Lecture 25

The utility of the Laplace transform in solving ODE's

Recall that if f is a function on $[0, \infty)$ which is piecewise continuous and is "a-nice," i.e., $|f(t)| \leq ce^{at}$, for large enough t, for a positive constant c, then its Laplace transform

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}dt \qquad \left(= \lim_{A\to\infty} \int_0^A f(t)e^{-st}dt\right)$$

is well defined. (In fact, $F(s)$ makes sense for any complex number s as long as $Re(s) > a$.) Note:

$$\begin{aligned}\mathcal{L}(1) &= \int_0^\infty e^{-st}dt = \lim_{A\to\infty} \int_0^A e^{-st}dt = \lim_{A\to\infty} \left(\frac{e^{-st}}{-s}\right)\Bigg|_0^A \\ &= \lim_{A\to\infty} \left(-\frac{e^{-sA}}{s} + \frac{1}{s}\right) \\ &= \frac{1}{s}, \text{ since } e^{-sA} \to 0 \text{ as } A \to \infty, \text{ for } s > 0.\end{aligned}$$

An oft-used function is the Heaviside function u_c attached to any $c > 0$:

$$u_c(t) = \begin{cases} 1, & \text{if } t \geq c \\ 0, & \text{if } 0 \leq t < c \end{cases}$$

The Laplace transform of t^n can be calculated by using induction and integration by parts (using $e^{-st}dt = -\frac{1}{s}d(e^{-st})$):

$$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}dt = -\lim_{A\to\infty} \frac{1}{s}\left(t^n e^{-st}\Big|_0^A - n\int_0^A t^{n-1}e^{-st}dt\right) = \frac{n}{s}\mathcal{L}(t^{n-1}).$$

$f(t)$	$F(s) = \mathcal{L}(f(t))$	
1	$\frac{1}{s}$	(for $s > 0$)
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$u_c(t)$	$\frac{e^{-at}}{s}$	(for $s > 0$)
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	(for $s > 0$)
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$(s > 0)$
t^n $(n \geq 0)$	$\frac{n!}{s^{n+1}}$	$(n > 0)$

Important properties of $\mathcal{L}$

1) $\mathcal{L}$ is linear, i.e., $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$, for all constants a, b

2) $\mathcal{L}(f(t)) = \frac{1}{c}F(\frac{s}{c})$, if $c > 0$ and $F(s) = \mathcal{L}(f(t))$

3) $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$ (check this!)

4) If f is *continuous* (not just piecewise continuous) *and* $F(s) = 0$ for all $s > M$, for some $M > 0$, then $f(t) = 0$ for all t.

5) Suppose f is n-times differentiable with $f^{(n)}$ being piecewise continuous and a-nice, then

$$\mathcal{L}(f^{(n)}(t)) = s^n\mathcal{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

Consequence of 4) **and** 1):

Given *continuous* functions f, g on $[0, \infty)$, if $F(s) = G(s)$ for all $s > c$ for some $c > 0$, then $f(t) = g(t)$, for all t.

Reason: By 1),

$$\mathcal{L}(f(t) - g(t)) = \mathcal{L}(f(t)) - \mathcal{L}(g(t)) = F(s) - G(s),$$

which is 0 by hypothesis. So by 4),

$$f(t) - g(t) = 0, \quad \forall\, t.$$

Thanks to the consequence above, $\mathcal{L}$ is a one-to-one linear transformation on the vector space of all continuous functions which are a-nice for some $a > 0$. Hence, it has an inverse, called the *inverse Laplace transformation*, and denoted $\mathcal{L}^{-1}$.

$\mathcal{L}^{-1}$ is also linear.

Example

Consider the 1st order inhomogeneous ODE:

$$\frac{dy}{dt} - by = e^t, \;\; y(0) = 0, \, b \neq 1 \tag{*}$$

(i) Recall: Before using $\mathcal{L}$, let us review how we solved this by our earlier methods. The homogeneous equation

$$\frac{dy}{dt} - by = 0$$

which has eigenvalue $\lambda = b$, has the general solution

$$y_c(t) = ce^{bt}, \;\text{ with } c: \text{ a constant.}$$

So the general solution of $(*)$ is:

$$y(t) = y_c(t) + y_p(t)$$

where $y_p(t)$ is a particular solution of $(*)$. Try

$$\begin{aligned} y_p(t) &= Ae^t \\ \Longrightarrow \;\; Ae^t - bAe^t &= e^t \\ &\Rightarrow A = \frac{1}{1-b}, \;\; b \neq 1. \end{aligned}$$

So the general solution of $(*)$ is given by

$$y_p(t) = ce^{bt} + \frac{1}{1-b}e^t.$$

The initial condition is

$$\begin{aligned} y(0) &= 0 \\ \Rightarrow 0 &= c + \frac{1}{1-b} \\ c &= \frac{1}{b-1} \\ \Longrightarrow \; y(t) &= \frac{1}{1-b}(e^{bt} - e^t) \end{aligned}$$

(ii) Now let's see if we get the same result by using $\mathcal{L}$, assuming that y' is continuous. (f is continuous since it is differentiable.) Applying $\mathcal{L}$ to both sides of $(*)$, we get, by the linearity of $\mathcal{L}$,

$$\begin{aligned} \underbrace{\mathcal{L}(y')}_{sY(s)-y(0)} - b\underbrace{\mathcal{L}(y)}_{Y(s)} &= \underbrace{\mathcal{L}(e^t)}_{\frac{1}{s-1},\text{ for } s>1} \\ \Rightarrow (s-b)Y(s) &= \frac{1}{s-1} \\ \Rightarrow Y(s) &= \frac{1}{(s-1)(s-b)} \end{aligned}$$

Applying $\mathcal{L}$ to both sides, we get

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)(s-b)}\right)$$

Idea: Use partial fractions!

$$\begin{aligned} \frac{1}{(s-1)(s-b)} &= \frac{c_1}{s-1} + \frac{c_2}{s-b} \\ \Rightarrow 1 &= c_1(s-b) + c_2(s-1) \\ c_2 = -c_1 &= \frac{1}{b-1} \\ \frac{1}{(s-1)(s-b)} &= \frac{1}{b-1}\left(\frac{1}{s-b} - \frac{1}{s-1}\right) \\ y(t) &= \frac{1}{b-1}\left[\mathcal{L}^{-1}\left(\frac{1}{s-b}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right] \Longrightarrow y(t) = \frac{1}{b-1}(e^{bt} - e^t), \end{aligned}$$

since $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{-at}$.

Post's formula for $\mathcal{L}^{-1}$

$F(s) = \mathcal{L}(f(t))$

$$\Rightarrow \mathcal{L}^{-1}(F(s)) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right)$$

Example

Let $F(s) = \frac{1}{s-a}$. Let us make sure that this formula gives us the answer we know, namely $f(t) = e^{at}$.

$$\begin{aligned} F'(s) &= \frac{1}{(s-a)^2}, \ldots, F^{(n)}(s) = \frac{(-1)^n n!}{(s-a)^{n+1}} \\ \Rightarrow f(t) &= \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \frac{(-1)^n n!}{(\frac{n}{t}-a)^{n+1}} \\ &= \lim_{n\to\infty} \frac{1}{(1-\frac{at}{n})^{n+1}} \\ \ln f(t) &= at \quad \Rightarrow \quad f(t) = e^{at}. \end{aligned}$$

Lecture 26

We will discuss two more topics related the the Laplace transform method of solving ODE's:

(i) Discontinuous forcing: This pertains to *inhomogeneous* ODE's of the form

$$y'' + p(x)y' + q(x) = F(x),$$

where F is not continuous, but piecewise continuous, for example a step function;

(ii) Convolution of functions, which allows us to calculate the inverse Laplace transform of a product of two or more functions of s.

Review of some basic and useful formulae:

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(u_c(t)) = \frac{e^{-cs}}{s}$$

$$\mathcal{L}(u_c f(t-c)) = e^{-cs}\mathcal{L}(f(t))$$

$$\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin(bt) \tag{a}$$

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos(bt) \tag{b}$$

Example:

$$u'' + \frac{1}{4}u' + u = f(t), \quad u(0) = u'(0) = 0, \tag{*}$$

where

$$f(t) = u_{3/2}(t) - u_{5/2}(t).$$

Apply $\mathcal{L}$ to both sides of $(*)$ to get (by the linearity of $\mathcal{L}$):

$$\mathcal{L}(u'') + \frac{1}{4}\mathcal{L}(u') + \mathcal{L}(u) = \mathcal{L}(f(t))$$

$$\mathcal{L}(u') = s\underbrace{\mathcal{L}(u)}_{=U(s)} - \underbrace{u(0)}_{=0}$$

$$\mathcal{L}(u'') = s^2\mathcal{L}(u) - \underbrace{su'(0) - u(0)}_{=0}$$

Get

$$U(s)\left(s^2+\frac{1}{4}s+1\right)=F(s),$$

where

$$F(s)=\mathcal{L}(f(t))=\mathcal{L}(u_{3/2}(t))-\mathcal{L}(u_{5/2}(t))=\frac{e^{-3s/2}-e^{-5s/2}}{s}.$$

$$\Rightarrow U(s) = \frac{e^{-3s/2}-e^{-5s/2}}{s(s^2+\frac{1}{4}s+1)}, \quad \text{if} \quad s^2+\frac{1}{4}s+1\neq 0.$$

The roots of $s^2+\frac{1}{4}s+1$ are $s=-\frac{1}{8}\pm\frac{1}{2}\sqrt{\frac{1}{16}-4}$, which are not real! Hence $s^2+\frac{1}{4}s+1$ cannot be 0 for any real s.

Use partial fractions:

$$G(s):=\frac{1}{s(s^2+\frac{1}{4}s+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+\frac{1}{4}s+1}.$$

Clearing the denominator,

$$\begin{aligned}1&=As^2+\frac{1}{4}As+A+Bs^2+Cs\\ 1&=(A+B)s^2+(\frac{1}{4}A+C)s+A\\ \Rightarrow A&=1,\ B=-A=-1,\ C=-\frac{1}{4}A=-\frac{1}{4}\end{aligned}$$

Hence

$$G(s)=\frac{1}{s}-\frac{s+\frac{1}{4}}{s^2+\frac{1}{4}s+1}.$$

Recall that

$$\mathcal{L}^{-1}(e^{cs}G(s))=u_c(t)g(t-c),$$

so that

$$\begin{aligned}u(t)=\mathcal{L}^{-1}(U(s))&=\mathcal{L}^{-1}(e^{-3s/2}G(s))-\mathcal{L}^{-1}(e^{-5s/2}G(s))\\ &=u_{3/2}(t)f(t-\frac{3}{2})-u_{5/2}(t)f\left(t-\frac{5}{2}\right)\end{aligned}$$

Need to find f, and for this we use formulae (a) and (b) from the previous page. It is left as an exercise to get the final explicit answer.

Convolution

Suppose $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$ in $s > a \geq 0$. Put

$$H(s) = F(s)G(s).$$

Question: Can we find a function $h(t)$ st $H(s) = \mathcal{L}(h(t))$ for $s > a$? In other words,

$$\textit{What is } \mathcal{L}^{-1}(F(s)G(s))\textit{?}$$

Answer: Yes. $h(t)$ is the "convolution" of f and g, denoted $f * g$.

Definition:

$$h(t) = (f * g)(t) = \int_0^t f(t-u)g(u)du$$

which also equals

$$\int_0^t f(u)g(t-u)du.$$

These two integral expressions are equal because we can use change of variables: $v = t - u$, $dv = -du$, yielding

$$\int_0^t f(t-u)g(u)du = -\int_t^0 f(v)g(t-v)dv = \int_0^t f(v)g(t-v)dv$$

$\Rightarrow f * g = g * f$ (commutativity of convolution)

Examples:

1. $(f * 1)(t) = \int_0^t f(t-u)du = \int_0^t f(u)du.$
   In particular, $(1 * 1)(t) = t \neq 1$
2. $f = \sin t$,
   $(f * 1)(t) = \int_0^t \sin u du = -\cos u|_0^t = 1 - \cos t$
3. $f * f$ need not be ≥ 0. For example, if $f = \sin(t)$, then

$$\begin{aligned}(f * f)(t) &= \int_0^t \sin(t-u)\sin(u)du \\ &= \frac{1}{2}(\sin t - t\cos t),\end{aligned}$$

   which is negative at $t = 2n\pi$, for any positive integer n.

Reason for importance of convolution to the solving of ODE's

Given

$$y'' + ay' + by = f(x),\ y(0) = y'(0) = 0,$$

with a, b constants, apply $\mathcal{L}$ to both sides to get

$$Y(s)(s^2 + as + b) \ = \ F(s),$$

so that for s large enough so that $s^2 + as + b \neq 0$,

$$Y(s) = F(s)G(s),\ \text{ where } G(s) = \frac{1}{s^2 + as + b}.$$

By using partial fractions and the formulae from an earlier page, we can find $g(t)$ such that $G(s)$ is the Laplace transform of $g(t)$. This implies that

$$y(t) \ = \ \mathcal{L}^{-1}(F(s)G(s)) \ = \ (f * g)(t).$$

This method is also useful, if applied with care, when a, b are not constants.

Example:

$$y'' + 2y' + 2y = \sin(\omega t),\ y(0) = y'(0) = 0,$$

where ω is called the *forcing frequency* (or external frequency).

Apply $\mathcal{L}$ to both sides, and use the linearity of $\mathcal{L}$, to obtain, with $f(t) = \sin \omega t$,

$$\begin{aligned}
Y(s)(s^2 + 2s + 2) &= \mathcal{L}(\sin \omega t) = F(s) \\
s^2 + s + 2 &= (s^2 + 2s + 1) + 1 \\
&= (s + 1)^2 + 1 \geq 1 \\
\Rightarrow Y(s) &= F(s)G(s),\ \text{ with } G(s) = \frac{1}{(s+1)^2 + 1} \\
g(t) &= \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2 + 1}\right) = e^{-t} \sin t \\
\Rightarrow y(t) &= \text{convolution of } f(t) = \sin \omega t \ \text{ and } g(t) = e^{-t} \sin t \\
y(t) &= \int_0^t f(t - u)g(u)du = \int_0^t \sin(\omega(t - u))e^{-u} \sin u du.
\end{aligned}$$

The integral on the right can be explicitly calculated using the addition formula

$$\sin A \sin B \;=\; \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right],$$

and integration by parts.

Note that we did not use here our prior knowledge that $F(s) = \frac{\omega}{s^2+\omega^2}$.