Lecture 22

Power series solutions for 2nd order linear ODE's

(not necessarily with constant coefficients)

Recall a few facts about power series:

$$\sum_{n=0}^{\infty} a_n z^n$$

This series in z is centered at $z = 0$.

Here z can be real or complex. Denote b $|z|$ its absolute value, which is a non-negative real number.

The following questions arise:

(Q1) When does this converge absolutely, i.e., when does $\sum_n a_n |z|^n$ converge? Moreover, is there a real number $R > 0$ such that $\sum a_n z^n$ converge absolutely $\forall z$ with $|z| < R$, but diverges for $|z| > R$?

(Q2) When can we differentiate $\sum a_n z^n$ term by term, i.e., when do we have the equality
$$\left(\sum a_n z^n\right)' = \sum n a_n z^{n-1} \,?$$

(Q3) Given a function $f(z)$ when can we express it as a power series $\sum a_n b z^n$, for z in some region, say for $|z| < R$?

Given $\sum_{n=0}^{\infty} a_n z^n$, we look at

$$\begin{aligned} L &= \lim_{n\to\infty} \left(\frac{|a_{n+1}| \cdot |z|^{n+1}}{|a_n| \cdot |z|^n} \right) \\ &= \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \right) |z| \end{aligned}$$

If L makes sense, it will be a non-negative real number.

A *basic result* (cf. Chapter 2 of the Notes for Ma1a) is the following:

Theorem 1 $\sum a_n z^n$ *converges absolutely iff* $L < 1$.

Definition *The radius of convergence* $R \geq 0$ *is the largest non-negative real number such that*

⋆ $\sum a_n z^n$ *converges absolutely for* $|z| < R$, *and*

⋆ $\sum a_n z^n$ *diverges for* $|z| > R$.

We will allow R *to be* ∞, *in which case* $\sum a_n z^n$ *converges* $\forall z$.

By this definition,

$$|z| < R \iff \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|z| = 1.$$

So $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. This answers Q1.

For Q2, here is another basic result (cf. Chap. 2 of the Notes for Ma1a):

Theorem 2 *For* $|z| < R$, *we have*

$$(\sum a_n z^n)' = \sum a_n n z^{n-1}.$$

If $|z| = R$, *anything can happen.*

To understand Q3 better, recall (from Ma1a) that if f is infinitely differentiable at $z = 0$, then we have the associated **Taylor series**

$$\sum_{n\geq 0} a_n z^n, \text{ with } a_n = \frac{f^{(n)}(0)}{n!}.$$

Examples of infinitely differentiable functions at $z = 0$:

e^z, $\sin(z), \cos(z)$, polynomials, and rational functions $P(z)/Q(z)$, with P, Q polynomials and $Q(0) \neq 0$.

A Subtle Fact:

The Taylor series of a function f *(around* $z = 0$*) may not equal* $f(z)$, *for* f *infinitely differentiable.*

The simplest example of such a function is $\varphi(x)$ which is $e^{-1/x}$ if $x > 0$ and 0 if $x \leq 0$. You may check that φ is differentiable to any order at $x = 0$, but with $\varphi^{(n)}(0) = 0$ for all $n \geq 0$, making the Taylor expansion around 0 to be identically 0. But $\varphi(x)$ is > 0 for $x > 0$, however small, and so cannot be represented by the Taylor expansion.

However, pathology like this does not happen if there is an $R > 0$ such that for all z with $|z| < R$, $f(z)$ has a(n absolutely convergent) power series expansion

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

In such a case, f is said to be an **analytic function** at $z = 0$, in fact in $|z| < R$. Since we can differentiate under the sum sign for $|z| < R$ (by Theorem 2), it follows (by evaluating each series expression for $f^{(n)}(z)$ at $z = 0$) that we must have

$$f: \text{ analytic } \implies a_n = \frac{f^{(n)}(0)}{n!}.$$

Examples of analytic functions

(1)

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}.$$

Applying the Ratio test,

$$L = \left(\lim_{n\to\infty} \frac{a_{n+1}}{a_n}\right) |z| < 1, \ \forall z \text{ in } \mathbb{C}$$

$$\lim_{n\to\infty} \left(\frac{1}{(n+1)!}\frac{n!}{1}\right) = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\Rightarrow R = \infty$, i.e., e^z is represented by this power series for all z; so $f(z) = e^z$ is analytic $\forall z$, and $\frac{1}{n!} = \frac{f^{(n)}(0)}{n!}$.

(2)

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \ldots$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)! + \ldots}$$

(3) $f(z) = \frac{1}{1-z} = \sum^{\infty} z^n$, a rational function regular at $z = 0$, i.e., the denominator of $f(z)$ is non-zero at $z = 0$.

$$L = \underbrace{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}_{1, \text{ as } a_n = 1, \ \forall\ n} |z| = |z|,$$

implying that $L < 1$ iff $|z| < 1$. So $R = 1$, and our function is analytic in $|z| < 1$.

(4) $f(z) = \ln(z)$, z real and positive.

As $z \to 0$, $\ln(z) \to -\infty$. This is nevertheless analytic near $z = 1$, since we can write

$$f(z) = \sum_n a_n (z-1)^n,$$

which is a convergent power series around $z = 1$. In fact we *claim* that that $R = 1$ in this case. In particular, the series expansion is valid for any small z, as long as it is non-zero.

To check the Claim, write $\log(z) = f(z)$ as $\ln(1 + (z - 1))$. Then, with $u = z - 1$, we have

$$f'(z) = \frac{1}{z} = \frac{1}{1+(z-1)} = \frac{1}{1+u}, \quad \text{and}$$
$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 - \ldots,$$

which is valid, by (3), for $|u| < 1$ (as $R = 1$). The series expression for $\frac{1}{1+u}$ can, by Theorem 2, be integrated term by term for $|u| < 1$. This way we get a power series for $\ln(1 + u)$, namely

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \ldots,$$

valid in $|u| < 1$. Since $f(z) = -\ln(1 + u)$, we get

$$\ln z = \sum_{n=1}^{\infty} (-1)^n \frac{(z-1)^n}{n},$$

which holds for all z with $|z - 1| < 1$. Hence the Claim.

Ordinary points and Singular points

Suppose we have a *linear ODE of 2nd order*:

$$(*) \qquad c_1(x)\frac{d^2u}{dx^2} + c_2(x)\frac{du}{dx} + c_3(x)u = 0,$$

where the coefficients are not (necessarily) constant! The fact that $(*)$ has second order means $c_1(x)$ is not identically zero. However, it could be zero for special values of x, where we need to exercise some care.

The values of x where $c_1(x) = 0$ are called the **singular points** of the ODE, while those x where $c_1(x) \neq 0$ are called the **ordinary points**.

Often there are only a finite number of singular points, but this is not always the case, as seen by looking at the example $c_1(x) = \sin x$, which vanishes at $n\pi$ for any integer n.

At the *ordinary points*, one can divide $(*)$ by $c_1(u)$ to get a "monic" ODE:

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u \;=\; 0,$$

with

$$p(x) = \frac{c_2(x)}{c_1(x)}, \text{ and } q(x) = \frac{c_3(x)}{c_1(x)}.$$

We will look for power series solutions!

Lecture 23

Last time we ended by looking at the general 2nd order linear homogeneous ODE

$$c_1(x)\frac{d^2y}{dx^2} + c_2(x)\frac{dy}{dx} + c_3(x)y = 0, \qquad (*)$$

with $c_1(x)$ not identically zero. We are interested in the case when $c_1(x), c_2(x), c_3(x)$ are not constants.

Definition. *A point x_0 is an* **ordinary point** *for $(*)$ iff $c_1(x) \neq 0$. Otherwise x_0 is called a* **singular point**.

More precisely, when $c_1(x) \neq 0$ we can divide by it, so $(*)$ becomes

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \;\; p(x) = \frac{c_2(x)}{c_1(x)}, q(x) = \frac{c_3(x)}{c_1(x)}.$$

To say x_0 is an ordinary point means p, q are analytic at x_0, i.e.,

$$p(x) = \sum_{n=0}^{\infty} \alpha_n (x - x_0)^n, \;\; q(x) = \sum_{n=0}^{\infty} \beta_n (x - x_0)^n,$$

valid in an interval around x_0.

Recall the following key *examples of analytic functions* at x_0:

1) Polynomials in x
2) Rational functions $\frac{f(x)}{g(x)}$, f, g polynomials, with $g(x_0) \neq 0$
3) Exponential functions $e^{xt}, e^{x^2+2}, \ldots$
4) $\sin x$, $\cos x$
(5) $\tan x$, if $x_0 \neq (2n+1)\frac{\pi}{2}$

The Basic Principle: Suppose x_0 is an ordinary point for

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \qquad (*)$$

Then it has a basis of (analytic) solutions of the form

$$y_1(x) = 1 + b_2(x - x_0)^2 + \cdots = 1 + \sum_{n=2}^{\infty} b_n(x - x_0)^n,$$

$$y_2(x) = (x - x_0) + c_2(x - x_0)^2 + \cdots = (x - x_0) + \sum_{n\geq 2}^{\infty} c_n(x - x_0)^n.$$

Hence the general solution is of the form

$$y = a_0y_1 + a_1y_2 = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where a_0, a_1 are arbitrary constants, and for $n \geq 2$, $a_n = a_0b_n + a_1c_n$.

Examples

(1)

$$\frac{d^2y}{dx^2} + y = 0$$

This ODE even has constant coefficients, every point is an ordinary point for this ODE. We already know that a basis of solutions is given by $\{\sin x, \cos x\}$, but we want to derive this using power series.

Try a power series solution for y at $x_0 = 0$:

$$y = \sum_{n=0}^{\infty} a_nx^n$$

If the radius of convergence $R > 0$, then we can differentiate term by term, i.e.,

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{d}{dx}(a_nx^n) = \sum_{n=0}^{\infty} a_nnx^{n-1} = \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m,\ m = n - 1,$$

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \sum_{m=1}^{\infty} m(m+1)a_{m+1}x^{m-1}\\ &= \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k.\end{aligned}$$

So

$$\begin{aligned}\frac{d^2y}{dx^2} + y &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n\\ &= \sum_{n=0}^{\infty}\left((n+1)(n+2)a_{n+2} + a_n\right)x^n\end{aligned}$$

For the above sum to be 0 at all points in an open interval around 0, we must have

$$(n+1)(n+2)a_{n+2} + a_n \;=\; 0,\; \forall\, n \geq 0.$$

In other words, to have such a power series solution for the ODE, we must satisfy the **recursive formula** for all $n \geq 0$:

$$a_{n+2} \;=\; \frac{-1}{(n+1)(n+2)}a_n.$$

$$\begin{aligned}\implies\quad a_2 &= -\frac{1}{2}a_0,\; a_4 = -\frac{1}{1\cdot 2}a_2 = \frac{1}{2\cdot 3\cdot 4}a_0,\\ a_6 &= -\frac{1}{5\cdot 6}a_4 = \frac{1}{5\cdot 6\cdot 4\cdot 3}a_2 = -\frac{1}{6!}a_0,\; \ldots\end{aligned}$$

Hence

$$a_{2n} \;=\; \frac{(-1)^n}{(2n)!}a_0,\;\; \forall\, n \geq 0.$$

Similarly, $a_3 = \frac{-1}{2\cdot 3}a_1, a_5 = \frac{1}{2\cdot 3\cdot 4\cdot 5}a_1,\; \ldots,$ implying

$$a_{2n+1} \;=\; \frac{(-1)^n}{(2n+1)!}a_1,\;\; \forall\, n \geq 0.$$

Thus

$$
\begin{aligned}
y = \sum_{n=0}^{\infty} a_n x^n &= \sum_{n \text{ even } \geq 0} a_n x^n + \sum_{n \text{ odd } \geq 0} a_n x^n \\
&= \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} \\
\Rightarrow \ y \ &= \ a_0 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}}_{=\cos x} + a_1 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}}_{=\sin x}
\end{aligned}
$$

So the general solution is (as expected) of the form

$$y = a_0 \cos x + a_1 \sin x,$$

where a_0, a_1 are arbitrary constants. In the notation of the *Basic Principle*, $y_1 = \cos x$ and $y_2 = \sin x$. Finally, since we know that the power series expressions are valid for all x (with $R = \infty$), the power series solutions we found are also valid everywhere.

(2)

$$y'' + xy' + y \ = \ 0$$

This is a linear ODE with *non-constant coefficients*.

Since $p(x) = x$, $q(x) = 1$, they are analytic everywhere, and so any x_0 is an ordinary point. Let $x_0 = 0$. Try $y = \sum_{n=0}^{\infty} a_n x^n$ and assume $R > 0$. Then

$$
\begin{aligned}
x\frac{dy}{dx} &= x \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n \\
\frac{d^2y}{dx^2} &= \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} x^m
\end{aligned}
$$

So

$$y'' + xy' + y \ = \ \sum_{n=0} ((n+1)(n+2)a_{n+2} + n a_n + a_n) x^n,$$

which will be 0 iff all the coefficients are 0, giving rise to the recursive formula:

$$(n+1)(n+2)a_{n+2} = -(n+1)a_n$$
$$\implies a_{n+2} = \frac{-a_n}{n+2} \text{ for all } n \geq 0$$

(We are able to cancel $n+1$ from both sides because it is non-zero for $n \geq 0$.) Explicitly,

$$a_2 = \frac{-a_0}{2},\ a_4 = \frac{-a_2}{4} = \frac{a_0}{4 \cdot 2},\ a_6 = \frac{-a_0}{6 \cdot 4 \cdot 2}, \ldots,$$

yielding

$$a_{2m} = \frac{(-1)^m a_0}{(2m)(2m-2)(2m-4)\ldots 4(2)} = \frac{(-1)^m a_0}{2^m m!}.$$

The odd coefficients all depend on a_1. If we put $a_1 = 0, a_0 = 1$, we get one fundamental solution, and if we put $a_0 = 0, a_1 = 1$, we get the other fundamental solution

Let $a_1 = 0, a_0 = 1$. Then

$$y_1 = \sum_{m=0}^{\infty} a_{2m} x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m m!}$$
$$= \sum_{m=0}^{\infty} \left(\frac{-x^2}{2}\right)^m \frac{1}{m!}$$

So

$$y_1 = e^{-x^2/2}.$$

Note that when a linear ODE has constant coefficients, the fundamental solutions involve terms like $e^{\lambda x}$, but here we have a quadratic exponent of the exponential, caused by the ODE having non-constant coefficients. It could even be more complicated in that a fundamental solution may not be simply expressible - at all (as with the case of y_2 below) in terms of an exponential function.

Let $a_1 = 1, a_0 = 0$. We get

$$y_2 = \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1}, \text{ with}$$

$$a_3 = \frac{-a_1}{3},\ a_5 = \frac{-a_3}{5} = \frac{-a_1}{5 \cdot 3},\ \ldots,\ a_{2m+1} = \frac{(-1)^m}{(2m+1)(2m-1)(2m-3)\ldots(3)}, \ldots$$

Thus

$$a_{2m+1} = \frac{(-1)^m(2m)(2m-2)\dots 4(2)}{(2m+1)!} = \frac{(-2)^m m!}{(2m+1)!},$$

and

$$y_2 \;=\; \sum_{m=0}^{\infty} \frac{(-2)^m m! x^{2m+1}}{(2m+1)!}.$$

It will be left as an exercise to check, as we did in Example 1, that the power series representing y_1 and y_2 have positive radii of convergence. (Is $R = \infty$ in each case?)

Regular Singular Points

Definition: *A singular point x_0 for an ODE*

$$y'' + p(x)y' + q(x) = 0$$

is called a **regular singular point** *iff $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at $x = x_0$.*

Example:

$$x^2 y'' + \sin x y' + y \;=\; 0,$$

which obviously has $x = 0$ as the unique singular point. Now

$$p(x) = \frac{\sin x}{x^2},\;\; q(x) = \frac{1}{x^2}$$

$$xp(x) = \frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}\dots}{x}$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1)!}$$

Since $x^2 q(x) = 1$, it is obviously analytic (everywhere). We need to check that the power series for $xp(x)$ converges around $x = 0$. For this we apply the Ratio test:

$$\lim_{n\to\infty} \frac{(-1)^{n+1}x^{2(n+1)}/(2(n+1)+1)!}{(-1)^n x^{2n}/(2n+1)!} \;=\; \lim_{n\to\infty} \frac{-x^2}{(2n+3)(2n+2)} \;=\; 0,$$

for any x. So $R = \infty$, and $x^2 p(x)$ is analytic everywhere, not just around $x = 0$.

So $x = 0$ is a regular singular point of this ODE, and every other point is ordinary!