_2\}$, and so $\beta_{\tilde{w}_j}$ is forced to be odd for each j. In other words, β is a totally odd, algebraic cusp form on $\mathrm{GL}(2)/L$, and by [BRo1], [Ta1] and [Jar], it corresponds to a 2-dimensional, irreducible, odd Galois representation r. Then the induction R, say, of r to the absolute Galois group $\mathfrak{G}_E$ of E corresponds by functoriality to π_E. Moreover, the cuspidality of π implies that β, and hence r, is not θ-invariant, where θ denotes the non-trivial automorphism of L/E. Then R is irreducible by Mackey, and must be the restriction of ρ to $\mathfrak{G}_E$; this can be checked, for example, at almost all places and then deduced by Tchebotarev. In any case, it implies that ρ itself must be irreducible.

$\square$

Note that in this section we did not need π to be quasi-regular, only that it is totally odd.

2   Proof of Theorem A

Fix a totally real number field F. In view of the Lemmas of the previous section, we may assume that $n = 4$ with π satisfying the hypotheses of Theorem A with associated ρ, and that π remains cuspidal upon base change to any finite solvable Galois extension K which is totally real.

Suppose we have a decomposition of the form (1.1), i.e., with ρ^{ss} being the direct sum of two 2-dimensional p-adic representations τ_1, τ_2 of $\mathfrak{G}_F$, with $\omega_j = \det(\tau_j)$, which we will also view (by class field theory) as idele class characters ω_j of F. The Hodge-Tate hypothesis on ρ implies that as a p-adic character, ω_j is locally algebraic for $j = 1, 2$, and is thus associated to an algebraic Hecke character, again denoted ω_j, of the idele class group C_F of (the totally real) F, which must be an integral power of the norm character times a finite order character ν_j. It follows that as a p-adic character,

$$\omega_j = \chi^{a_j} \nu_j,$$

where χ is the p-adic cyclotomic character. Clearly, $\omega = \det(\rho) = \omega_1 \omega_2$. Note that τ_j is even relative to some c iff a_j and ν_j have the same parity at an archimedean place.

Lemma 2.1 *We have $a_1 = a_2$. Also, ν_1 and ν_2 have the same parity.*

Proof We may, and we will, assume that $a_1 \geq a_2$. A twist of π by $|\cdot|^t$ is, for some $t \in \mathbb{R}$, unitary; call this representation π^u. Since π is algebraic, t lies in $\frac{1}{2}\mathbb{Z}$. Then at any finite place v where π is unramified, we know by the Rankin-Selberg theory that the inverse roots $\alpha_{j,v}$, $1 \leq j \leq 4$, defining the local factor of π^u, satisfy $|\alpha_{j,v}| < (Nv)^{1/2}$. Consequently, any inverse root of $\Lambda^2(\pi_v^u)$ is strictly bounded in absolute value by Nv. Now,

$$\Lambda^2(\pi^u) \simeq \Lambda^2(\pi \otimes |\cdot|^t) \simeq \Lambda^2(\pi) \otimes |\cdot|^{2t}.$$

So $|a_j - 2t| < 1$ for $j = 1, 2$, and since $a_j, 2t \in \mathbb{Z}$, $a_1 = 2t = a_2$ (as desired).

Consequently, the central character of π, which is associated to $\omega = \det(\rho)$, equals $\nu_1\nu_2|\cdot|^{2a}$, where $a = a_1 = a_2$. Since ω is even, ν_1 and ν_2 have the same parity.

$\square$

Assume the following:

For $j = 1, 2$, ν_j has the same parity as a at some archimedean place v_0.
(2.1)

We will obtain a contradiction below, leading to Theorem A.

Lemma 2.2 *Let K be a finite abelian extension of F in which v_0 splits. Then $\omega_{j,K}$ ($= \omega_j \circ N_{K/F}$) cannot occur, for either j, in the isobaric sum decomposition of $\Lambda^2(\pi_K)$. In particular, $L(s, \Lambda^2(\pi) \otimes \omega_{j,K}^{-1})$ has no pole at $s = 1$.*

It should be noted that this crucial Lemma will be false if ω_j were to be totally odd; in that case $L(s, \pi; \mathrm{sym}^2 \otimes \omega_{j,K}^{-1})$ will be regular at $s = 1$.

Proof First consider when π is regular at the archimedean place v_0, so that the associated 4-dimensional representation of $W_\mathbb{R}$ is of the form

$$\sigma_{v_0} \simeq I(\xi) \oplus I(\xi'),$$

for distinct, non-conjugate characters $\xi : z \mapsto z^p \overline{z}^{w-p}$ and $\xi' : z \mapsto z^r \overline{z}^{w-r}$ of $\mathbb{C}^*$, where I denotes the induction from $\mathbb{C}^*$ to $W_\mathbb{R}$. Then $I(\xi) \otimes I(\xi')$ decomposes as $I(\xi\xi') \oplus I(\xi\overline{\xi}')$. It follows that

$$\Lambda^2(\sigma_\infty) \simeq I(z^{p+r}\overline{z}^{2w-p-r}) \oplus I(z^{p+w-r}\overline{z}^{w-p+r}) \oplus sgn^{w+1}|\cdot|^w \oplus sgn^{w+1}|\cdot|^w,$$
(2.2)

where w is the archimedean weight of π (with $w - a_j \in \mathbb{Z}$). Since K/F is abelian, the base change π_K (of π) makes sense ([AC]), and for any archimedean place $\tilde{v_0}$ of K above v_0, $K_{\tilde{v_0}} \simeq \mathbb{R}$ (since v_0 splits in K by assumption) and $\sigma_{\tilde{v_0}} \simeq \sigma_{v_0}$ is the parameter of $\pi_{K,\tilde{v_0}}$ if $\tilde{v_0} \mid v_0$. So the occurrence of $\omega_{j,K}$ (for either j) in the isobaric decomposition of $\Lambda^2(\pi_K)$ implies that $\omega_{j,K,\tilde{v_0}}$ will occur in $\Lambda^2(\sigma_\infty)$, and so must equal $sgn^{w+1}|\cdot|^w$. Now since ω_{j,v_0} is of the form $\omega_{j,v_0}|\cdot|^a$, we see that $w = a$ and that the parity of ω_j at v_0 is the opposite of that of a, a contradiction!

Next look at when π_{v_0} is quasi-regular, but not regular. Then we must have

$$\sigma_{v_0} \simeq I(\xi) \oplus (1 + \mathrm{sgn})|\cdot|^{w/2},$$

where again ξ is given as above. In this case, since $I(\xi)$ is twist invariant under sgn, the sign character, we obtain

$$\Lambda^2(\sigma_\infty) \simeq (I(\xi) \oplus \mathrm{sgn}|\cdot|^w)^{\oplus 2}. \tag{2.3}$$

Again, we see that $\omega_{j,\tilde{v_0}}$ cannot be a summand on the right because of parity.

$\square$

As the proof shows, a is necessarily the archimedean weight of π. Let

$$\alpha = \nu_1/\nu_2, \ \nu = \nu_1\nu_2, \quad \text{and} \quad E = F(\alpha, \nu),$$

the compositum of the cyclic extensions of F cut out by α and ν. Then by Lemma 2.1, $\alpha = \omega_1/\omega_2$, and $\omega = \nu|\cdot|^{2a}$. Moreover, the total evenness of α and ν implies that E is totally real. By construction, $\nu_{1,E} = \nu_{2,E}$ and $\nu_E = 1$, so if we put

$$\mu := \nu_{1,E}|\cdot|_E^a,$$

then μ^2 is $|\cdot|_E^{2a}$ on C_E.

The reason for considering E is that π becomes essentially selfdual over E:

Lemma 2.3 *We have*

(a) $\pi_E^\vee \simeq \pi_E \otimes \mu^{-1}$;

(b) *The incomplete L-function $L^T(s, \pi_E; \mathrm{sym}^2 \otimes \mu^{-1}))$ has a pole at $s = 1$ of order 1, where T is a finite set of places of E containing the archimedean and ramified primes;*

(c) $\Lambda^2(\pi_E \otimes |\cdot|_E^{-a})$ *is selfdual.*

Proof (a) By assumption, ρ^{ss} is $\tau_1 \oplus \tau_2$, and for either j, $\tau_j^\vee \simeq \tau_j \otimes \omega_j^{-1}$. So with μ being the restriction of ω_j to E, then

$$\rho_E^{\mathrm{ss}\vee} \simeq \rho_E^{\mathrm{ss}} \otimes \mu^{-1}.$$

Comparing L-functions, we get

$$L^T(s, \pi_E^\vee) = L^T(s, \pi_E \otimes \mu^{-1}).$$

It follows by the strong multiplicity one theorem for isobaric automorphic representations ([JS1]) that $\pi_E^\vee$ is isomorphic to $\pi_E \otimes \mu^{-1}$.

(b) We have

$$L^T(s, \pi_E \times \pi_E \otimes \mu^{-1}) = L^T(s, \pi_E, \Lambda^2 \otimes \mu^{-1}))L^T(s, \pi_E, \mathrm{sym}^2 \otimes \mu^{-1})).$$

One knows by the Rankin-Selberg theory that the L-function on the left has a pole at $s = 1$ of order ≥ 1, and so the assertion follows in view of Lemma 2.2 (with $K = E$).

(c) This follows from (a).

$\square$

For any isobaric automorphic representation Π of $\mathrm{GL}(6)/F$ and a character ξ of F, let $L^T(s, \Pi; \Lambda^3 \otimes \xi)$ denote, for a finite set T of places, the incomplete ξ-twisted *exterior cube* L-function of Π of degree 20. One knows — see [Sh1], Corollary 6.8, that this L-function admits a meromorphic continuation to the whole s-plane and satisfies a standard functional equation.

Proposition 2.4 *Let $\Pi = \Lambda^2(\pi)$, and T a sufficiently large finite set of places of E containing the archimedean and ramified places as well as those dividing p. Then for any character ξ of E,*

(a) $L^T(s, \Pi_E; \Lambda^3 \otimes \xi) = L^T(s, \pi_E; \mathrm{sym}^2 \otimes \xi)^2$;

(b) $L^T(s, \Pi_E; \Lambda^2 \otimes \xi)L^T(s, \xi) = L^T(s, \pi_E \times \pi_E \otimes \xi\mu)$.

Corollary 2.5

(a) $-\mathrm{ord}_{s=1}L^T(s, \Pi_E; \Lambda^3 \otimes \mu) \geq 2$;

(b) $\mathrm{ord}_{s=1}L^T(s, \Pi_E; \Lambda^2 \otimes \mu^{-2}) = 0$.

Clearly, part (a) (resp. (b)) of the Corollary follows from part (b) (resp. (a)) of Lemma 2.3 and part (a) (resp. (b) of Proposition 2.4.

To prove the Proposition we need the following

Lemma 2.6 *Let $\sigma = \sigma_1 \oplus \sigma_2$ be a representation of a group G, with each σ_j being of dimension 2 with determinant α_j. Then $\mathrm{sym}^2(\sigma)$ is $\sigma_1 \otimes \sigma_2 \oplus \mathrm{sym}^2(\sigma_1) \oplus \mathrm{sym}^2(\sigma_2)$. Moreover,*

$$\Lambda^3(\Lambda^2(\sigma)) \simeq$$
$$(\sigma_1 \otimes \sigma_2 \otimes \alpha_1\alpha_2)^{\oplus 2} \oplus \mathrm{sym}^2(\sigma_1) \otimes (\alpha_1\alpha_2 \oplus \alpha_2^2) \oplus \mathrm{sym}^2(\sigma_2) \otimes (\alpha_1\alpha_2 \oplus \alpha_1^2),$$
$$\sigma^{\otimes 2} \simeq \mathrm{sym}^2(\sigma_1) \otimes \alpha_1 \oplus \mathrm{sym}^2(\sigma_2) \otimes \alpha_2 \oplus (\sigma_1 \otimes \sigma_2)^{\oplus 2},$$

and

$$\Lambda^2(\Lambda^2(\sigma)) \simeq (\sigma_1 \otimes \sigma_2)^{\oplus 2} \oplus \mathrm{sym}^2(\sigma_1) \otimes \alpha_2 \oplus \mathrm{sym}^2(\sigma_2) \oplus \alpha_1.$$

In particular, if $\alpha_1 = \alpha_2 = \alpha$, we have

(a) $\Lambda^3(\Lambda^2(\sigma)) \otimes \alpha^{-2} \simeq \mathrm{sym}^2(\sigma)^{\oplus 2}$

    and

(b) $\Lambda^2(\Lambda^2(\sigma)) \oplus \alpha^2 \simeq \sigma \otimes \sigma \otimes \alpha.$

Proof (of Lemma 2.6) The first identity (involving the symmetric square) is evident. For the second, observe that since $\Lambda^2(\sigma)$ is $\sigma_1 \otimes \sigma_2 \oplus \alpha_1 \oplus \alpha_2$,

(i) $\Lambda^3(\Lambda^2(\sigma)) \simeq \Lambda^3(\sigma_1 \otimes \sigma_2) \oplus \Lambda^2(\sigma_1 \otimes \sigma_2) \otimes (\alpha_1 \oplus \alpha_2) \oplus \sigma_1 \otimes \sigma_2 \otimes \alpha_1\alpha_2.$

    We have

(ii) $\Lambda^2(\sigma_1 \otimes \sigma_2) \simeq \mathrm{sym}^2(\sigma_1) \otimes \Lambda^2(\sigma_2) \oplus \Lambda^2(\sigma_1) \otimes \mathrm{sym}^2(\sigma_2).$

Moreover, the non-degenerate G-pairing

$$\sigma_1 \otimes \sigma_2 \times \Lambda^3(\sigma_1 \otimes \sigma_2) \to \Lambda^4(\sigma_1 \otimes \sigma_2) = \det(\sigma_1 \otimes \sigma_2) = \alpha_1^2\alpha_2^2$$

identifies $\Lambda^3(\sigma_1 \otimes \sigma_2)$ with $(\sigma_1 \otimes \sigma_2)^{\vee} \otimes \alpha_1^2\alpha_2^2$, which is isomorphic to $\sigma_1 \otimes \sigma_2 \otimes \alpha_1\alpha_2$. The second identity and part (a) of the Lemma now follow by putting these together.

    Moreover,

$$\Lambda^2(\Lambda^2(\sigma)) \simeq \Lambda^2(\sigma_1 \otimes \sigma_2) \oplus \alpha_1\alpha_2 \oplus (\sigma_1 \otimes \sigma_2 \otimes (\alpha_1 \oplus \alpha_2)).$$

Applying (ii), we also get the third identity and part (b).

<div align="right">□</div>

Proof (of Proposition 2.4) Since ρ is associated to π, there is a finite set S of places of F containing the ramified places and ∞, such that for all $u \notin S$,

$$L(s, \pi_u) = L(s, \rho_u^{\mathrm{ss}}),$$

which is just an equality of 4-tuples of inverse roots defining the respective L-factors. It follows using (1.3) that for any such u,

$$L(s, \Lambda^2(\pi)_u) = L(s, \tau_{1,u} \otimes \tau_{2,u})L(s, \omega_{1,u})L(s, \omega_{2,u}).$$

Since by construction, ω_1 and ω_2 become the same over E, it follows that at any place v of E outside the inverse image T of S, we can write

$$\sigma_v(\pi_E) \simeq \sigma_{1,v} \oplus \sigma_{2,v},$$

with $\sigma_{1,v}$, $\sigma_{2,v}$ both being 2-dimensional of the same determinant. Here, $\sigma_v(\pi_E)$ is the 4-dimensional representation of the Weil group of $F_{1,v}$ attached to $\pi_{E,v}$. Now we are done by appealing to Lemma 2.6.

$\square$

Proof (of Theorem A (contd.))

As noted at the beginning of this section, we have already reduced to the case when π remains cuspidal over any solvable totally real extension of F, in particular over E. Recall that π_E is essentially selfdual relative to the character μ, and we may replace π by its twist by a power of $|\cdot|$ so that Π_E is selfdual.

Suppose $\Pi_E = \Lambda^2(\pi_E)$ is cuspidal. Then by a result of Ginzburg and Rallis (cf. [GR], Theorem 3.2) we know that $L^T(s, \Pi_E; \Lambda^3 \otimes \xi)$, for any character ξ, will have at most a simple pole at $s = 1$, for any finite set T of places of E containing the archimedean and ramified places. (If Π_E has a supercuspidal component, this also follows from [KSh], where one finds such a result even for the complete L-function.) But this contradicts Corollary 2.5, part (a).

So Π_E is not cuspidal, and we may write

$$\Pi_E \simeq \boxplus_{j=1}^m \beta_j, \ m > 1, \tag{2.4}$$

where each β_j is a cuspidal automorphic representation of $\mathrm{GL}(n_j, \mathbb{A}_E)$ with $\sum_j n_j = 6$ and $n_i \leq n_j$ if $i \leq j$. Note that each β_j will necessarily be algebraic.

It remains to get a contradiction when Π_E is Eisenteinian. Note that since π_E has trivial central character, $\Lambda^2(\pi_E)$ is selfdual.

Lemma 2.7 $n_1 \geq 2$ and $m \leq 3$.

Proof (of Lemma 2.7) First note that if some $n_j = 1$, then twisting by the inverse of the corresponding character β_j, we get a pole at $s = 1$ of the L-function

$$L^T(s, \pi_E \times \pi_E \otimes \beta_j^{-1}) = L^T(s, \Pi_E \otimes \beta_j^{-1}) L^T(s, \pi_E, \mathrm{sym}^2 \otimes \beta_j^{-1}),$$

because $\prod_{j \geq 2} L^T(s, \beta_j)$ and $L(s, \pi_E; \mathrm{sym}^2)$ has no zero at $s = 1$ (cf. [Sh1]). Consequently, by the cuspidality of π_E and [JS1], we obtain

$$\pi_E^\vee \simeq \pi_E \otimes \beta_j^{-1} \ (\text{if } n_j = 1).$$

Moreover, at any archimedean place v of E lying above above u (of F), the expressions (2.2) and (2.3) (for the possible shapes of) $\Lambda^2(\sigma_u(\pi))$ imply, since u splits in E, that *at most two of the n_j could be* 1, and the corresponding β_j must be totally odd. Suppose $n_1, n_2 = 1$. Since π is cuspidal,

$$\gamma := \beta_1 \beta_2^{-1} \neq 1,$$

and π admits a self-twist by this totally even character, π must be induced from the totally real quadratic extension $L = E(\gamma)$ of E; so π_L is not cuspidal. But this cannot happen, as we have earlier reduced (see the first paragraph of this section) to the situation where the base change of π to any finite solvable Galois, totally real extension remains cuspidal. Hence $n_2 \geq 2$ and $m \leq 3$.

If $n_1 = 1$, then there are then two possibilities for the type of Π_E, namely $(1, 5)$ and $(1, 2, 3)$. In the former case,

$$L^T(s, \Pi_E; \Lambda^3) = L^T(s, \beta_2; \Lambda^2 \otimes \beta_1) L^T(s, \beta_2; \Lambda^3).$$

The second L-function can be identified with $L^T(s, \Lambda^2(\beta_2)^\vee \otimes \lambda)$, where λ is the central character of β_2. So each factor on the right of (5.13) is an abelian twist of the exterior square L-function of a cusp form on $\mathrm{GL}(5)/F$, and by a theorem of Jacquet and Shalika ([JS2]), it admits no pole at $s = 1$ (because 5 is odd). This contradicts part (a) of Corollary 2.5, and so the case $(1, 5)$ cannot happen.

Suppose $(n_1, n_2, n_3) = (1, 2, 3)$. Here each β_j must be selfdual. In particular, the square of the idele class character β_1 is 1; similarly for the square of the central characters δ_j of β_j for $j = 2, 3$. But β_1 must be non-trivial, because the L-function of Π_E has no pole at $s = 1$. Similarly, $L^T(s, \delta_2)$ divides $L^T(s, \Pi_E; \Lambda^2)$, which (by part (b) of Corollary 2.5 is invertible at $s = 1$, implying that δ_2 is non-trivial. Consequently, $\beta_2 \simeq \beta_2^\vee = \beta_2 \otimes \delta_2$ and so β_2 is dihedral. Moreover, the selfduality of the $\mathrm{GL}(3)$-cusp form β_3 implies (cf. [Ra5], for example) that it is of the form $\mathrm{sym}^2(\eta) \otimes \xi$ for a cusp

form η on $\mathrm{GL}(2)/E$ and a character ξ. A direct computation yields (using $\Lambda^2(\beta_3) \simeq \beta_3$ by selfduality):

$$L^T(s, \Pi_E; \Lambda^3) =$$
$$L^T(s, \beta_1 \times \delta_2) L^T(s, \beta_1 \otimes \beta_2 \times \mathrm{sym}^2(\eta) \otimes \xi) L^T(s, (\beta_1 \delta_3^{-1}) \otimes \beta_3) \times$$
$$L^T(s, \delta_2 \otimes \beta_3) L(s, \xi_3).$$

Since the left hand side equals $L^T(s, \pi_E; \mathrm{sym}^2)^2$, the incomplete L-function $L^T(s, \beta_1 \xi \otimes \beta_2 \times \mathrm{sym}^2(\eta))$ must be the square of a degree 3 automorphic L-function $L(s)$. This forces η to be dihedral, which is not possible since $\mathrm{sym}^2(\eta)$ is not cuspidal. Hence the $(1, 2, 3)$ case is not possible either.
$$\square$$

End of proof of Theorem A

To recap, we have already established Theorem A when Π_E is cuspidal, and we may assume by Lemma 2.7 that Π_E is an isobaric sum as in (2.4) with $n_1 \geq 2$ and $m \leq 3$. As above, we may also replace π by its twist by a power of $|\cdot|$ to assume that Π_E is selfdual.

Suppose Π_E is of *type* $(2, 2, 2)$, i.e., an isobaric sum of three algebraic cusp forms $\beta_1, \beta_2, \beta_3$ on $\mathrm{GL}(2)/E$, with α_j being the central character of β_j, we get (for suitable finite set T of places containing the archimedean ones):

$$L^T(s, \Pi_E; \Lambda^3) = L^T(s, (\beta_1 \boxtimes \beta_2) \times \beta_3) \prod_{i \neq j} L^T(s, \beta_i \otimes \alpha_j). \qquad (2.5)$$

We know by part (b) of Corollary 2.5 that this exterior cube L-function of Π_E has a pole of order ≥ 2 at $s = 1$. Since each $L^T(s, \beta_i \otimes \alpha_j)$ is invertible at $s = 1$ (as β_i is cuspidal), we deduce from (2.5) that $L^T(s, (\beta_1 \boxtimes \beta_2) \times \beta_3)$ must have at least a double pole, which implies ([Ra3]) that $\beta_1 \boxtimes \beta_2$ must be of type $(2, 2)$ and contain $\beta_2^\vee$ as an isobaric summand with multiplicity 2. This forces (see [PRa]) each β_i to be dihedral, in fact corresponding to a character χ_i of a common quadratic extension $\tilde{E}/E$. Then each β_i is associated to an irreducible 2-dimensional representation τ_i of $\mathfrak{G}_E$. It follows that the restriction of $\Lambda^2(\rho^{ss})$ to $\mathfrak{G}_E$ must be $\tau_1 \oplus \tau_2 \oplus \tau_3$, which contradicts the fact that the (supposed) decomposition (1.1) results in $\Lambda^2(\rho^{ss})$ having one-dimensional summands. Hence Π_E cannot be of type $(2, 2, 2)$.

Next suppose Π_E is of *type* $(2, 4)$, i.e., with β_1, resp. β_2, being a selfdual cusp form on $\mathrm{GL}(2)/E$, resp. $\mathrm{GL}(4)/E$. Let ξ_j be the central character of β_j whose square is 1. Then $L^T(s, \xi_1)$ divides $L^T(s, \Pi_E, \Lambda^2)$, which has no pole at $s = 1$; thus $\xi_1 \neq 1$, implying that the selfdual β_1 must be dihedral,

say of the form $I_L^E(\chi)$ for a character χ of a quadratic extension L of E. We obtain

$$L^T(s, \Pi_E; \Lambda^3) = L^T(s, \beta_1 \times \Lambda^2(\beta_2))L^T(s, \xi_1 \otimes \beta_2)^2. \qquad (2.6)$$

The occurrence of at least a double pole then forces the isobaric decomposition

$$\Lambda^2(\beta_2) \simeq (\beta_1 \otimes \xi_1)^{\boxplus 2} \boxplus I_L^E(\chi'), \qquad (2.7)$$

with $I_L^E(\chi\chi') \simeq I_L^E(\chi^\theta \chi')$, where θ is the non-trivial automorphism of L/E. This forces χ' to be θ-invariant, forcing $I_L^E(\chi')$ to be Eisensteinian and of the form $\lambda \boxplus \lambda\delta$, where λ is the restriction of χ' to C_E and δ is the quadratic character attached to L. Since β_2 is selfdual, it follows that it admits a self-twist by δ and is hence induced from L; say $\beta_2 = I_L^E(\eta)$, for a cusp form η on $GL(2)/L$ with central character γ. Then we have

$$\Lambda^2(\beta_2) \simeq \mathrm{As}_{L/E}(\eta) \otimes \delta \boxplus I_L^E(\gamma), \qquad (2.8)$$

where $\mathrm{As}_{L/E}(\eta)$ is the Asai transfer of η to a form on $GL(4)/E$ ([Ra4]). Comparing (2.7) and (2.8) we see that this Asai representation is not cuspidal and of type $(2, 2)$. Then by Theorem B of [PRa], we see that η must be dihedral, induced by a character λ, say, of a biquadratic extension N of E containing L. Thus

$$\beta_2 \simeq I_L^E(\eta) \simeq I_N^E(\lambda),$$

with λ algebraic. So we may attach to β_2 an irreducible 4-dimensional representation σ obtained by Galois induction from the p-adic Galois character λ associated to λ in [Se]. Since $\Lambda^2(\pi_E) = \beta_1 \boxplus \beta_2$, we obtain (by Tchebotarev) an isomorphism of $\mathfrak{G}_E$-modules

$$\Lambda^2(\rho^{\mathrm{ss}}) \simeq \mathrm{Ind}_L^E(\chi) \oplus \mathrm{Ind}_N^E(\lambda), \qquad (2.9)$$

with the two summands on the right being irreducible. This contradicts the fact that (1.1) implies that the left hand side of (2.9) admits one-dimensional summands, already over F. Hence the type $(2, 4)$ is not possible in our case.

Finally, suppose Π_E is of *type* $(3, 3)$, and write

$$\Lambda^2(\pi_E) \simeq \beta \boxplus \beta',$$

with β, β' cusp forms on $GL(3)/E$. Since E is a totally real extension of F, the archimedean type is the same as over F. It follows from (2.2), (2.3) that β, β' must both be *regular* algebraic. Either they are both selfdual or duals of each other. In the former situation, one has known since [Pic] how

to attach irreducible 3-dimensional p-adic representations σ, σ' to them; see also [Cℓ2], [PL]. In the non-selfdual case, one still has such a construction in the recent work [HLTT]. Combining this with the exterior square of (1.1), we get (by Tchebotarev) the following as $\mathfrak{G}_E$-modules:

$$\tau \otimes \tau' \oplus \omega \oplus \omega' \simeq \Lambda^2(\rho^{\mathrm{ss}}) \simeq \sigma \oplus \sigma',$$

which gives the needed contradiction.

$\square$

3   The $(3,1)$-case

The following provides a step towards establishing Theorem B:

Proposition D   *Let π be an isobaric, algebraic representation of $GL_4(\mathbb{A}_F)$ which is not of type $(3,1)$, with associated 4-dimensional Hodge-Tate, $\overline{\mathbb{Q}}_p$-representation ρ of $\mathfrak{G}_F$. Then ρ^{ss} is not of type $(3,1)$. Conversely, assuming in addition that π is quasi-regular, if the decomposition type of ρ is not $(3,1)$ for a sufficiently large p, then the isobaric type of π is not $(3,1)$ either.*

Proof   Now suppose we have a decomposition as $\mathfrak{G}_F$-modules over $\overline{\mathbb{Q}}_p$:

$$\rho^{\mathrm{ss}} \simeq \tau \oplus \chi, \tag{3.1}$$

with τ (resp. χ) of dimension 3 (resp. 1). Since every subrepresentation of a crystalline, resp. Hodge-Tate, representation is crystalline, resp. Hodge-Tate, χ is crystalline. It follows that χ is locally algebraic and by [Se], it is defined by an algebraic Hecke character χ. Put

$$\nu = \det(\tau). \tag{3.2}$$

Then ν is also Hodge-Tate and corresponds to an algebraic Hecke character ν.

Taking the contragredients of both sides of (3.1), then twisting by $\nu\chi^{-1}$, and noting that these processes commute with taking semisimplification, we obtain

$$(\rho^{\vee} \otimes \nu\chi^{-1})^{\mathrm{ss}} \simeq (\tau^{\vee} \otimes \nu\chi^{-1}) \oplus \nu\chi^{-2}. \tag{3.3}$$

Appealing to hypothesis (b) of Theorem A, we see that outside a finite set S of places, the decompositions (3.1) and (3.3) imply the following identity of L-functions:

$$L^S(s,\pi)L^S(s,\pi^{\vee}\otimes\nu\chi^{-1}) = L^S(s,\tau)L^S(s,\tau^{\vee}\otimes\nu\chi^{-1})L^S(s,\chi)L^S(s,\nu\chi^{-2}). \tag{3.4}$$

Lemma 3.1 *Assuming* (3.1), *we have*

$$\Lambda^2(\rho^{ss}) \simeq (\tau^{\vee} \otimes \nu) \oplus (\tau \otimes \chi).$$

Proof Using (3.1) and the fact that χ is one-dimensional and hence has trivial exterior square, we get

$$\Lambda^2(\rho^{ss}) \simeq \Lambda^2(\tau) \oplus \tau \otimes \chi. \tag{3.5}$$

So the Lemma will be proved if we verify that

$$\Lambda^2(\tau) \simeq \tau^{\vee} \otimes \nu. \tag{3.6}$$

By the Tchebotarev density theorem, it suffices to check this easy identity at the primes where all the representations are unramified. Let P such a prime, and let $\{\alpha_P, \beta_P, \gamma_P\}$ denote the inverse roots of ϕ_P, the Frobenius at P, acting on τ. Then the inverse roots on $\Lambda^2(\tau)$ are given by the set

$$\{\alpha_P\beta_P, \alpha_P\gamma_P, \beta_P\gamma_P\} = \nu(\phi_P)\{\alpha_P^{-1}, \beta_P^{-1}, \gamma_P^{-1}\}.$$

Here we have used the fact that ν is the determinant of τ. The identity (3.6) now follows because the inverse roots of ϕ_P on $\tau^{\vee}$ are given by the inverses of those on τ. Done.

$\square$

Proposition 3.2 *Let ρ, π satisfy the hypotheses of Theorem A. Then the decomposition* (3.1) *cannot hold.*

Proof Combining Lemma 3.1 with the identity (3.4), we get

$$L^S(s, \pi)L^S(s, \pi^{\vee} \otimes \nu\chi^{-1}) = L^S(s, \Lambda^2(\rho^{ss}) \otimes \chi^{-1})L^S(s, \chi)L^S(s, \nu\chi^{-2}). \tag{3.7}$$

Now we appeal to a beautiful recent theorem of H. Kim ([K]) which establishes the automorphy in GL(6) of the exterior square of any cusp form on GL(4). Applying this to our form π, we get an isobaric automorphic representation $\Lambda^2(\pi)$ which is functorial at all the unramified primes. Using this information in (3.7), and twisting by χ^{-1}, we get

$$L^S(s, \pi \otimes \chi^{-1})L^S(s, \pi^{\vee} \otimes \nu\chi^{-2}) = L^S(s, \Lambda^2(\pi) \otimes \chi^{-1})\zeta^S(s)L^S(s, \nu\chi^{-3}). \tag{3.8}$$

(This identity slightly corrects (7.10) in [Ra1], where χ^{-2} in the second L-function on the left turns up mistakenly as χ^{-1}, which however does not affect the proof.)

One knows by Jacquet and Shalika ([JS1]) that for any isobaric automorphic representation Π_E of GL(n)/$\mathbb{Q}$ for any $n \geq 1$, the incomplete

L-function $L^S(s, \Pi_E)$ has no zero at $s = 1$. Consequently, $L^S(s, \Lambda^2(\pi) \otimes \chi^{-1})L^S(s, \nu\chi^{-3})$ is non-vanishing at $s = 1$. Since $\zeta^S(s)$ has a pole at $s = 1$, we see then that the right hand side, and hence the left hand side, of (3.8) admits a pole at $s = 1$. On the other hand, since π and $\pi^\vee$ are cusp forms on $\mathrm{GL}(4)/\mathbb{Q}$, the left hand side of (3.8) does not have a pole at $s = 1$. So we get a contradiction. The only possibility is that the decomposition of ρ^{ss} given by (3.1) cannot hold.

$\square$

Now let us prove the converse direction in Proposition D. Suppose we have an isobaric decomposition

$$\pi \simeq \eta \boxplus \nu, \tag{3.9}$$

where η is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ and ν an idele class character of F, necessarily with η, ν algebraic. Moreover, the quasi-regularity hypothesis on π implies that η is regular. Hence by the recent powerful theorem of Harris, Lan, Thorne and Taylor ([HLTT]), one may associate to η a semisimple, 3-dimensional $\overline{\mathbb{Q}}_p$-representation β. Since ν is algebraic, it also corresponds to an abelian p-adic representation ν of $\mathfrak{G}_F$. On the other hand, ρ is associated to π. It follows by Tchebotarev that

$$\rho^{\mathrm{ss}} \simeq \beta \oplus \nu \tag{3.10}$$

Hence the Proposition will be proved if (the semisimple representation) β is irreducible. Suppose not. If β is a direct sum of three p-adic character, each of which is necessarily Hodge-Tate, we will deduce, by the strong multiplicity one theorem, that η is isomorphic to an isobaric sum of three idele class characters of F, contradicting the cuspidality of η. So we may assume that we have

$$\beta \simeq \sigma \oplus \mu, \tag{3.11}$$

with σ (resp. μ) irreducible of dimension 2 (resp. 1). Let μ be the Hecke character attached to the (Hodge-Tate) p-adic character μ.

Now the regularity of η implies the same about the Hodge-Tate type of β. It follows that σ ha distinct Hodge-Tate weights and by Taylor ([Ta2, Ta3]), there is a finite solvable Galois extension E of F such that the restriction σ_E of σ to $\mathfrak{G}_E$ is modular, i.e., is associated to a cusp form π_0 of $\mathrm{GL}_2(\mathbb{A}_E)$, which is regular algebraic. One can, by well known arguments via cyclic layers, descend π_0 to any subfield M of E with E/M Galois and solvable and obtain a cusp form π_0^M on $\mathrm{GL}(2)/M$ which base changes to π_0 over E and in fact corresponds to the restriction σ_M of σ to $\mathfrak{G}_M$.

As in [Ta2], we now appeal to Brauer's theorem. By the inductive nature of L-functions, we have

$$L^{S^E}(s, \sigma_E \otimes \mu_E^{-1}) = L^S(s, \sigma \otimes \mu^{-1} \otimes \mathrm{Ind}_E^F(1_E)), \qquad (3.12)$$

where $\mathrm{Ind}_E^F(1_E))$ is the representation of $\mathrm{Gal}(\overline{F}/F)$ induced by the trivial representation 1_E of $\mathrm{Gal}(\overline{F}/E)$, S is a finite set of places of F containing the archimedean and ramified places as well as the ones above p, and S^E the finite set of places of E above S. We can write

$$\mathrm{Ind}_E^F(1_E)) \simeq 1_F \oplus a_{E/F}, \qquad (3.13)$$

for a unique representation $a_{E/F}$ of $\mathrm{Gal}(\overline{F}/F)$ called the *augmentation representation*. Using Brauer's theorem we can write it as a virtual sum of monomial representations. More precisely,

$$a_{E/F} \simeq \oplus_{j=1}^r n_j \mathrm{Ind}_{E_j}^F(\alpha_j), \qquad (3.14)$$

where for each j, n_j is in $\mathbb{Z}$, E_j a subfield of E with E/E_j cyclic, and α_j a finite order character of $\mathrm{Gal}(\overline{F}/E_j)$ with trivial restriction to $\mathrm{Gal}(\overline{F}/E)$. Consequently, using (3.13), (3.14), the inductive nature and additivity,

$$L^S(s, \sigma \otimes \mu^{-1} \otimes \mathrm{Ind}_E^F(1_E)) = L^S(s, \sigma \otimes \mu^{-1}) \prod_{j=1}^r L^{S_j}(s, \sigma_{E_j} \otimes \mu_{E_j}^{-1} \otimes \alpha_j)^{n_j},$$
$$(3.15)$$

where S_j is the set of places of E_j above S. Now we can appeal to (3.12) and the modularity of σ_{E_j} ($\forall\, j$) to obtain

$$L^S(s, \sigma \otimes \mu^{-1}) = \prod_{j=0}^r L^{S_j}(s, \pi_0^{E_j} \otimes \mu_{E_j}^{-1} \alpha_j)^{-n_j}, \qquad (3.16)$$

where $n_0 = -1$, $E_0 = E$ and $\alpha_0 = 1$.

Since $\pi_0^{E_j}$ is cuspidal, $L^{S_j}(s, \pi_0^{E_j} \otimes \mu_{E_j}^{-1} \alpha_j)$ has no zero or pole at the right edge $s = 1$. It then follows by (3.16) that

$$\mathrm{ord}_{s=1}\, L^S(s, \sigma \otimes \mu^{-1}) = 0. \qquad (3.17)$$

Now applying (3.11) and the fact that β is associated to η, we get

$$L^S(s, \eta \otimes \mu^{-1}) = L^S(s, \sigma \otimes \mu^{-1}) \zeta_F^S(s). \qquad (3.18)$$

In view of (3.17), the right hand side of (3.18) has a pole at $s = 1$, which yields a contradiction as the left hand side (of (3.18)) is entire, η being a cusp form on $\mathrm{GL}(3)/F$.

This finishes the proof of Proposition D.

$\square$

4   The $(2,2)$-case; End of Proof of Theorem B

We have just proved that π is of type $(3,1)$ iff ρ is also of the same type, under the running hypothesis that ρ is crystalline and p sufficiently large, i.e., with each of the Hodge-Tate weights of ρ being $< 2(p-1)$. By a similar argument we can also prove that for π Eisensteinian, ρ^{ss} has the same type as π. So we may *assume from here on that π is cuspidal*, which is the key case. We already know that ρ^{ss} is not of type $(3,1)$, and it is easy to see that it cannot also be of type $(1,1,1,1)$ as then π would, by the strong multiplicity one theorem, an isobaric sum of four Hecke characters.

Hence we will be done that it is impossible for ρ^{ss} to be of type $(2,2)$ or $(2,1,1)$ when π is cuspidal. Suppose not. Then we have

$$\rho^{\mathrm{ss}} \simeq \sigma \oplus \sigma', \tag{4.1}$$

with $\dim(\sigma) = \dim(\sigma') = 2$ and σ irreducible. Thanks to Theorem A, both σ and σ' are odd. The Hodge-Tate types of σ and σ' are regular and they are crystalline as ρ is by hypothesis. By Taylor ([Ta2, Ta3]), we know that there is a finite Galois, totally real extension E of F such that σ is modular over E, i.e., associated to a cusp form π_0 on $\mathrm{GL}(2)/E$.

Suppose σ' is reducible, i.e., of the form $\nu \oplus \nu'$ with $\dim(\nu) = \dim(\nu') = 1$, then there are associated Hecke characters ν, ν' of F, and by the argument at the end of the previous section, $L^S(s, \sigma \otimes \nu^{-1})$ is invertible at $s = 1$. Consequently, the cuspidal L-function $L^S(s, \pi \otimes \nu^{-1})$, which equals $L^S(s, \sigma \otimes \nu^{-1})\zeta_F^S(s)L^S(s, \nu'/\nu)$, has a pole at $s = 1$, which is a contradiction.

Hence we may assume that σ, σ' are both irreducible and crystalline with p sufficiently large. Since π evidently satisfies the hypotheses of Theorem A as well, we may apply Lemma 1.2 and assume that π remains cuspidal when base changed to any finite solvable normal extension which is totally real. Moreover, since the determinants ν, ν' of σ, σ' respectively are both odd by section 2, we may replace F by an abelian, totally real extension over which $\nu = \nu'$, still with π cuspidal. Appealing to Taylor's theorem for σ' as well, one gets a finite Galois, totally real extension E' of F and a cuspidal automorphic representation π_0' of $\mathrm{GL}_2(\mathbb{A}_{E'})$ which is associated to $\sigma'_{E'}$. By Gelbart-Jacquet ([GJ]), there is an isobaric automorphic representation $\mathrm{sym}^2(\pi_0)$, resp. $\mathrm{sym}^2(\pi_0')$, of $\mathrm{GL}_3(\mathbb{A}_E)$, resp. $\mathrm{GL}_3(\mathbb{A}_{E'})$ whose standard L-function equals $L^S(s, \pi_0; \mathrm{sym}^2)$, resp. $L^S(s, \pi_0'; \mathrm{sym}^2)$. Applying Brauer's theorem as in the previous section, and the modularity of σ, resp. σ', over any sub $M \subset E$, resp. $M' \subset E'$, with $[E : M]$, resp. $[E' : M']$, solvable and normal, we get the analogue of (3.17):

$$\mathrm{ord}_{s=1} L^S(s, \mathrm{Ad}(\tau)) = 0, \ \text{ for } \tau \in \{\sigma, \sigma'\}, \tag{4.2}$$

where $\operatorname{Ad}(\tau) = \operatorname{sym}^2(\tau) \otimes \nu^{-1}$. In addition, $L^S(s, \operatorname{Ad}(\tau))$ is meromorphic on the s-plane.

Applying the exterior square to (4.1) and twisting by ν^{-1}, we obtain

$$L^S(s, \Lambda^2(\pi) \otimes \nu^{-1}) = L^S(\sigma \otimes \sigma' \otimes \nu^{-1})\zeta_F^S(s)^2, \qquad (4.3)$$

which also gives the meromorphic continuation for $L^S(\sigma \otimes \sigma' \otimes \nu^{-1})$. Moreover, since π is cuspidal, the left hand side of (4.3) can have at most a simple pole at $s = 1$. So it follows that

$$\operatorname{ord}_{s=1} L^S(\sigma \otimes \sigma' \otimes \nu^{-1}) = 1. \qquad (4.4)$$

On the other hand, applying the symmetric square to (4.1) and twisting, we get

$$L^S(s, \pi, \operatorname{sym}^2 \otimes \nu^{-1}) = L^S(\sigma \otimes \sigma' \otimes \nu^{-1})L^S(s, \operatorname{Ad}(\sigma))L^S(s, \operatorname{Ad}(\sigma')), \quad (4.5)$$

where the properties of the left hand side are given in [BuG] and [Sh1]. Moreover, by [BuG], $L^S(s, \pi, \operatorname{sym}^2 \otimes \nu^{-1})$ has no zero at $s = 1$. On the other hand, the right hand side of (4.5) must have a zero at $s = 1$ by the conjunction of (4.2) and (4.4). This gives the requisite contradiction to the decomposition (4.1).

We have now proved that ρ is irreducible when π is cuspidal satisfying the hypotheses of Theorem B.

$\square$

5 Proof of Theorem C

Let σ, σ' be 2-dimensional odd, semisimple, crystalline representations as in Theorem C, with $\sigma \oplus \sigma'$ associated to an isobaric automorphic form π on $\operatorname{GL}(4)/F$. If σ, σ' are both reducible, they correspond to sums of algebraic Hecke characters, and the assertion of Theorem C follows by Hecke. More generally, if σ, σ' become reducible over a finite Galois extension, then they are both of Artin-Hecke type, and the claim is well known (see [De]). So we may, and we will, assume that at least one of them, say σ, remains irreducible over any finite extension of F.

Lemma 5.1 *Under the hypothesis of irreducibility of σ, we must have*

$$\pi \simeq \eta \boxplus \eta',$$

where η, η' are isobaric automorphic forms on $\operatorname{GL}(2)/F$, with η cuspidal. Moreover, η' is cuspidal iff σ' is irreducible.

Before proving this Lemma, let us note that if we knew π to be algebraic and quasi-regular, we can conclude by the results in the earlier sections that π cannot be of type $(3, 1)$, $(2, 1, 1)$ or $(1, 1, 1, 1)$. Similarly, if π were regular and algebraic, we can also rule out π being cuspidal.

Proof (of Lemma 5.1) First suppose σ' is reducible. Then evidently, since σ is irreducible, $\sigma^\vee \otimes \sigma'$ has no Galois invariants. Moreover, applying the potential automorphy result of Taylor ([Ta1, Ta2]) to the crystalline, odd 2-dimensional σ, whose Hodge-Tate type satisfies his hypothesis relative to p, we deduce by Brauer (as it was done earlier) the following for any character μ of F:

$$\mathrm{ord}_{s=e} L^S(s, \sigma \otimes \mu) \,=\, 0, \tag{5.1}$$

where e denotes the right edge ("Tate point"), which is $1 + w/2$ if μ is of finite order. For suitable finite set S of places of F, $L^S(s, \pi \otimes \mu)$ is a product of two abelian twists of the L^S-function of σ, which implies, by (5.1), that π must be an isobaric sum of the form $\eta \boxplus \mu \boxplus \mu'$, with η a form on $\mathrm{GL}(2)/F$ and μ, μ' idele class characters of F. Moreover, η must be cuspidal, for otherwise π will be an isobaric sum of four characters, yielding an associated abelian 4-dimensional representation of the Weil group of F, contradicting the irreducibility of σ. Putting $\eta' = \mu \boxplus \mu'$, we get, as asserted, $\pi \simeq \eta \boxplus \eta'$, with η cuspidal and η' not cuspidal.

We may now assume that σ and σ' are both irreducible, still with σ remaining irreducible upon restriction to any open subgroup. Since they have the same parity and weight, we may consider the cyclic, totally real extension E of F such that their determinants are the same over E, say $\chi^w \nu$ with ν of finite order. Then the unitarily normalized π_E, the base change of π, has central character $\omega_E = \nu^2$ and satisfies $\pi_E^\vee \simeq \pi_E \otimes \nu^{-1}$. Comparing L-functions we get, for a finite set T of places of E,

$$L^T(s, \Lambda^2(\pi_E) \otimes \nu^{-1}) \,=\, L^T(s, \sigma_E^\vee \otimes \sigma') \zeta_E^T(s)^2. \tag{5.2}$$

Suppose π_E were cuspidal. Then the left hand side of (5.1) will have at most a simple pole at $s = 1$, implying that $L^T(s, \sigma_E^\vee \otimes \sigma')$ must have a zero at $s = 1$. On the other hand, we have the identity (4.5) over E, which implies that $L^T(s, \mathrm{Ad}(\sigma_E))$ or $L^T(s, \mathrm{Ad}(\sigma'))$ must have a pole at $s = 1$, since $L^T(s, \pi_E, \mathrm{sym}^2 \otimes \nu^{-1})$ does not have a zero at $s = 1$ by Shahidi ([Sh1, Sh2]). Now we apply Taylor's theorem ([Ta1, Ta2]) to conclude as before that σ, σ' are potentially automorphic over finite Galois, totally real extensions M, M' respectively of F, and we may deduce as before that for any character μ of F, and for any finite set S of places of F,

$$\mathrm{ord}_{s=1} L^S(s, \mathrm{Ad}(\tau) \otimes \mu) \,=\, 0, \ \ \text{for } \tau \in \{\sigma, \sigma'\}. \tag{5.3}$$

More precisely, we see that as $L^T(s, \sigma^\vee \otimes \sigma' \otimes \mu)$ has no zero at $s = 1$,

$$-\mathrm{ord}_{s=1} L^T(s, \Lambda^2(\pi_E) \otimes \nu^{-1}) = -\mathrm{ord}_{s=1} L^T(s, \sigma_E^\vee \otimes \sigma_E') + 2 \geq 2. \quad (5.4)$$

Similarly,

$$-\mathrm{ord}_{s=1} L^T(s, \pi_E; \mathrm{sym}^2 \otimes \nu^{-1}) = -\mathrm{ord}_{s=1} L^T(s, \sigma_E^\vee \otimes \sigma_E') \geq 0, \quad (5.5)$$

implying

$$-\mathrm{ord}_{s=1} L^T(s, \pi_E^\vee \times \pi_E) = -2\mathrm{ord}_{s=1} L^T(s, \sigma_E^\vee \otimes \sigma_E') + 2 \geq 2. \quad (5.6)$$

In particular, π_E is not cuspidal. We claim that π itself is not cuspidal. Suppose not. Then there exists a quadratic extension M/L with $F \subset L \subset M \subset E$ such that π_L is cuspidal, but π_M is not, implying that π_L admits a self twist by the quadratic character δ, say, of L associated to M. Then $\sigma_L \oplus \sigma_L'$ must also be isomorphic to its twist by δ. Since σ remains irreducible under restriction to any open subgroup, σ remain irreducible, which forces an isomorphism $\sigma_L' \simeq \sigma_L \otimes \delta$. Then for non-trivial automorphism α of L/F, if any, $\sigma_L \simeq \sigma_L \otimes (\delta/\delta^\alpha)$, which implies that δ must be α-invariant (since otherwise σ_L would become reducible over $L(\delta/\delta^\alpha)$). It follows that already over F, $\sigma' \simeq \sigma \otimes \delta_0$, where δ_0 is a descent of δ to F. Then σ and σ' have the same determinant, say $\nu \chi^w$, and (the assumption of cuspidality of π implies)

$$\sigma^\vee \otimes \sigma' \simeq (\mathrm{Ad}(\sigma) \otimes \delta) \oplus \delta \simeq \sigma'^\vee \otimes \sigma.$$

This implies by (5.3) that $L^S(s, \sigma^\vee \otimes \sigma')$ does not vanish at $s = 1$. Consequently, $L^S(\pi^\vee \times \pi)$, which equals $L^S(s, (\sigma \oplus \sigma')^\vee \otimes (\sigma \oplus \sigma'))$, has a pole of order > 1 at $s = 1$, which contradicts (by [JS1]) the cuspidality assumption on π. Thus π is not cuspidal. Moreover, if π is of type $(3, 1)$, then no abelian twist of its exterior square L-function can admit any pole at $s = 1$. Thus π must be of type $(2, 2), (2, 1, 1)$ or $(1, 1, 1, 1)$. As above we can eliminate $(1, 1, 1, 1)$, and we can eliminate $(2, 1, 1)$ as well, for otherwise $L^S(s, \sigma \otimes \mu) L^S(s, \sigma' \otimes \mu)$ will have a pole at $s = e$ for a suitable character μ, contradicting (5.1) (which holds for σ' as well). Hence the assertion of the Lemma holds when σ' is irreducible as well, with both η, η' cuspidal.

$\square$

Proof of Theorem C (contd.) Comparing the exterior square L-functions of $\pi = \eta \boxplus \eta'$ and of $\sigma \oplus \sigma'$, we see that for any character μ of F,

$$L^S(s, \sigma^\vee \otimes \sigma' \otimes \mu) = L^S(s, \eta^\vee \boxtimes \eta' \otimes \mu), \quad (5.7)$$

where $\eta^\vee \boxtimes \eta'$ is the isobaric automorphic form on $\mathrm{GL}(4)/F$ associated to the pair $(\eta^\vee, \eta')$ in [Ra3].

Suppose σ' is reducible. Then $\sigma \otimes \sigma'$ has no Galois invariants, since σ is irreducible. On the other hand, as the right hand side of (5.7) is a product of two abelian twists of the standard L-function of the cusp form η, the order of $L^S(s, \sigma^\vee \otimes \sigma)$ at $s = 1$ is 0. So Theorem C is proved in this case.

It remains to consider when both σ and σ' are irreducible, in fact upon restriction to any open subgroup. Then $\sigma^\vee \otimes \sigma'$ has Galois invariants iff $\sigma \simeq \sigma'$, in which case the dimension of $\mathfrak{G}_F$-invariants is 1. On the other hand, by Lemma 5.1, η and η' are both cuspidal on $\mathrm{GL}(2)/F$. Then $L^S(s, \eta^\vee \boxtimes \eta')$ has a pole at $s = 1$ iff η is isomorphic to η', in which case the pole is of order 1. We get the same assertion for $L^S(s, \sigma^\vee \otimes \sigma')$ by (5.7). So we have only to prove the *claim* that $\sigma \simeq \sigma'$ iff $\eta \simeq \eta'$. Arithmetically normalizing both sides, an isomorphism on either side furnishes an equality (at good primes v of norm q_v) of the form:

$$\{\alpha_v, q_v^w \alpha_v^{-1}, \beta_v, q_v^w \beta_v^{-1}\} = \{\gamma_v, q_v^w \gamma_v^{-1}, \gamma_v, q_v^w \gamma_v^{-1}\}.$$

Up to renaming α_v as β_v and switching γ_v with $q_v^w \gamma_v^{-1}$, we may assume that $\alpha_v = \gamma_v$. Then $q_v^w \alpha_v^{-1}$ equals $q_v^w \gamma_v^{-1}$, resulting in the equality $\{\beta_v, q_v^w \beta_v^{-1}\} = \{\gamma_v, q_v^w \gamma_v^{-1}\}$. The claim follows.

$\square$

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