Lecture 5

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(The odd numbered lectures are given by D. Ramakrishnan, and the even ones by R. Tanner.)

0.1 Equilibria, Nullclines

As we saw in Lecture 3, subject to the algebraic relations

$$c = e_0 - e,$$

$$v = p'(t) = k_3 c,$$

and

$$s + c + p = s_0,$$

the basic enzyme kinetic equations can be reduced to the following of first order, non-linear equations:

$$s'(t) = -k_1 s(e_0 - c) + k_2 c \tag{1}$$

$$c'(t) = k_1 s(e_0 - c) - (k_2 + k_3)c, \tag{2}$$

which we will write in vector form as

$$\frac{d}{dt}\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} -k_1 e_0 s + (k_1 s + k_2)c \\ k_1 e_0 s - (k_1 s + k_2 + k_3)c \end{pmatrix}. \tag{3}$$

The *equilibria* or *stationary points* occur at the points where the right hand side is zero, i.e., where

$$s'(t) = c'(t)\, 0.$$

From the equations (1) and (2) we see that

$$c'(t) = -s'(t) - k_3 c.$$

Hence the only equilibrium points are where $c = 0$ and $s = 0$ (by using (1)). At time $t = 0$, $c = 0$ but $s = s_0 > 0$. As the reaction proceeds, c rises to a maximum bounded above by s_0, and then decreases. As t goes to infinity, s

falls to 0, and so does c. Hence there is a **unique equilibrium point**, which happens at $(s, c) = (0, 0)$, which happens at infinite time (in the limit). (This is another reason why we sometimes prefer to plot $v = p'(t)$ as a function of s.)

The *nullclines* are the curves defined by setting just one of the coordinates on the right of (3) equal to zero. Since we have a 2×2-system, there are two nullclines, and the equilibria occur at the intersection of the two.

The s**-nullcline** is given by the set of solutions of (3) where $s'(t) = 0$, i.e., where

$$c = \frac{k_1 e_0 s}{k_1 s + k_2},$$

which can be rewritten as

$$v = k_3 c = \frac{V s}{s + K_e}, \text{ where } V := k_3 e_0,\ K_e := \frac{k_2}{k_1}.$$

The graph of this is a *hyperbola*.

The c**-nullcline** is given by the set of solutions of (3) where $c'(t) = 0$, i.e., where

$$c = \frac{k_1 e_0 s}{k_1 s + k_2},$$

which can be rewritten as

$$v = k_3 c = \frac{V s}{s + K_m}, \text{ where } K_m := \frac{k_2 + k_3}{k_1}.$$

The graph of this is also a *hyperbola*, which meets the other nullcline at $(s, c) = (0, 0)$.

0.2 Stability, Linearization

Now consider a point close to the stationary point. We are interested in *stable solutions* (s, c) to (3), i.e., which approach the limit $(0, 0)$ as $t \to \infty$ (or equivalently, as $s \to 0$). One way to analyze the behavior near the origin is to *linearize* the situation. Roughly speaking, we look at only the first derivative of the vector differential in the expansion at a small (s, c). Write

$$\frac{d}{dt}\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} f(s, c) \\ g(s, c) \end{pmatrix},$$

with

$$f(s,c) \;=\; -k_1e_0s + (k_1s + k_2)c$$

and

$$g(s,c) \;=\; k_1e_0s - (k_1s + k_2 + k_3)c.$$

Vector Calculus gives us the approximate expression (for (s,c) close to $(0,0)$)

$$f(s,c) \;=\; f(0,0) + \frac{\partial f}{\partial s}(0,0)s + \frac{\partial f}{\partial c}(0,0)c$$

and

$$g(s,c) \;=\; g(0,0) + \frac{\partial g}{\partial s}(0,0)s + \frac{\partial g}{\partial c}(0,0)c.$$

Explicitly,

$$\begin{pmatrix} \frac{\partial f}{ds} & \frac{\partial f}{\partial c} \\ \frac{\partial g}{ds} & \frac{\partial g}{\partial c} \end{pmatrix} = \begin{pmatrix} -k_1e_0 + k_1c & k_2 \\ k_1e_0 - k_1c & -k_2 - k_3 \end{pmatrix}.$$

Consequently, remembering that $f(0,0) = g(0,0) = 0$, we get the linearized equations:

$$f(s,c) \;=\; -k_1e_0s + k_2c,$$

$$g(s,c) \;=\; k_1e_0s - (k_2 + k_3)c.$$

(Note that the process of linearization has eliminated the non-linear terms involving sc.)

Indeed, as $(s,c) \to (0,0)$, these linearized solutions tend to the zero solution, indicating stability.

Try, as an exercise, to see what happens if we use the actual solution, which cannot be explicitly derived, and tend to the equilibrium point (to be discussed a bit in class).

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