Lecture 16

For any $m \times m$-matrix

$$M = (m_{ij}),\ 1 \le i, j \le m,$$

the exponential of M is defined by

$$\exp(M) = e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!},$$

when the right hand side converges, with $M^0 = I$. Notation:

$$\sum_{n=0}^{\infty} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} a_n & \sum_{n=0}^{\infty} b_n \\ \sum_{n=0}^{\infty} c_n & \sum_{n=0}^{\infty} d_n \end{pmatrix}$$

Why is this relevant for us?

Reason: If we have $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_m}{dt} \end{pmatrix}$, and A an $m \times m$-matrix with constant coefficients, a basic set of solutions is given by the columns of the matrix

$$\Phi(t) = e^{At}$$

Note: $\Phi'(t) = Ae^{At} = A\Phi(t)$. A set of basic solutions are given by the columns of $\Phi(t) = \left(x^{(1)}\, x^{(2)}\, \dots\, x^{(m)}\right)$, with

$$x^{(i)}(t) = \begin{pmatrix} x_1^{(i)}(t) \\ x_2^{(i)}(t) \\ \vdots \\ x_m^{(i)}(t) \end{pmatrix}, \quad \forall\ i \le m.$$

We have mostly looked so far at the case where A has *distinct* eigenvalues. A modification is necessary for *repeated eigenvalues*!

Now we will look at the exponential of some simple 2×2-cases:

(i) A is *diagonal*: $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$A^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_1^2 \end{pmatrix}, \dots, A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

So

$$\begin{aligned} e^A &= A^0 + \frac{A^1}{1!} + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} + \dots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \frac{\lambda^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2}{2!} \end{pmatrix} + \dots + \begin{pmatrix} \frac{\lambda_1^n}{n!} & 0 \\ 0 & \frac{\lambda_2^n}{n!} \end{pmatrix} + \dots \\ &= \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \dots + \frac{\lambda_n^n}{n!} & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{n!} + \dots + \frac{\lambda_2^n}{n!} + \dots \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \end{aligned}$$

Conclusion:

$$\exp \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

Note:

$$\exp \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} e^a & e^b \\ e^c & e^d \end{pmatrix}!$$

(ii) A: *upper triangular*

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 3\alpha \\ 0 & 1 \end{pmatrix}, \dots, A^n = \begin{pmatrix} 1 & n\alpha \\ 0 & 1 \end{pmatrix} \text{ (by induction)}$$

Thus

$$\begin{aligned} e^A &= \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{1}{n!} & \frac{n\alpha}{n!} \\ 0 & \frac{1}{n!} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} & \sum_{n=0}^{\infty} \frac{n\alpha}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} \end{pmatrix} \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$$\alpha \sum_{n=1}^{\infty} \frac{n}{n!} = \alpha \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \alpha \sum_{k=0}^{\infty} \frac{1}{k!} = \alpha e$$

$$\implies e^{A} = \begin{pmatrix} e & \alpha e \\ 0 & e \end{pmatrix}$$

Check: For $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$,

$$e^{At} = \begin{pmatrix} e^t & \alpha e^t \\ 0 & e^t \end{pmatrix}.$$

(iii)

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$A^2 = -\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} \quad A^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix} \ldots A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

Remembering that $A^0 = I$,

$$e^{A} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} & \sum_{n=0}^{\infty} \frac{n\lambda^{n-1}}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \end{pmatrix}$$

Note that for any t,

$$\sum_{n=0}^{\infty} \frac{n\lambda^{n-1}t^n}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}t^n}{(n-1)!} = t \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} = te^{\lambda t}$$

Putting $t = 1$,

$$e^{A} = \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix}$$

Check: For $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$,

$$e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

It appears hopeless to get a nice expression for e^M for an arbitrary $m \times m$ matrix $M = (m_{ij})$, even for $m = 2$. We tackle this problem by appealing to *similarity of matrices* (also called *conjugacy*).

One says that 2 matrices M, M' are **similar** (or *conjugate*) iff we can find an *invertible* matrix B, meaning a nonsingular matrix, such that

$$M' \;=\; B^{-1}MB.$$

In this case,

$$\begin{aligned} \det(M') &= \underbrace{\det(B^{-1})\det(M)\det(B)}_{=1} \\ \Rightarrow \det M' &= \det M \end{aligned}$$

Powers of M' and the exponential:

$$\begin{aligned} M' &= B^{-1}MB \\ (M')^2 &= (B^{-1}MB)(B^{-1}MB) \\ &= B^{-1}MIMB \\ &= B^{-1}M^2B \text{ so } (B^{-1}MB)^2 = B^{-2}M^2B \end{aligned}$$

Similarly $(B^{-1}MB)^3 = B^{-1}M^3B$

So in general, $(B^{-1}MB)^n = B^{-1}M^nB$ for any $n \geq 0$.

Hence

$$\begin{aligned} \exp(B^{-1}MB) &= \sum_{n=0}^{\infty} \frac{(B^{-1}MB)^n}{n!} = \sum_{n=0}^{\infty} \frac{B^{-1}M^nB}{n!} \\ &= B^{-1} \underbrace{\left(\sum_{n=0}^{\infty} \frac{M^n}{n!} \right)}_{\exp(M)} B \end{aligned}$$

In other words,

$$(*) \qquad e^{B^{-1}MB} = B^{-1}e^MB$$

A natural question which arises is to know whether every $m \times m$-matrix M is similar to a matrix M' for which we can compute its exponential, thereby allowing us to know e^M as well by the identity $(*)$. Camille Jordan gave a nice positive answer to this, and for $m = 2$, here is the statement, which can be checked by hand using matrix multiplication.

Theorem *Suppose* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *then* M *is similar, we can find an invertible matrix* B *st* $B^{-1}MB$ *is of one of the following form*

(i) $B^{-1}MB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

(ii) $B^{-1}MB = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

In particular, if $M' = B^{-1}MB$, then we can explicitly calculate $e^{M'}$, and hence also determine e^M using $M = BM'B^{-1}$, because

$$e^M = Be^{M'}B^{-1}.$$

In case (i) of this Theorem, called the *Jordan decomposition* (for $m = 2$), M is called *diagonalizable*, but the eigenvalues λ_1, λ_2 may or may not be distinct.

Lecture 17

More on repeated eigenvalues and the Jordan decomposition:

Recall that $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, has two independent solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ given by

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix} \\
\text{Check : } \frac{d\mathbf{x}^{(1)}}{dt} &= \begin{pmatrix} \lambda e^{\lambda t} \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix} = A\mathbf{x}^{(1)} \\
\mathbf{x}^{(2)} &= \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} te^{\lambda t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda t} \\
\text{Check : } \frac{d\mathbf{x}^{(2)}}{dt} &= \begin{pmatrix} e^{\lambda t} + \lambda te^{\lambda t} \\ \lambda e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix} = A\mathbf{x}^{(2)}
\end{aligned}
$$

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is chosen first because it is an eigenvector for $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with eigenvalue λ, i.e., $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or equivalently

$$(A - \lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Here $A - \lambda$ means $A - \lambda I$, and $\mathbf{0}$ is the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Note: To get independent solutions, we would need to choose a second vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ carefully. Note that

$$A\mathbf{v} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a + b \\ \lambda b \end{pmatrix} \neq \lambda\mathbf{v} \text{ if } b \neq 0.$$

Of course, $b = 0$ iff $\mathbf{v}$ is proportional to the first eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In particular, we cannot choose $\mathbf{v}$ to be the other unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Nevertheless, we have

$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow (A-\lambda)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow (A-\lambda)^2\begin{pmatrix}0\\1\end{pmatrix} = (A-\lambda)(\underbrace{(A-\lambda)\begin{pmatrix}0\\1\end{pmatrix}}_{\begin{pmatrix}1\\0\end{pmatrix}}) = \mathbf{0}$$

Conclusion:

(i) $\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector for A with eigenvalue λ, i.e.,

$$(A-\lambda)\begin{pmatrix}1\\0\end{pmatrix} = 0.$$

(ii) $(A-\lambda)\begin{pmatrix}0\\1\end{pmatrix} \neq 0$, so $\begin{pmatrix}0\\1\end{pmatrix}$ is not an eigenvector but $(A-\lambda)^2\begin{pmatrix}0\\1\end{pmatrix} = 0$.

We call $\begin{pmatrix}0\\1\end{pmatrix}$ a *generalized eigenvector* for A relative to λ.

General definition: *For any positive integer n, given an $n \times n$-matrix $A = (a_{ij})$, a generalized eigenvector for A relative to an eigenvalue λ is a non-zero vector v such that $(A-\lambda)^k = 0$, for some k, with $1 \leq k \leq n$.*

Recall that if A has *distinct eigenvalues* $\lambda_1, \lambda_2, \dots \lambda_n$, then we can find a basis of the n-space consisting of eigenvectors for A. If this basis is $v^{(1)}, v^{(2)}, \dots, v^{(n)}$ with $Av^{(j)} = \lambda_j v^{(j)}$, we can put $B = (v^{(1)}, v^{(2)} \dots v^{(n)})$. Then $B^{-1}AB = \begin{pmatrix}\lambda_1 & & & & 0\\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda_n\end{pmatrix}$.

Having distinct eigenvalues implies that the matrix A in question is diagonalizable, but the converse is not necessarily true. For example, look at $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, which is diagonal but has 1 as a repeated eigenvalue.

Note: There are many matrices with repeated eigenvalues that are not diagonalizable,

$$\text{e.g.,}\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A general fact: (without proof)

For any matrix A, we can find a basis consisting of generalized eigenvectors.

Example 1:

$$n = 2,\ A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad \text{Basis:} \quad \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{generalizedeigenvectors}}$$

Example 2:

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

Compute eigenvalues

$$\begin{aligned} \det(A - \lambda) &= \det \begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (3-x)(1-\lambda)+1 \\ &= 4 - 4\lambda + \lambda^2 \\ &= (\lambda - 2)^2 = 0 \end{aligned}$$

So $\lambda = 2$ is the unique eigenvalue with multiplicity 2. There is only one eigenvector up to scaling. To see this, we solve:

$$\begin{aligned} (A - \lambda) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= 0 \\ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x_1 - x_2 &= 0 \\ x_1 - x_2 &= 0 \\ x_1 &= x_2 \end{aligned}$$

So every eigenvector is a multiple of $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To get the second independent vector, which can only be a *generalized* eigenvector, not a true one, we solve:

$$\begin{aligned}
(A-\lambda)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{\mathbf{w}} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
y_1 - y_2 &= 1 \\
y_1 - y_2 &= 1
\end{aligned}$$

So we may choose $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as our generalized eigenvector, since $(A-\lambda)^2\mathbf{w} = 0$, and $\mathbf{w}$ is linearly independent from $\mathbf{v}$.

Thus we have found a basis of $\mathbb{R}^2$ made up of generalized eigenvectors for $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, namely $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Put

$$\begin{aligned}
B &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \ \det(B) = -1 \\
B^{-1} &= \frac{1}{\det(B)} \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \\
B^{-1}AB &= \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \ \text{with } \lambda = 2.
\end{aligned}$$

This is a special case of the decomposition asserted by Jordan for 2×2-matrices.

Jordan decomposition for $(\mathbf{n} \times \mathbf{n})$-matrices:

Given any $A = (a_{ij}) 1 \le i, j \le n$, with eigenvalues $\lambda_1 \dots \lambda_k$ having multiplicities $m_1, m_2, \dots m_k$ respectively, we can find an invertible $n \times n$-matrix

B, and positive integers $n_1, n_2, \ldots, n_r$ with $\sum_{j=1}^{r} n_j = n$, such that

$$B^{-1}AB = \begin{pmatrix} M_1 & & & & 0 \\ & M_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & M_2 \end{pmatrix},$$

which is a block diagonal matrix, with each block M_j being an $(n_j \times n_j)$-matrix of the form

$$M_j = \begin{pmatrix} \lambda_j & 1 & & & 0 \\ & \lambda_j & 1 & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & \lambda_j \end{pmatrix},$$

where the only non-zero entries are either those on the diagonal, which are all λ_j, an eigenvalue of A, and those on the super-diagonal, i.e., the line parallel to the diagonal and just above it, which are all 1's. Moreover, every eigenvalue is some λ_j, but the λ_j's may not all be distinct.

Note that $B^{-1}AB$ is block diagonal, but is not diagonal unless all the n_j's are 1. Furthermore, if λ is an eigenvalue of A, then its multiplicity m_λ is given by

$$m_\lambda = \sum_{j \in J_\lambda} n_j,$$

where J_λ is the set of $j \leq r$ such that $\lambda_j = \lambda$.

Recall (from the previous lecture) that for $n = 2$, there are two Jordan forms, namely

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ and } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

For $n = 3$, there are four Jordan forms:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

and

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Lecture 18

Consider the linear system of first order ODE's with constant coefficients:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad (*)$$

Suppose we have found an invertible matrix B such that $B^{-1}AB$ is a diagonal matrix $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$. To get a solution to $(*)$, define a new (vector) variable $\mathbf{y} = B^{-1}\mathbf{x}$, which implies $\mathbf{x} = B\mathbf{y}$. Then $\mathbf{y}' = B^{-1}\mathbf{x}' = B^{-1}A\mathbf{x} = B^{-1}AB\mathbf{y}$, and so $\mathbf{y}$ satisfies the simpler ODE::

$$(**) \qquad \mathbf{y}' = D\mathbf{y}.$$

A *Special solution matrix* for $(**)$ is obtained by exponentiating Dt:

$$\Phi = e^{Dt} = \exp\begin{pmatrix} \lambda_1 t & & 0 \\ & \lambda_2 t & \\ 0 & & \lambda_n t \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

The special set of independent solutions are

$$\mathbf{y}^{(1)} = \begin{pmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{y}^{(2)} = \begin{pmatrix} 0 \\ e^{\lambda_2 t} \\ \vdots \\ 0 \end{pmatrix}, \ \ldots \quad, \mathbf{y}^{(n)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{pmatrix}.$$

The corresponding solutions for $\mathbf{x}$ satisfying $(*)$ are

$$\mathbf{x}^{(j)} = B\mathbf{y}^{(j)}, \quad \forall\ j = 1, 2, \ldots n.$$

Any time we are given $\mathbf{x}' = A\mathbf{x}$ with A a constant $n \times n$ matrix we can get a matrix of solutions as:

$$\Phi(t) = e^{At}$$

We can get a basis of solutions if $\det[\Phi(t)] \neq 0$.

Problem: It is hard to compute e^{At}, when there is no eigenbasis.

A way out is to use the Jordan decomposition for A (discussed last time), which allows us to find a similar matrix in block diagonal form of a specific type, allowing us to compute the exponential.

A remark on linear homogeneous systems with nonconstant coefficients

Suppose we have to solve $\mathbf{x}' = A(t)\mathbf{x}$, with $A(t)$ a non-constant matrix. A special solution matrix is given by

$$\Phi(t) = e^{\int_0^t A(u)du},$$

and to make sense of this, we need to know that $A(t)$ is integrable; a sufficient condition, which we will often assume, is to have $A(t)$ be a continuous (matrix) function of t. We have, by the *Fundamental Theorem of calculus*,

$$\Phi'(t) = \frac{d}{dt}\left(\int_0^t A(u)du)\Phi(t)\right) = A(t)\Phi(t).$$

Example

$$n = 2,\ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \mathbf{x}' = A(t)\mathbf{x},\ A(t) = \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix}$$

$$\Phi(t) = e^{\int_0^t A(u)du}$$

$$\int_0^t A(u)du = \int_0^t \begin{pmatrix} \frac{1}{2}t^2 & 0 \\ 0 & \frac{1}{3}t^3 \end{pmatrix}$$

$$\Rightarrow \Phi(t) = e^{\begin{pmatrix} \frac{1}{2}t^2 & 0 \\ 0 & \frac{1}{3}t^3 \end{pmatrix}} = \begin{pmatrix} e^{\frac{1}{2}t^2} & 0 \\ 0 & e^{\frac{1}{3}t^3} \end{pmatrix}$$

Inhomogeneous linear systems with constant coefficients

Homogeneous: $\mathbf{x}' = A\mathbf{x}$ $\mathbf{x}' = \frac{d}{dt}\mathbf{x}$
Non-homogeneous: $\mathbf{x}' = A\mathbf{x} + \mathbf{G}(t)$,
where $\mathbf{G}(t)$ purely a vector function of t.

Even the case $\mathbf{n = 1}$ is not trivial for inhomogeneous equation

$$\frac{dx}{dt} = ax + g(t),\ a \neq 0$$

If $g(t) = c$, a constant, then $x = -c/a$ is the equilibrium solution, and for $x + \frac{c}{a} \neq 0$,we have $\int \frac{dx/dt}{x+c/a} dt = \int a dt$, yielding $\ln \left| x + \frac{c}{a} \right| = at$

General solution:

$$x = -\frac{c}{a} + Be^{at}, \quad B \in \mathbb{R},$$

with $B = 0$ giving the equilibrium solution.

It is a bit more subtle when $g(t)$ is not a constant. Given

$$\frac{dx}{dt} = ax + g(t), \ a \neq 0,$$

with $g(t)$ integrable, one tries

$$x = h(t)e^{at},$$

where $h(t)$ will be suitably chosen differentiable function. We have

$$\frac{dx}{dt} = ax + h'(t)e^{at}.$$

So we need to choose $h(t)$ such that

$$h'(t) = e^{-at}g(t).$$

Note that this fixes $h(t)$ up to adding an arbitrary constant.

We may choose

$$h(t) = \int_b^t e^{-au}g(u)du,$$

where b is a real number; a particular solution is obtained by taking $b = 0$, for example. Any change of b only adds a constant to $h(t)$, as expected.

So the general solution is obtained as

$$x(t) = e^{at} \int_b^t e^{-au}g(u)du,$$

for an arbitrary $b \in \mathbb{R}$. We can also write this as

$$x(t) = Be^{at} + e^{at} \int_0^t e^{-au}g(u)du,$$

where B is an arbitrary constant. These two representations are equivalent, and one can pass from the first to the second, for example, by setting $B = -\int_0^b e^{-au}g(u)du$.

For an arbitrary $n \times n$ system:

Consider

$$\mathbf{x}' = A\mathbf{x} + G(t), \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; G(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

If A is a diagonal matrix, i.e.,

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix},$$

then

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 + g_1(t) \\ \lambda_2 x_2 + g_2(t) \\ \vdots \\ \lambda_n x_n + g_n(t) \end{pmatrix},$$

which gives n independent linear equations. So we may apply in this case the solution we obtained above in the $n = 1$ case to each row, and obtain the general solution to the linear system of ODE's as

$$\mathbf{x} = \begin{pmatrix} e^{\lambda_1 t} \int_{b_1}^{t} e^{-\lambda_1 u} g_1(u) du \\ e^{\lambda_2 t} \int_{b_2}^{t} e^{-\lambda_2 u} g_2(u) du \\ \vdots \\ e^{\lambda_n t} \int_{b_n}^{t} e^{-\lambda_n u} g_n(u) du \end{pmatrix},$$

where $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is an arbitrary (constant) vector in $\mathbb{R}^n$.

This method can be extended to solve the inhomogeneous system if A is diagonalizable, i.e., when $M^{-1}AM$ is, for a non-singular matrix M, a diagonal matrix D.

Sometimes one can solve the inhomogeneous equations without trying to diagonalize the matrix:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + e^{rt}\mathbf{z}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix},$$

where z is a constant vector, i.e., independent of t. Try:

$$\mathbf{x} = e^{rt}C,$$

where C is a constant vector. Then $\mathbf{x}' = re^{rt}C$, which equals $Ae^{rt}C + e^{rt}\mathbf{z}$ iff we have

$$C = (r - A)^{-1}\mathbf{z}.$$

Hence the general solution is

$$\mathbf{x} = e^{rt}(r - A)^{-1}\mathbf{z}.$$