Lecture 13

Fundamental Matrices

If we have a linear system $\mathbf{x}' = A\mathbf{x}$, with A an $n \times n$ matrix with constant coefficients a_{ij}, $1 \leq i, j \leq n$, a *fundamental set*, or *basis*, of solutions is given by $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ such that *every* solution is of the form

$$\mathbf{x} \;=\; c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)},$$

for suitable constants $c_1, c_2, \ldots, c_n$.

We've seen how to get a fundamental set of solutions $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots \mathbf{x}^{(n)})$ when A has n distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ wth (column) eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots \mathbf{v}^{(n)} \neq 0$ (i.e., $A\mathbf{v}^{(i)} = \lambda_j \mathbf{v}^{(j)}$):

$$\mathbf{x}^{(j)} = \mathbf{v}^{(j)} e^{\lambda_i t} \quad \mathbf{x}^{(j)}(0) = \mathbf{v}^{(j)}$$

The associated fundamental matrix is given by

$$\Psi(t) = (\mathbf{x}^{(1)} \; \mathbf{x}^{(2)} \ldots \mathbf{x}^{(n)})$$

What is important here is not that A has distinct eigenvalues, but that there is a basis of n-space consisting of eigenvectors for A. In general there is no eigenbasis, and what we do know is that when the eigenvalues are all distinct, then there is definitely such a basis.

Note: In general, there are many fundamental sets of solutions, and so $\Psi(t)$ depends on the particular choice of $\mathbf{x}^{(1)}, \ldots \mathbf{x}^{(n)}$. We have, for all j,

$$\frac{d\mathbf{x}^{(j)}}{dt} = A\mathbf{x}^{(j)}$$

$$\Rightarrow \frac{d\Psi}{dt} = \left(\frac{d\mathbf{x}^{(1)}}{dt}, \ldots, \frac{d\mathbf{x}^{(n)}}{dt} \right) \leftarrow n \times n \text{matrix}$$

Thus Ψ satisfies the matrix differential equation

$$\frac{d\Psi}{dt} \;=\; A\Psi.$$

Note: $\Psi(0) = (\mathbf{x}^{(1)}(0) \ldots \mathbf{x}^{(n)}(0)) = (\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots \mathbf{v}^{(n)})$. For example, one could have $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ then $\mathbf{v}^{(1)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and

$\mathbf{v}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\lambda_1 = 1$ and $\lambda_2 = 2$. In this case,

$$\Psi = \begin{pmatrix} 2e^t & -e^t \\ -e^t & e^t \end{pmatrix}, \quad \Psi(0) = \begin{pmatrix} \mathbf{v}^{(1)} \, \mathbf{v}^{(2)} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Recall: We say that a fund matrix Ψ is in *special form* if $\Psi(0)$ is the identity matrix I_n. In this case, the eigenvectors are

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1, \mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = e_2, \mathbf{v}^{(n)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = e_n.$$

When we have a special fundamental matrix, it is customary to denote it by $\Phi(t)$ instead of $\Psi(t)$.

$\mathbf{n = 3}$:

Suppose A is a diagonal matrix: $\begin{pmatrix} 1 & 6 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$\begin{aligned} \text{Eigenvalues : } \det(\lambda I_3 - A) &= \det \begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda + 2 & 0 \\ 0 & 0 & \lambda - 3 \end{pmatrix} \\ &= (\lambda - 1)(\lambda + 2)(\lambda - 3) \\ \implies & \lambda \in \{1, -2, 3\} \end{aligned}$$

For $\lambda_1 = 1$, $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = -2$: $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 3$: $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Psi = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}, \quad \Psi(0) = I_3$$

For any n, the general solution $\mathbf{x}$ is a linear combination of fundamental ones. We can solve for the constants $c_1, c_2, c_3 \dots c_n$, hence obtain a particular solution, if we are given the initial value $\mathbf{x}(0) = \begin{pmatrix} \vdots \end{pmatrix}$. Indeed,

$$c_1\mathbf{x}^{(1)}(0) + c_2\mathbf{x}^{(2)}(0) + \cdots + c_n\mathbf{x}^{(n)}(0) = \mathbf{x}(0)$$

is a system of n linear equations in n unknowns. We can rewrite this in the matrix form

$$(\mathbf{x}^{(1)}(0)\,\mathbf{x}^{(2)}(0)\,\dots\,\mathbf{x}^{(n)}(0)) = \Psi(0)$$

$$\underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\underbrace{\Psi(0)}_{n\times n \text{ matrix}} \quad \underline{c} = \underbrace{\mathbf{x}(0)}_{\text{column vector of size } n} \qquad (0)$$

So we can multiply on the left by $\Psi(0)^{-1}$ (on both sides) to get

$$\underline{c} = \Psi(0)^{-1}\mathbf{x}(0)$$

If Ψ were a special (or standard) fundamental matrix x of solutions, i.e., if $\Psi = \Phi$, then $\Phi(0) = I_n$, so $\Phi(0)^{-1} = I_n$.

Conclusion: If Ψ is in special form then

$$\underline{c} = \mathbf{x}(0).$$

Since $\mathbf{x} = \Psi(t)\underline{c}$, we get

$$\mathbf{x} = \Psi(t)\Psi(0)^{-1}x(0).$$

Example for $\mathbf{n = 2}$: Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$. Then we know that the eigenvalues are $\lambda = 1, -1$, with corresponding eigenvectors $\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Fundamental solutions:

$$\mathbf{x}^{(1)} = \mathbf{v}^{(1)} e^t = \begin{pmatrix} et \\ 0 \end{pmatrix}$$
$$\mathbf{x}^{(2)} = \mathbf{v}^{(2)} e^{-2t} = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}$$
$$\Rightarrow \Phi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

In this case, $At = \begin{pmatrix} t & 0 \\ 0 & -2t \end{pmatrix}$ and $\Phi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}$, which can be obtained by "exponentiating" At.

If we have a diagonal $n \times n$-matrix:

$$A = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda_n \end{pmatrix},$$

then we write

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & & & 0 \\ & e^{\lambda_2 t} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & e^{\lambda_n t} \end{pmatrix}.$$

In particular, its value at $t = 0$ is just the identity matrix I_n.

Idea: Try to define e^{At} for any A, and put $\Phi(t) = e^{At}$ so that the column vectors of e^{At} give solutions $\mathbf{x}^{(1)}(t) \dots \mathbf{x}^{(n)}(t)$ of the linear system of first order ODE's. We are justified in writing Φ (denoting a special matrix of sultions), because $\Phi(0) = e^0 = I_n$.

Why should such a $\Phi(t)$, defined as e^{At}, give a solution to $\Phi(t) = A\Phi(t)$? The reason is this: Since A is a constant matrix,

$$\frac{d}{dt}(At) = A$$
$$\Longrightarrow \frac{d}{dt}(e^{At}) = e^{At}\left(\frac{d(At)}{dt}\right) = Ae^{At}$$

A natural Question: Can we define the exponential matrix in a satisfactory way in general? Yes, at least if A is conjugate to a *triangular matrix*, i.e., $A = (a_{ij})$, with $a_{ij} = 0$ if $i < j$ (*upper triangular*) or with $a_{ij} = 0$ if $i > j$ (*lower triangular*). When $n = 2$, A triangular iff we have e.g.,

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

A possible definition of e^M:

For M: any $n \times n$-matrix, put

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!},$$

when the infinite series of matrices converges. Let us check this definition in a known case (for $n = 2$):

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \to e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 1^n & 0 \\ 0 & (-2)^n \end{pmatrix}$$
$$= \begin{pmatrix} e & 0 \\ 0 & e^{-2} \end{pmatrix}$$

More generally,

$$M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \implies e^M = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

Lecture 14

Today's topics: Repeated eigenvalues (especially for $n = 2, 3$), the exponential of a matrix

Repeated eigenvalues

Start with the following simple example

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues of A: solve for $|\lambda I_2 - A| = 0$, i.e.,

$$\det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^2 = 0$$

There is only 1 eigenvalue namely $\lambda = 1$. In this case we say $\lambda = 1$ appears as an eigenvalue with multiplicity 2 for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We can find one eigenvector $\mathbf{v} \neq 0$ for $\lambda = 1$ easily:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

This holds iff $v_2 = 0$. So we may take $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow$, which is a standard unit vector. So we get a solution to $\mathbf{x}' = A\mathbf{x}$ by setting

$$\mathbf{x}^{(1)} = \mathbf{v}e^t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

$$\frac{d\mathbf{x}^{(1)}}{dt} = \begin{pmatrix} e^t \\ 0 \end{pmatrix} = A\mathbf{x}^{(1)}$$

Question: Is there a second solution to $\mathbf{x}' = A\mathbf{x}$? If so, how can we find it? Will it be linearly independent of $\mathbf{x}^{(1)}$?

Note: We cannot write the 2nd solution in the form $\mathbf{v}^{(2)}e^t$ because, for it to be a solution, $\mathbf{v}^{(2)}$ would need to be an eigenvector for $\lambda = 1$, and then $\mathbf{v}^{(2)}$ would in this case be a scalar multiple of $\mathbf{v}^{(1)}$. Then $\mathbf{x}^{(2)}$ would in turn be proportional to of $\mathbf{x}^{(1)}$, which is not what we want. In this case the Wronskian $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ is 0.

Moral: If there's a 2nd solution $\mathbf{x}^{(2)}$ linearly independent from $\mathbf{x}^{(1)}$, it cannot be of the form

$$\begin{pmatrix} \text{a vector indep} \\ \text{of } t \end{pmatrix} e^t.$$

Idea: Make the first term vector dependent on t.

Try:

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} t \\ 1 \end{pmatrix} e^t = \begin{pmatrix} te^t \\ e^t \end{pmatrix} \\ \Rightarrow \frac{d}{dt}\mathbf{x}^{(2)} &= \begin{pmatrix} e^t + te^t \\ e^t \end{pmatrix}, \\ A\mathbf{x}^{(2)} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} te^t \\ e^t \end{pmatrix} = \begin{pmatrix} te^t + e^t \\ e^t \end{pmatrix} \end{aligned}$$

So $\frac{d}{dx}\mathbf{x}^{(2)} = A\mathbf{x}^{(2)}$, showing that $\mathbf{x}^{(2)}$ is a second solution of $\mathbf{x}' = A\mathbf{x}$. But we want to know if $\mathbf{x}^{(2)}$ is linearly independent of $\mathbf{x}^{(1)}$. The answer is YES for this choice of $\mathbf{x}^{(2)}$, because

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \det \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} = e^{2t} \neq 0,$$

for any t.

Summary: *When* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, *which has* $\lambda = 1$ *as a repeated eigenvalue, two linearly independent solutions of* $\mathbf{x}' = A\mathbf{x}$ *are given by*

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t, \ \mathbf{x}^{(2)} = \begin{pmatrix} t \\ 1 \end{pmatrix} e^t.$$

A slight variation of this example:

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \text{any scalar}$$

Eigenvalues: $\det(\mathbf{v}I_2 - A) = 0$

$$\det \begin{pmatrix} r - \lambda & -1 \\ 0 & r - \lambda \end{pmatrix} = (r - \lambda)^2 = 0$$

$\Rightarrow \lambda$ is the only eigenvalue (with multiplicity 2).

Let $\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then,

$$A\mathbf{v}^{(1)} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \lambda \mathbf{v}^{(1)}$$

$\Rightarrow \mathbf{v}^{(1)}$ is an eigenvector

So one solution to $\mathbf{x}' = A\mathbf{x}$ is just $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix}$.

Second solution?

Try $\mathbf{x}^{(2)} = \begin{pmatrix} te^{t\lambda} \\ e^{t\lambda} \end{pmatrix}$ again. Then

$$\frac{d\mathbf{x}^{(2)}}{dt} = \begin{pmatrix} \lambda te^{t\lambda} + e^{t\lambda} \\ \lambda e^{t\lambda} \end{pmatrix}$$

$$A\mathbf{x}^{(2)} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \lambda te^{\lambda t} + e^{\lambda t} \\ \lambda e^{\lambda t} \end{pmatrix}$$

So it works!

What about the Wronskian?

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \det \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} = e^{2\lambda t} \neq 0,$$

for any t. Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent solutions (for all t).

Another example:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Key fact: *The eigenvalues of a triangular matrix are just the diagonal entries.*

So in this case, $\lambda = 1$ is the only eigenvalue, so with multiplicity 3 (as A is a 3×3-matrix.

Check: $\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue 1

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So one solution to the ODE is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}.$$

By analogy with the $n = 2$ case, try

$$\mathbf{x}^{(2)} = \begin{pmatrix} te^t + e^t \\ e^t \\ 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \mathbf{x}^{(2)} = \begin{pmatrix} te^t + e^t \\ e^t \\ 0 \end{pmatrix},$$

and

$$\frac{d}{dt}\mathbf{x}^{(2)} = \begin{pmatrix} te^t + e^t \\ e^t \\ 0 \end{pmatrix} = A\mathbf{x}^{(2)}.$$

What about a third independent solution?

Try

$$\mathbf{x}^{(3)} = \begin{pmatrix} \frac{1}{3}t^2e^t \\ te^t \\ e^t \end{pmatrix}.$$

Then

$$\tfrac{d\mathbf{x}^{(3)}}{dt} = \begin{pmatrix} \frac{1}{2}t^2e^t + te^t \\ te^t + e^t \\ e^t \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \mathbf{x}^{(3)} = \begin{pmatrix} \frac{1}{2}t^2e^t + te^t \\ te^t + e^t \\ e^t \end{pmatrix}.$$

So $\mathbf{x}^{(3)}$ is also a solution. To check linear independence of these three solutions, we need to evaluate the Wronskian and check that it is non-zero:

$$W = \det \begin{pmatrix} e^t & te^t & \frac{1}{2}t^2e^t + te^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = e^{3t} \neq 0, \ \ \forall t.$$

The associated (special) fundamental matrix is

$$\Phi = \begin{pmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}, \text{ with } \Phi(0) = I_3.$$

Back to the exponential of a matrix

Why do we need it?

Reason: Given $\mathbf{x}' = A\mathbf{x}$ for any $n \times n$ matrix A, we can find a "canonical" or "special" set of solutions as the column vectors of $\Phi(t) = e^{At}$ (when it makes sense), with

$$\Phi(0) = e^{A(0)} = e^0 = I$$

One defines the exponential of any $n \times n$ matrix B by the infinite series

$$e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}; \quad B^0 = I$$

Does this make sense?

Example: $n = 2$, $\mathbf{x}' = A\mathbf{x}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$e^{At} =? \quad A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

By induction, $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, for any $n \geq 0$. Hence

$$(At)^n = \begin{pmatrix} t^n & nt^n \\ 0 & t^n \end{pmatrix}, \ \forall\ n \geq 0,$$

and

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} t^n & nt^n \\ 0 & t^n \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{t^n}{n!} & \frac{nt^n}{n!} \\ 0 & \frac{t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{pmatrix}$$

$$\Rightarrow e^{At} = \begin{pmatrix} \underbrace{\begin{matrix} e^t \\ 0 \end{matrix}}_{\mathbf{x}^{(1)}} & \underbrace{\begin{matrix} te^t \\ e^t \end{matrix}}_{\mathbf{x}^{(2)}} \end{pmatrix}$$

Check the following for any $n \times n$-matrix B, and any invertible $n \times n$-matrix M:

$$e^{M^{-1}BM} = M^{-1}e^{B}M.$$

(*Hint*: What is $(M^{-1}BM)^{n}$?)

Lecture 15

Review of some earlier topics for the midterm:

First order ODE:

$$\frac{dy}{dt} = f(t, y) \tag{*}$$

t: independent variable (typically, $t \geq 0$)
y: dependent variable $= y(t)$

Note: An equation like $(\frac{dy}{dt})^2 = t + y$ is also a first order ODE but it is not linear.

x is *linear* iff $f(t, y) = a(t)y + b(t)$, i.e., the ODE is linear iff f is linear in y. A linear ODE $(*)$ has *constant coefficients* iff $a(t), b(t)$ are independent of t.

Typically we work with $(*)$ where f is at least continuous. If f is continuously diffferentiable, then we can find a solution by a limiting process (of Picard), but not necessarily in closed form.

Example 1

$$\frac{dy}{dt} = g(t),\ g(t) \text{ integrable}$$
$$y(t) = \int g(t)dt + c$$

For example, if $\frac{dy}{dt} = 2t$, the function $2t$ is continuous, so integrable, and $y = t^2 + C$.

Can evaluate C if given an initial condition. In this example $C = y(0)$.

Example 2 (*separation of variables*)

$$\frac{dy}{dt} = \frac{y-1}{\mathbf{x}^2+1}$$

Note that $\mathbf{x}^2 + 1 \neq 0$ in $\mathbb{R}$.

Equilibrium solution occurs when $y = 1$, but how do we solve the ODE for $y \neq 1$? The equation is rewritten as

$$\frac{\frac{dy}{dx}}{y-1} = \frac{1}{\mathbf{x}^2+1} \text{ when } y \neq 1.$$

Integrating both sides with respect to x, we get

$$\begin{aligned}\int \frac{\frac{dy}{dx}}{y-1} &= \int \frac{dx}{\mathbf{x}^2+1} + C \\ \ln|y-1| &= \arctan x + C \\ |y-1| &= e^{\arctan x + C} \\ y &= 1 \pm e^C e^{\arctan x}.\end{aligned}$$

General solution: $y = 1 + Be^{\arctan x}$, for any constant B, with $B = 0$ corresponding to the equilibrium solution.

Example 3 (*change of variables*)

$$\frac{du}{dt} = u - t$$

Equilibrium solution is by definition a solution u where $\frac{du}{dt} = 0$. It's a bit subtle here, as u' is zero when $u = t$, but $u = t$ is not a solution of the ODE! Indeed, u' would be 1 if $u = t$. not a solution! $\Rightarrow \frac{du}{dt} = 1$.

We solve this ODE by changing variables. Put $y = u - t$. Then

$$\frac{dy}{dt} = \frac{du}{dt} - 1 = \underbrace{u - t}_{y} - 1.$$

So we have converted to a new differential equation

$$\frac{dy}{dt} = y - 1.$$

As in example 2, we can write, for $y \neq 1$,

$$\begin{aligned}\int \frac{\frac{dy}{dt}}{y-1} dt &= \int 1 dt \\ \ln|y-1| &= t + c \\ y - 1 &= e^t e^c \\ y &= 1 \pm e^t e^c \\ \Rightarrow y &= 1 \pm Be^t, \quad B \neq 0 \\ \Rightarrow u = y + t &= 1 + t + Be^t,\end{aligned}$$

where B is any constant, with $B = 0$ corresponding to the equilibrium solution $y = 1$ (for the ODE in y). Suppose $u(0) = 0$. Then we can evaluate B:

$$1 + B = 0 \implies B = -1$$

Final solution:

$$u = 1 + t - e^t, \text{ if } u(0) = 0.$$

Stability

Note that in Example 2, the general solution to $\frac{dy}{dt} = \frac{y-1}{\mathbf{x}^2+1}$ was $y = 1+Be^{\arctan x}$, for any constant B, with $B = 0$ corresponding to the equilibrium solution $y = 1$. It is of interest to know the asymptotic behavior of such a solution as x goes to infinity. We have

$$\lim_{x\to\infty} 1 + Be^{\arctan x} = 1 + Be^{\pi/2}$$

If y_1 is a particular solution of $\frac{dy}{dt} = f(t, y)$, then we say that y_1 is **stable** if for any other solution y_2 which starts out being close to $_1$ at $t = 0$, we will have $y(t)$ close to $y_1(t)$ for all $t > 0$.

More precisely, for all $\epsilon > 0$, $\exists\, \delta > 0$ such that if $|y = (0) - y_1(0)| < \delta$, then $|y_2(t) - y_1(t)| < \epsilon$.

We are specifically interested in the *stability of equilibrium solutions.*

An important variant: An equilibrium solution $y_{eq}(t)$ is *asymptotically stable* if for any y_2 starting out near y_{eq} at $t = 0$,

$$\lim_{t\to\infty} |y_2(t) - y_{eq}(t)| = 0$$

When f is continuous, then asymptotic stability implies stability.

Example

$$\frac{dy}{dx} = \frac{y-1}{\mathbf{x}^2+1}$$

Equilibrium solution: $y = 1$

General solution: $y_B = 1 + Be^{\arctan(x)} \overset{x\to\infty}{\longrightarrow} 1 + Be^{\frac{\pi}{2}}$. If $B \neq 0$, then $1 + Be^{\frac{\pi}{2}} \neq 1$. So if $B \neq 0$, then $\lim_{x\to\infty} |1 - y_B| = Be^{\frac{\pi}{2}} \neq 0$. So $y = 1$ is not asymptotically stable.

On the other hand, If B is close to 0, then this limit is close to $y = 1$, but the limit is not exactly 1 for $B \neq 0$, however small. Thus the (unique) equilibrium solution $y = 1$ in this example is stable, but not asymptotically stable.

A useful criterion for asymptotic stability

Suppose we have $\frac{dy}{dt} = f(t,y)$, with $f(t,y)$ depending on only y. So we may write $f(t,y) = \varphi(y)$ which does not depend on t. Suppose in addition that φ is a differentiable function of y. Let y_1 be any equilibrium solution (y_1), so that $\varphi(y_1) = 0$. Then

(i) If $\varphi'(y_1) < 0$, then y_1 is asymptotically stable;

(ii) If $\varphi'(y_1) > 0$ then y_1 not asymptotically stable .

If $\varphi'(y_1) = 0$, then nothing can be said.

Note that this criterion works only if we can check two things, namely that $f(t,y)\varphi(y)$ is *independent* of t, and that $\varphi(y)$ is a differentiable function of y.

Example: The Logistic equation:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}),\ r, K > 0,$$

whose right hand side $\varphi(P)$, say, is differentiable bing a polynomial; it is also independent of t. So we may apply the criterion. Note that

$$\varphi'(P)\ -\ r(1 - \frac{2P}{K}).$$

Equilibrium Points: $P = 0$ and $P = K$

$$\varphi'(P) < 0 \text{ at } P = K$$
$$\varphi(P) > 0 \text{ at } P = 0$$

So $P = K$ is asymptotically stable, while $P = 0$ is not, by the stability criterion.