# Lecture 19

## Linear ODE's of higher order

Solutions can be obtained by solving linear system of first order ODE's.

**Idea**: Replace one equation by many equations, thereby reducing the order of the ODE to just first order.

$n \geq 1$: A $n$-th order linear ODE is of the form

$$\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1}u}{dx^{n-1}} + a_2 \frac{d^{n-2}u}{dx^{n-2}} + \cdots + a_{n-1}\frac{du}{dx} + a_n u = f(x) \qquad (*)$$

Here the coefficients $a_j$ could depend on $x$. We say that $(*)$ has constant coefficients iff each $a_j$ is independent of $x$.

$(*)$ is homogeneous if $f(x) = 0$. First, look at the homogeneous case:

Define a vector $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ in $n$-space where

$$y_1 = u,\ y_2 = \frac{du}{dx},\ y_3 = \frac{d^2u}{dx^2},\ \ldots,\ y_n = \frac{d^{n-1}u}{dx^{n-1}}.$$

Then $(*)$ can be rewritten suggestively as

$$\begin{aligned}
\frac{dy_1}{dx} &= y_2 \\
\frac{dy_2}{dx^2} &= y_3 \\
&\vdots \\
\frac{dy_{n-1}}{dx^{n-1}} &= y_n \\
\frac{dy_n}{dx^n} &= -a_n y_1 - a_{n-1} y_2 \ldots \quad \ldots - a_1 y_n
\end{aligned}$$

This can be viewed as a matrix of linear equations, and so gives rise to the linear system:

$$\frac{d}{dx}\mathbf{y} = A\mathbf{y}, \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1 & & & & 0 \\ & 0 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 0 & 1 \\ -a_n & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{pmatrix} \qquad (**)$$

Whether or not $A$ has constant coefficients, a solution matrix is given by

$$\Phi(x) \;=\; e^{\int_0^x A(t)dt},$$

as long as $A$ is integrable as a function of $x$, which is the case, for example, when $A$ is continuous. (If $A$ is not defined at 0, then we can take some other lower limit where $A$ is defined; the effect of changing the lower limit is to multiply $\Phi$ by a constant.)

Of course, *when $A$ is a constant matrix*, $\int_0^x A(t)dt$ is just $Ax$, and we get the more familiar expression

$$\Phi(x) \;=\; e^{Ax}.$$

If $\Phi(x) = \left(\mathbf{y}^{(1)}\,\mathbf{y}^{(2)}\,\ldots\,\mathbf{y}^{(n)}\right)$, then the $\mathbf{y}^{(j)}$ are the different solution vectors:

$$\mathbf{y}^{(j)} = \begin{pmatrix} y_1^{(j)} \\ y_1^{(j)} \\ \vdots \\ y_n^{(j)} \end{pmatrix}$$

What we seek is a set of independent solutions to the original ODE $(*)$.

If $A$ has constant coefficients, then for every $j \le n$, $y_2^{(j)} = \frac{dy_1^{(j)}}{dx}, \ldots, y_n^{(j)} = \frac{dy_{n-1}^{(j)}}{dx}$.

**Useful fact 1**: If $A$ is a constant matrix with $n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$, then a basis of solution for $(**)$ is given by

$$\mathbf{y}^{(1)} = \mathbf{v}_1 e^{\lambda t},\; \mathbf{y}^{(2)} = \mathbf{v}_2 e^{\lambda_2 t},\; \ldots\;, \mathbf{y}^{(n)} = \mathbf{v}_n e^{\lambda t}.$$

If $\mathbf{v}^{(j)} = \begin{pmatrix} v_1^{(j)} \\ \vdots \\ v_n^{(j)} \end{pmatrix}$, then a basis of solutions of the $n$-th order ODE $(*)$ is given by

$$v_1^{(1)} e^{\lambda_1 t},\ v_1^{(2)} e^{\lambda_2 t},\ \ldots,\ v_1^{(n)} e^{\lambda_n t}$$

**Useful fact 2**: Recall that The eigenvalues of $A$ are determined by solving the characteristic equation: $|\lambda - A| = 0$.

$n = 2$: $\quad A = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_2 \end{pmatrix}$, $\lambda - A = \begin{pmatrix} \lambda & -t \\ a_2 & \lambda + a_1 \end{pmatrix}$

$$\Rightarrow |\lambda - A| = \lambda^2 + a_1\lambda + a_2$$

*General formula*:

$$|\lambda - A| = \lambda^2 = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

As we started with the $n$th order linear, homogeneous ODE:

$$\frac{d^2y}{dx} + a_1\frac{d^{n-1}u}{dx^{n-1}} + \ldots a_{n-1}\frac{du}{dx} + a_n u = 0,$$

we can read off the characteristic equation directly from the ODE.

**2nd order linear ODE's**

Many ODE's occurring in nature are of second order, and it is essential to understand them thoroughly, at a minimum when they are linear with constant coefficients.

A prime example (which contains all the subtleties) is the base case of a "mass on a spring", where $u$ = displacement, $t$ = time, and $k$ = spring constant:

$$m\frac{d^2u}{dt^2} = mg - k(L + u) - \underbrace{\gamma\frac{du}{dt}}_{\text{damping force}} + \underbrace{F_{\mathrm{e}}(t)}_{\text{external force}}$$

Dividing by the mass $m > 0$, and noting that $mg = kL$ by Hooke's Law, we may rewrite this as

$$\frac{d^2u}{dt^2} + \frac{\gamma}{m}\frac{du}{dt} + \frac{k}{m}u \;=\; F_{\mathrm{e}}(t),$$

which is a linear ODE with constant coefficients. It is homogeneous iff there is no external force, i.e., $F_{\text{ext}}(t) = 0$.

This spring vibrates, and we want to find an expression for $u$ as a function of $t$. If $F_{\text{e}}(t) = 0$, we say that we have *free vibrations*.

One says the vibrations are *undamped* if $\gamma = 0$, i.e., there is no damping force.

**Undamped free vibrations**:

$u''(t) + \frac{k}{m}u(t) = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}$ : **frequency**

*Characteristic equation*: $\lambda^2 + \frac{k}{m} = 0$

We have two distinct eigenvalues

$$\lambda_{\pm} = \pm i\omega_0, \; \omega_0 = \sqrt{\frac{k}{m}},$$

both of which are non-real. (They are complex conjugates of each other.)

The corresponding linear system is

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} u \\ u' \end{pmatrix},$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}$$

*Eigenvectors*: We want to find $\mathbf{v}^{\pm} \neq 0$ st

$$A\mathbf{v}^{\pm} = \pm i\omega_0 \mathbf{v}^{\pm}, \; \mathbf{v}^{\pm} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

So need $\begin{pmatrix} y_2 \\ -\frac{k}{m}y_1 \end{pmatrix} = \begin{pmatrix} \pm i\omega_0 y_1 \\ \pm i\omega_0 y_2 \end{pmatrix} \Leftrightarrow y_2 = \pm i\omega_0 y_1$.

We can take

$$\mathbf{v}^{+} = \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix}, \; \mathbf{v}^{-} = \begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix}.$$

Hence a basis of solutions of the linear system is given by

$$\begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} e^{i\omega_0 t}, \; \begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix} e^{-i\omega_0 t},$$

implying that a *basis of complex solutions of the second order ODE* is given by $\begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix} e^{-i\omega_0 t}$ is

$$z_1 = e^{i\omega_0 t}, \ z_2 = e^{-i\omega_0 t}.$$

The general complex solution is

$$C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t},$$

for arbitrary (complex) constants $C_1, C_2$.

What we want, however, are **real solutions**!

To get a basis of real solutions,we look at the real and imaginary parts of $z_1, z_2$. Since $z_2$ is the complex conjugate of $z_1$, we need only look at $z_1$. Thus a *basis of real solutions* is given by

$$\cos t, \ \sin t.$$

Hence the **general real solution** is

$$u(t) \ = \ A\cos(\omega_0 t) + B\sin(\omega_0 t), \quad A, B \in \mathbb{R}.$$

It is often convenient to express this as

$$u(t) \ = \ R\cos(\omega_0 t - \delta),$$

with $R$ denoting the *amplitude* and $\delta$ denoting the phase.

# Lecture 20

**Second Order Linear ODE's with constant coefficients**

*Typical example*: Equation of "mass on a spring"

$$u''(t) + \underbrace{au'(t)}_{\text{dampingforce}} + bu(t) = \underbrace{F_{\mathrm{e}}(t)}_{\text{“external force''}},$$

where

$$a = \frac{\gamma}{m}, \quad b = \frac{k}{m} > 0$$

In the process of analyzing $(*)$, we will introduce some key basic terms which are also used in many other contexts. such as vibrations (free or forced), frequency, period, phase, quasi-frequency, quasi period, damping, amplitude, and resonance.

## I Undamped Free Oscillations

"Vibration": trajectory of a solution to $(*)$

"forced" means $F_{\mathrm{e}} \neq 0$

"free" means $F_{\mathrm{e}} = 0 \quad \Leftrightarrow \quad (*)$ is homogeneous

$\gamma =$ damping constant

"undamped" means $\gamma = 0$

So the ODE $(*)$ becomes in this case:

$$u'' + \frac{k}{m}u = 0,$$

with the associated linear system being

$$\mathbf{y}' = A\mathbf{y}, \ \mathbf{y} = \begin{pmatrix} u \\ u' \end{pmatrix}, \ A = \begin{pmatrix} 0 & -1 \\ -w & 0 \end{pmatrix}$$

The characteristic equation is $\lambda^2 + \frac{k}{m} = 0$, so the eigenvalues are $\lambda_{\pm} = \pm i\omega_0$. This is a case of distinct eigenvalues. It's common to write $\omega_0 = \sqrt{\frac{k}{m}}$ : "frequency of vibrations".

We discussed this case thoroughly last time, and found that a basis of complex solutions of the linear system $(**)$ is

$$z^{(1)} = \begin{pmatrix} e^{i\omega_0 t} \\ i\omega_0 e^{i\omega_0 t} \end{pmatrix}, \; z^{(2)} = \begin{pmatrix} e^{-i\omega_0 t} \\ -i\omega_0 e^{-i\omega_0 t} \end{pmatrix}$$

Let $z_1, z_2$ be the first entries of $z^{(1)}, z^{(2)}$ respectively. Since the first coordinate of $\mathbf{y}$ is $u$, it follows that a basis of (complex) solutions of $(*)$ is given by

$$z_1 = \underbrace{e^{i\omega_0 t}}_{\cos(\omega_0 t)+i\sin(\omega_0 t)}, \quad z_2 = \underbrace{e^{-i\omega_0 t}}_{\cos(\omega_0 t)-i\sin(\omega_0 t)} = \overline{z_1}.$$

A basis of **real solutions** to $(*)$ is given by

$$x_1 = \cos(\omega_0 t), \;\; y_1 = \sin(\omega_0 t),$$

which are the real and imaginary parts of $z_1$.

Hence the **general real solution** to $(*)$ is

$$u(t) = b_1 \cos(\omega_0 t) + b_2 \sin(\omega_0 t).$$

A better (more geometric) way to write the general real solution can be expressed as

$$u(t) = R\cos(\omega_0 t - \delta)$$

where $R$ is the amplitude, and $\delta$ the "phase" of the vibration.

*Recall from Trigonometry*:

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2,$$

so that

$$R\cos(\omega_0 t - \delta) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t).$$

This allows us to go back and forth between the two expressions for the general real solution $u(t)$.

## II Damped free oscillations

$$u'' + \frac{\gamma}{m}u' + \frac{k}{m}u = 0 \tag{*}$$

Characteristic equation: $\lambda^2 + \frac{\gamma}{m}\lambda + \frac{k}{m} = 0$

*Eigenvalues*: $\lambda_\pm \; = \; -\frac{\gamma}{2m} \pm \frac{1}{2}\sqrt{\frac{\gamma^2}{m^2} - \frac{4k}{m}}$

There three cases, depending on whether the eigenvalues are real and distinct, or non-real (and complex conjugates of each other), or real and repeated.

**Case (i)**: $\boldsymbol{\gamma^2 > 4km}$

Here $\lambda\pm \in \mathbb{R}$ with $\lambda_+ \neq \lambda_-$.

*General solution*:

$$u(t) \; = \; b_1 e^{\lambda_+ t} + b_2 e^{\lambda_- t},$$

where $b_1, b_2$ are arbitrary real numbers.

*Note*: $1 - \frac{4km}{\gamma^2} < 1$, so $\lambda_+$ and $\lambda_-$ are both negative, so $e^{\lambda_- t} \to 0$ and $e^{\lambda_- t} \to 0$ as $t \to \infty$.

**Case (ii)**: $\boldsymbol{\gamma^2 < 4km}$

Here $\lambda_\pm$: not real, with $\lambda_-$ the complex conj of $\lambda_+$ (since characteristic equation has real coefficients).

*General solution*:

$$u(t) = e^{-\gamma t/2m} \; (b_1 \cos(\nu t) + b_2 \sin(\nu t)),$$

where

$$\nu \; = \; \frac{\sqrt{4km - \gamma^2}}{2m} > 0$$

If $\gamma = 0$, as if there is no damping, this is like **I** considered earlier.

We call $\nu$ the *quasi-frequency* of vibration, and $\frac{2\pi}{u}$ the *quasi-period*.

We can rewrite the general solution in the better form

$$u(t) = Re^{-\gamma t/2m} \cos(\nu t - \delta)$$

Even though $R$ is supposed to be the amplitude, it becomes evident by drawing the trajectory of $u(t)$ that the quantity $Re^{-\gamma t/2m}$ is the true amplitude (as it includes the effect of damping); it is sometimes called the "quasi-amplitude."

**Case (iii)**: $\boldsymbol{\gamma^2 = 4km}$ (*critically damped*)

There is a unique eigenvalue $\lambda = -\frac{\gamma}{2m}$ with multiplicity of 2. Recall from the earlier lectures that the fundamental solutions of the associated linear system for $\mathbf{y} = \begin{pmatrix} u \\ u' \end{pmatrix}$ are given by

$$\begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix}, \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix}.$$

Hence a *basis of solutions* of the second order ODE $(*)$ for $u$ is given by

$$e^{-\gamma t/2m}, \ te^{-\gamma t/2m}.$$

The *general solution* is

$$u(t) \ = \ (b_1 + tb_2)e^{-\frac{\gamma t}{2m}}.$$

## III Forced Vibrations

Here the external force $F_{\mathrm{e}}(t)$ is non-zero, so the ODE $(*)$ is *inhomogeneous*:

$$u'' + au' + bu \ = \ F_{\mathrm{e}}(t), \ a = \frac{\gamma}{m}, b = \frac{k}{m}.$$

*Important special case*: $F_{\mathrm{e}}(t)$ is **periodic**

*Example*: $F_{\mathrm{e}}(t) = F_0 \cos(wt), \quad w \neq \omega_0$

(Recall that $\omega_0 = \sqrt{\frac{k}{m}}$.)

**Idea**: *First find the general solution $u_c(t)$ of the homogeneous equation $u'' + au' + bu = 0$, and then add to it any (one) particular solution $U(t)$ of the inhomogeneous equation.*

This procedure in fact gives us all the solutions of the inhomogeneous equation, and this idea is useful even when $F_{\mathrm{e}}$ is not periodic.

*General solution*:

$$\underbrace{u(t)}_{\text{“forced response”}} = \underbrace{u_c(t)}_{\text{transient solution}} + \underbrace{U(t)}_{\text{particular solution}}$$

When $F_{\mathrm{e}}(t) = F_0 \cos(wt)$, the choice of $U(t) = A\cos(wt) + B\sin(wt)$ works for suitable $A$ and $B$. (Check this!)

When there's damping, i.e., $\gamma \neq 0$, $U_c(t) \to 0$ as $t \to \infty$.

*Key new effect*: **Resonance** when $w \approx \omega_0$

**Lecture 21**

## Basic results on linear, homogeneous ODE's of higher order

$$\frac{d^n u}{dt^n} + a_1 \frac{d^{n-1}u}{dt^{n-1}} + \cdots + a_{n-1}\frac{du}{dt} + a_n u = 0 \qquad (*)$$

*Eigenvalues*: Solutions of the characteristic equation $\lambda^n + a_1\lambda^{n-1} + \dots a_n = 0$. Let $\lambda_1, \lambda_2, \dots \lambda_k$ be the distinct roots (in $\mathbb{C}$). For each $x_j$, we get a set of basic solutions of $(*)$

(i) $\lambda_j$ real with multiplicity $m_j \geq 1$. The corresponding solutions are:

$$e^{\lambda_j t},\ te^{\lambda_j t}, \dots \frac{t^{m_j - 1}}{(m_j - 1)!}e^{\lambda_j t}.$$

(ii) $\lambda_j$: complex (not real), $\lambda_j = \alpha_j + i\beta_j$, $\alpha_j, \beta_j \in \mathbb{R}$.
The associated basic real solutions (to $(\lambda_j, \overline{\lambda}_j)$ are $e^{\lambda j} = e^{\alpha_j}(\cos(\beta_j) + 1\sin(\beta_j)$, if $m_j = 1$. For general $m_j$, a basis of real solutions is given by

$$e^{\alpha_j}\cos(\beta_j), \dots, \frac{t^{m_j} - 1}{(m_j - 1)!}e^{\alpha_j}\cos(\beta_j),\ e^{\alpha_j}\sin(\beta_j), \dots, \frac{t^{m_j} - 1}{(m_j - 1)!}e^{\alpha_j}\sin(\beta_j).$$

We then put together all these solutions, as $j$ ranges from 1 to $k$, to get a full basis of real solutions.

### Inhomogeneous equation

*Example 1*:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = e^{5t} \leftarrow \text{non-zero external force} \qquad (*)$$

Look at the associated homogeneous ODE:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = 0 \qquad (**)$$

Characteristic equation:

$$\lambda^2 + \lambda - 2 = 0 \;\Rightarrow\; (\lambda + 2)(\lambda - 1) = 0.$$

Hence the eigenvalues are

$$\lambda_1 = -2, \;\; \lambda_2 = 1.$$

*Note*: The general solution of $(*)$ is of form

$$u(t) = \underbrace{u_c(t)}_{\text{gen solution of } (**)} + \underbrace{U(t)}_{\text{particular solution of } (*)}$$

Indeed, $u_c(t)+Ut)$ is evidently a solution of $(*)$, and conversely, if we have two solutions $U_1(t), U_2(t)$ of $(*)$, then their difference $U_1(t) - U_2(t)$ is a solution of $(**)$.

Since $\lambda_1 = -2, \lambda_2 = 1$, we have

$$u_c(t) = b_1 e^{-2t} + b_2 e^t,$$

with $b_1, b_2 \in \mathbb{R}$.

Try

$$U(t) \;=\; ce^{5t}.$$

Then $U' = 5ce^{5t}$ and $U'' = 25ce^{5t}$, so that

$$U'' + U' - 2U \;=\; c(25 + 5 - 2)e^{5t} \;=\; 28ce^{5t}.$$

This must equal the right hand side of $(*)$, which is $e^{5t}$. So $U(t)$ will be a solution of $(*)$ if we take $c = \frac{1}{28}$.

So the general solution of $(*)$ is

$$u(t) = u_c(t) + U(t) \;=\; b_1 e^{-2t} + b_2 e^t + \frac{1}{28} e^{5t}.$$

*Remark*: What made this idea work is that $e^{5t}$ is not $e^{-2t}$ or $e^t$.

*Example 2*:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 2u = t^2 e^{5t},$$

where $u_c(t)$ is same as in Example 1. Try

$$U(t) \;=\; \phi(t) e^{5t},$$

where $\phi(t)$ is twice differentiable. Then

$$U' = (\phi'(t) + 5\phi(t))e^{5t},\ U'' = (\phi''(t) + 5\phi'(t) + 5(\phi'(t) + 5\phi(t)))\, e^{5t},$$

so that

$$\begin{aligned} U'' + U' - 2U &= (\phi''(t) + 11\phi'(t) + 28\phi(t))\, e^{5t} \\ &\stackrel{?}{=} t^2 e^{5t} \end{aligned}$$

We can get a particular solution $U$ of $(*)$ this way if we can solve the simpler ODE

$$\phi''(t) + 11(\phi)'(t) + 28\phi(t) = t^2$$

We use the *Method of Unknown Coefficients*:

Try $\phi(t) = a + bt + ct^2$, and solve for $a, b, c$. (Do the calculation!)

Once we solve for $\phi(t)$, it can be plugged back into the expression for $U(t)$.

**Resonance**

Consider

$$(*) \qquad u'' + w_0^2 u = A\cos(\alpha t),$$

where $\omega_0$ is the frequency of the homogeneous system. If $\alpha$ is close to $\omega_0$, then the (true) amplitude of the solution rises tremendously.

Characteristic equation: $\lambda^2 + w_0^2 = 0, \omega_0 > 0$

Eigenvalues: $\lambda = i\omega_0,\ \overline{\lambda} = -i\omega_0$

The basic *real* solutions of homogeneous equation

$$(**) \qquad u'' + w_0^2 u = 0$$

are $\cos(\omega_0 t),\ \sin(\omega_0 t)$, so that the general solution of the homogeneous equation is

$$u_c(t) \;=\; \underbrace{b_1\cos(\omega_0 t) + b_2(\omega_0 t)}_{R_0\cos(i\omega_0 - \delta)},\ b_1, b_2 \in \mathbb{R},$$

where $R$ is the amplitude and $\delta$ the phase (when there is no forcing).

**Case (i)**: $\alpha \neq \omega_0$ ("bounded response", no "resonance")

General solution of $(*)$ is

$$u(t) = u_0(t) + U(t),$$

with $U$ a particular solution.

Try

$$U(t) = C\cos(\alpha t) + D\sin(\alpha t).$$

In fact, we do not need the sine term here since the first derivative does not occur in $(*)$, so we may take $D = 0$. Then

$$\begin{aligned}
U' &= -C\alpha\sin(\alpha t),\ U'' = -C\alpha^2\cos(\alpha t) \\
\Rightarrow U'' + w_0^2 U &= c(w^2 - \alpha^2)\cos(\alpha t) \stackrel{?}{=} A\cos(\alpha t) \\
\Rightarrow C &= \frac{A}{w^2 - \alpha^2} \\
\Rightarrow U(t) &= \underbrace{R_0\cos(\omega_0 t - \delta)}_{u_c(t)} + \frac{A}{w^2 - \alpha^2}\cos(\alpha t)
\end{aligned}$$

The second term gets larger as $\alpha$ gets closer to $\omega_0$.

**Case (ii)**: $\alpha = \omega_0$ ("unbounded response", *ideal resonance*)

$$u'' + w_0^2 u = A\cos(\omega_0 t),\ \omega_0 > 0.$$

The basic real solutions of the homogeneous equation

$$(**) \qquad u'' + w_0^2 u = 0$$

are $\cos(\omega_0 t), \sin(\omega_0 t)$, so that the general solution of $(**)$ is

$$u_c(t) \ = \ b_1\cos(\omega_0 t) + b_2\sin(\omega_0 t), \ \text{ with } b_1, b_2 \in \mathbb{R}.$$

Try

$$U(t) = ct\sin(wt).$$

Then

$$\begin{aligned}
U' &= c\sin(\omega_0 t) + c\omega_0 t\cos(\omega_0 t) \\
U'' &= 2c\omega_0\cos(\omega_0 t) - cw_0^2 t\sin(\omega_0 t) \\
\Rightarrow U'' + w_0^2 U &= 2c\omega_0\cos(\omega_0 t) \stackrel{?}{=} A\cos(\omega_0 t)
\end{aligned}$$

We need: $C = \frac{A}{2\omega_0}$ to get a particular solution of $(*)$. Thus the general solution of $(*)$ is

$$\Rightarrow u(t) = \underbrace{R_0\cos(wt - \delta)}_{\text{stable}} + \underbrace{\frac{A}{2\omega_0}t\sin(\omega_0 t)}_{\text{unstable}}$$

**Quasi-Resonance** (under-damping)

For simplicity, let us assume $m = 1$, and consider

$$u'' + \gamma u' + w_0^2 u = A\cos(\alpha t),$$

with $\gamma > 0$ the damping constant. Assume

$$\gamma^2 \; < \; 4w_0^2,$$

which is the interesting case.

Characteristic equation: $\lambda^2 + \gamma\lambda + \omega_0^2 = 0$

Eigenvalues:

$$\lambda_1 = -\frac{\gamma}{2} \pm \underbrace{\sqrt{\frac{\gamma^2}{2} - 4\omega_0^2}}_{\substack{\text{imaginary since} \\ \gamma^2 < 4w_0^2}}$$

$$\Rightarrow \lambda_\pm = -\frac{\gamma}{2} + i\sigma.$$

Now

$$u_c(t) = e^{-\gamma t/2}(b_1\cos(\sigma t) + b_2\sin(\sigma t)), \quad b_1, b_2 \in \mathbb{R}.$$

$\sigma$ is called the *quasi-frequency* of the system.

Since $u(t) = u_c(t) + U(t)$, we need to find one particular solution $U(t)$.

Suppose we are in the case $\alpha \neq \omega_0$. Try

$$U(t) = C\cos(\alpha t) + D\sin(\alpha t).$$

Check that we need

$$C = \frac{A(\omega_0^2 - \alpha^2)}{(\omega_0^2 - \alpha^2) + (\gamma\alpha)^2}$$
$$D = \frac{A\gamma\alpha}{(\omega_0^2 - \alpha^2) + (\gamma\alpha)^2}$$

Amplitude: $\frac{A}{\sqrt{(\omega_0^2 - \alpha^2)} + \gamma^2\alpha^2}$

Since we are assuming that the damping constant $\gamma$ is non-zero, we may also look at when $\alpha = \omega_0$, in which case

$$C = 0, \;\text{ and }\; D = \frac{A}{\gamma\alpha},$$

so that

$$U(t) \;=\; \frac{A}{\gamma\alpha}\sin(\alpha t).$$

If we plot this for different values of $\gamma > 0$, then we see a narrowing and simultaneous spiking of the trajectory as $\gamma$ gets closer and closer to 0, whence the name "quasi-resonance".