Lecture 10

Linear Systems of First order ODEs

We've looked at $x' = f(t, x)$, for $x \in \mathbb{R}$ depending on an independent variable t.

Recall: If f and f_y are continuous, then we can find a solution (through Picard's iteration) which is unique if we fix an initial value x_0 at $t = 0$.

For special classes of f, one can say a lot. A particular case of interest is when the ODE is *linear*, which is the case when $f(t, x)$ is linear in x, i.e., $f(t, x) = a(t)x + b(t)$, for suitable functions $a(t)$ and $b(t)$. It is said to be *homogeneous* iff the following happens: *When x is a solution, any scalar multiple cx is again a solution*; in particular, 0 is a solution. In the linear case we just considered, it is homogeneous exactly when $b(t) = 0$.

We say that the linear ODE has *constant coefficients* if $a(t)$ and $b(t)$ are both independent of t. Hence we have homogeneity *and* constant coefficients iff $a(t)$ is independent of t and $b(t) = 0$; in other words, the ODE is of the form $x' = ax$, for some scalar a.

Recall: If $x' = ax$, a is constant, the set of all solutions is given by

$$x = \{Be^{at} \mid B \text{ any constant}\}$$

Indeed,

$$\begin{aligned}
x' = ax &\implies \left(\frac{dx}{dt} = 0 \Leftrightarrow x = 0 : \text{ equilibrium point}\right) \\
\frac{1}{x}\frac{dx}{dt} &= a, \text{ when } x \neq 0 \\
\implies \int \frac{(dx/dt)}{x}\, dt &= a \int dt \\
\implies \log|x| &= at + c \\
\implies |x| &= e^{at+c} \\
x &= Be^{at}, \text{ with } B = \pm e^c \neq 0
\end{aligned}$$

But $B = 0$ is also possible and corresponds to the equilibrium solution $x = 0$. Hence the claim above, that the general solution of $x' = ax$ is $x = Be^{at}$, where B is any constant.

We may think of x as a vector in a space of dimension 1. The set of all solutions is also a 1-dimensional vector space, with B as the coordinate.

(Brush up on the basics of vector spaces, linear maps, and properties of matrices - including eigenvalues, eigenvectors, and diagonalization - from Ma1b; it will also be good if you know about the Jordan decomposition, which we will discuss later.)

Generalization

Let t be an independent variable (as before), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. i.e., $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, whose derivative is another vector in $\mathbb{R}^n$:

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \begin{pmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{pmatrix}$$

We can look at a linear ODE in vector form:

$$\mathbf{x}' = A(t)\mathbf{x} \tag{*}$$

where $A(t)$ is an $n \times n$ matrix; $A(t) = (a_{ij}(t))$, $1 \le i, j \le n$. Explicitly $(*)$ means we have n ODEs

$$\begin{aligned} \tfrac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ \tfrac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ &\vdots \\ \tfrac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n \end{aligned}$$

called an $n \times n$ linear system of ODEs

We can say that this system has *constant coefficients* iff A is independent of t, i.e., each a_{ij} is a constant. From now on, assume that we are in the case of constant coefficients, and look for solutions $\mathbf{x}(t)$ of $\mathbf{x}' = A\mathbf{x}$. It will be of interest to consider the set of all solutions of such a homogeneous linear system of ODE's:

$$V = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \mathbf{x}' = A\mathbf{x},\ A = (a_{ij}) \right\}$$

Using the known fact(s) that differentiation and matrix multiplication are linear operations, we get the following

Properties of V:

(i) (existence of origin) $0 \in V$

(ii) (additivity) If $\mathbf{x}, \mathbf{y}$ *are both in* V*, then* $\mathbf{x} + \mathbf{y} \in V$

(iii) (homogeneity) If $\mathbf{x} \in V$*, then so is* $\alpha\mathbf{x}$ *for any scalar* α

Reason for (ii):

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \mathbf{y}' &= A\mathbf{y} \\ \mathbf{x}' + \mathbf{y}' &= A(\mathbf{x} + \mathbf{y}) \\ (\mathbf{x} + \mathbf{y})' &= A(\mathbf{x} + \mathbf{y}) \end{aligned}$$

Reason for (iii):

$$\alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}, \ \alpha\mathbf{x}' = \begin{pmatrix} \alpha x_1' \\ \vdots \\ \alpha x_n' \end{pmatrix} = \alpha A\mathbf{x} = A(\alpha\mathbf{x}),$$

$$\text{since} \quad \alpha A = A\alpha.$$

Conclusion

The solution set V of a linear, homogeneous system $\mathbf{x}' = A\mathbf{x}$ is a vector space. It is natural to expect V has dimension n.

Basic Questions: Can we guess a non-zero solution of $\mathbf{x}' = A\mathbf{x}$, for any $n \times n$ constant matrix A? If so, can we find all the solutions, i.e., write down a general solution like in the $n = 1$ case?

Here's a clever idea for any n: (in many cases, but not all, this furnishes all the solutions)

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Try:

$$\mathbf{x} = \mathbf{v}e^{\lambda t}, \quad \text{where} \quad \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq 0, \ \lambda \in \mathbb{R}.$$

Here $\mathbf{v}$ is independent of t. Then

$$\mathbf{x}' = \lambda \mathbf{v} e^{\lambda t}$$

But $\mathbf{x}' = A\mathbf{x}$, so we must have

$$A\mathbf{x} = \lambda \mathbf{v} e^{\lambda t}$$
$$\Rightarrow A\mathbf{v} \underbrace{e^{\lambda t}}_{\neq 0} = \lambda \mathbf{v} \underbrace{e^{\lambda t}}_{\neq 0}$$
$$\Rightarrow A\mathbf{v} = \lambda \mathbf{v}$$

Hence λ must be an eigenvalue (since $\mathbf{v}$ is a non-zero vector).

Conversely, if $\mathbf{x}' = A\mathbf{x}$, with λ an eigenvalue of A, i.e., with $A\mathbf{v} = \lambda\mathbf{v}$, for some non-zero vector $\mathbf{v}$, then

$$A\mathbf{v}e^{\lambda t} = \lambda \mathbf{v} e^{\lambda t} = \frac{d}{dt}(\mathbf{v}e^{\lambda t}).$$

So $\mathbf{x} = \mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{x}' = A\mathbf{x}$.

Recall from *Basic Linear Algebra* (Ma1b):

Given any $n \times n$ matrix A, we can always find all of its eigenvalues in $\mathbb{C}$. So we get an added complexity (no pun intended) on whether there are real eigenvalues. To elaborate further, the eigenvalues λ are solutions of the *characteristic equation*

$$\det(\lambda I_n - A) = 0,$$

which is a polynomial equation in λ of degree n. There are n complex roots but not necessarily all distinct. Even when A is a real matrix, some of the eigenvalues may be non-real. However, when A is a real matrix, if a complex, i.e., non-real, eigenvalue λ occurs, then its complex conjugate $\overline{\lambda}$ will also be an eigenvalue of A, which is evident from applying complex conjugation to the characteristic equation. Consequently, the complex eigenvalues come in conjugate pairs, and when n is odd, this forces the existence of at least one real eigenvalue. One of the basic results of Linear Algebra (Ma1b), which we will use at various places, is this:

If A is a real symmetric matrix, then all of its eigenvalues are real.

Recall that $A = (a_{ij})$ is symmetric iff $a_{ji} = a_{ij}$ for all $i, j \leq n$, i.e., iff A equals its *transpose* $A^t = (a_{ji})$. More generally, we say that a complex matrix A is *hermitian* iff A equals its *conjugate transpose*, i.e., $A = \overline{A}^t$. The general fact (hopefully discussed in Ma1b) is that the eigenvalues of a complex hermitian matrix are all real. Of course, a real matrix is hermitian iff it is symmetric, since $\overline{A} = A$ for real A.

Examples:

(i) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det(\underbrace{\lambda I_2}_{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} - A) = \det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 - 1$$

Eigenvalues: $\lambda = 1, -1$ since ± 1: roots of $\lambda^2 - 1 = 0$.

Eigenvectors:

$\lambda = 1$	$\lambda = -1$
$A\mathbf{v} = \mathbf{v}$	$A\mathbf{v} = -\mathbf{v}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} -x \\ -y \end{pmatrix}$
$\downarrow$	$\downarrow$
$x = y$	$x = -y$
Take $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	Take $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \det(\lambda I_2 - A) = \lambda^2 + 1$

Eigenvalues: $\lambda_\pm = \pm i$

Eigenvectors: $\mathbf{v}^\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ (no real eigenvector)

Suppose $\lambda_1 \neq \lambda_2$ are two real eigenvalues of A. Then we get two distinct solutions to $\mathbf{x}' = A\mathbf{x}$, namely

$$\mathbf{x}^{(1)} := \mathbf{v}^{(1)} e^{\lambda_1 t} \quad \text{and} \quad \mathbf{x}^{(2)} := \mathbf{v}^{(2)} e^{\lambda_2 t},$$

with $\mathbf{v}^{(1)}$ eigenvector of λ_1 and $\mathbf{v}^{(2)}$ eigenvector for λ_2. Indeed, for $j = 1, 2$, since $A\mathbf{v}^{(j)} = \lambda_j \mathbf{v}^{(j)}$,

$$A\mathbf{x}^{(j)} = \lambda_j \mathbf{v}^{(j)} e^{\lambda_j t} = \mathbf{v}^{(j)} \frac{d}{dt}\left(e^{\lambda_j t}\right) = \frac{d}{dt}\mathbf{x}^{(j)}.$$

Claim: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent solutions.

Proof: Suppose $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = 0$, for scalars c_1, c_2, not both zero. Putting $t = 0$, and noting that, by definition, $\mathbf{x}^{(j)}(0) = \mathbf{v}^{(j)}$ for $j \in \{1, 2\}$, we get the linear dependence relation

$$c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)} = 0, \tag{1}$$

not both constants c_1, c_2 being zero. So it suffices to check that $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ are linearly independent. This should be clear from the material covered in Ma1b, since these eigenvectors correspond to different eigenvalues, but in any case, here is the argument: To begin, since the eigenvectors are non-zero, if one constant, say c_1, is zero, then so is the other. So we may assume that both c_1 and c_2 are non-zero. Applying the matrix A to this relation (1), and using the fact that $A\mathbf{v}^{(j)} = \lambda_j \mathbf{v}^{(j)}$, we obtain

$$c_1\lambda_1\mathbf{v}^{(1)} + c_2\lambda_2\mathbf{v}^{(2)} = 0. \tag{1}$$

Multiplying (1) by λ_2 and subtracting it from (2),

$$c_1(\lambda_1 - \lambda_2)\mathbf{v}^{(1)} \; = \; 0,$$

which is impossible since c_1, $\lambda_1 - \lambda_2$, and $\mathbf{v}^{(1)}$ are all non-zero. This gives the necessary contradiction, and the Claim follows.

Lecture 11

Linear, homogeneous system with constant coefficients:

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{x}' = \frac{d\mathbf{x}}{dt}, \ A = (a_{ij})_{1\le i,j\le n} \qquad (*)$$

During the first 3 weeks we studied this for $n = 1$. In general, try to understand well the $n = 2$ and $n = 3$ cases. For $n = 2$, you should know how to draw various pictures, often called portraits, in the (x_1, x_2)-plane.

Equilibrium points are the solutions $\mathbf{x}$ for which $\mathbf{x}' = 0$, i.e., where $A\mathbf{x} = 0$. Important special case: *when A is an invertible matrix*, i.e., when the determinant of A, denotes as $\det(A)$ or just $|A|$, is nonzero. Then there exists an inverse matrix to A. Applying A^{-1} (in this case) to $A\mathbf{x} = 0$ on both sides, we see that $\mathbf{x} = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is the only equilibrium point (when A is invertible).

Check that in general, a matrix A is *singular*, i.e., not invertible, if and only if 0 is an eigenvalue of A.

General principle/ Theorem:

Consider $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

(a) If λ is an eigenvalue of A with eigenvector $\mathbf{v}$ $(\neq 0)$, then the function of t given by $\mathbf{x} = \mathbf{v}e^{\lambda t}$ is a non-zero solution of $(*)$.

(b) Suppose A has n distinct eigenvalues, say $\lambda_1, \lambda_2, \dots \lambda_n$ (with $\lambda_i \neq \lambda_j$ if $i \neq j$) with eigenvector $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots \mathbf{v}^{(n)}$, i.e.,

$$A\mathbf{v}^{(j)|} = \lambda_j \mathbf{v}^{(i)}$$

then every solution of $\mathbf{x}' = A\mathbf{x}$ is a linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)}$$

where the c_j are scalars and

$$\mathbf{x}^{(j)} = \mathbf{v}^{(i)}e^{\lambda_j t},$$

for each $j = 1, 2, \dots, n$.

Look at the case $\mathbf{n=2}$: When the eigenvalues are (real and) distinct, i.e., $\lambda_1 \neq \lambda_2$,

$$\mathbf{x}^{(1)} = \mathbf{v}^{(1)} e^{\lambda_1 t}, \quad \mathbf{x}^{(2)} = \mathbf{v}^{(2)} e^{\lambda_2 t}$$

$$\begin{aligned} W(\lambda^{(1)}, \lambda^{(2)}) &= \det({}^{(1)} e^{\lambda_1 t}, \mathbf{v}^{(2)} e^{\lambda_2 t} \\ &= \det \begin{pmatrix} a e^{\lambda_1 t} & c e^{\lambda_2 t} \\ b e^{\lambda_1 t} & d e^{\lambda_2 t} \end{pmatrix} \\ &= \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} e^{(\lambda_1 \lambda_2) t} \end{aligned}$$

Remarks:

(a) It is more subtle if the solutions are not real or not all distinct. Here when $\lambda_1 \dots \lambda_n$ are all real and distinct, all fundamental solutions $\mathbf{x}^{(1)} \dots \mathbf{x}^{(n)}$ are all real vectors, i.e., in $\mathbb{R}^n$.

(b) A key point to remember (from Ma1b) is that eigenvectors corresponding to distinct eigenvalues are linearly independent

(c) The matrix $\Psi = \begin{pmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \dots & \mathbf{x}^{(n)} \end{pmatrix}$ is called a *fundamental matrix.*

(d) If $\mathbf{y}^{(1)} \dots \mathbf{y}^{(n)}$ are n arbitrary solutions of $(*)$, one defines their

Wronskian determinant to be

$$W(\mathbf{y}^{(1)} \dots \mathbf{y}^{(n)}) = \det(\mathbf{y}^{(1)} \dots \mathbf{y}^{(n)}).$$

These $\mathbf{y}^{(j)}$'s give a fundamental set of solutions when $W(\mathbf{y}^{(1)} \dots \mathbf{y}^{(n)}) \neq 0$.

Clearly, there are at most n independent solutions.

Example:

(1) $n = 2,\ \mathbf{x}' = A\mathbf{x},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

We saw last time A has 2 eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, with corresponding eigenvectors $\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The two basic solutions of the linear system are

$$\mathbf{x}^{(1)} = \mathbf{v}^{(1)} e^t = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \;\; \mathbf{x}^{(2)} = \mathbf{v}(2) e^t = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t},$$

and the Wronskian is

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \det \begin{pmatrix} e^t & e^t \\ e^t & e^{-t} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.$$

Thus $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ are independent solutions. Of course we knew this already, because they correspond to distinct eigenvalues.

Slope field:

This is a plot in the (x_1, x_2)-plane, called the **phase plane**, where one chooses a grid and draws, at each point on the grid, a short arrow in the direction of the vector connecting the origin to the point determined by $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Note that since A is a constant matrix, $\mathbf{x}'(t)$, given by $A\mathbf{x}$, is independent of t, which is what allows us to draw the slope field on the phase plane (at all times t).

Asymptotics:

Suppose A is an $n \times n$-matrix with distinct (real) eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$. Then the general solution of $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x} \,=\, c_1 \mathbf{v}^{(1)} e^{\lambda_1 t} \cdots + c_n \mathbf{v}^{(n)} e^{\lambda_n t}.$$

Note that when $\lambda_j > 0$, $e^{\lambda_j t}$ goes to ∞ as $t \to \infty$ and goes to 0 when $t \to -\infty$. It follows that the term corresponding to the largest (positive) λ_j dominates the other terms as $t \to \infty$, while the largest (negative) λ_j dominates when $t \to -\infty$. This is because the eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots \mathbf{v}^{(n)}$ of A are *static*, i.e., do not vary with t (in our "constant coefficients" context). However, for each j with $\lambda_j \neq 0$, the corresponding basic solution $\mathbf{x}^{(j)} := \mathbf{v}^{(j)} e^{\lambda_j t}$ evolves as t varies.

Note: If A has a non-zero eigenvalue, then the equilibrium solution $\mathbf{x} = 0$ is not asymptotically stable (or even stable), since any solution which is near 0 goes to a (non-zero vector times) $\pm\infty$ either as t goes to ∞ or as $t \to -\infty$. If the only eigenvalue is 0, $\mathbf{x} = 0$ is stable as solutions near it will stay nearby for larger $|t|$, but is not asymptotically stable.

Trajectory:

To fix ideas, look at the example above with eigenvalues ± 1, and with general solution

$$\mathbf{x} = \phi(t) = c_1\mathbf{v}^{(1)}e^t + c_2\mathbf{v}^{(2)}e^{-t}.$$
$$\mathbf{x}_0 = \phi(0) = c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

If we sketch the evolution of $\phi(t)$ for any particular choice of c_1, c_2, we can represent it by a curve, called a *trajectory*, in the phase plane.

A **phase portrait** is just a sampling of different types of trajectories in the phase plane.

Trajectory of $\mathbf{x}^{(1)}(t)$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix}1\\1\end{pmatrix}e^t$$

Choose $t_1, t_2, \ldots t_m$ and plot $\mathbf{x}^{(1)}(t_j)$ for each j, and then join them:

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{x}^{(1)}(1) = \begin{pmatrix}e\\e\end{pmatrix},$$
$$\mathbf{x}^{(1)}(a) = \begin{pmatrix}e^a\\e^a\end{pmatrix}, \ \ldots$$

Lecture 12

Last time we discussed the example, in the plane:

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A three-dimensional example

$$A = \begin{pmatrix} 7 & -8 & 0 \\ 3 & -8 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solve $\mathbf{x}' = A\mathbf{x}$ subject to the initial condition: $\quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix}$.

Eigenvalues of A: Solve $\det(\lambda I_3 - A) = 0$.

$$\begin{aligned} \det \begin{pmatrix} \lambda - 7 & 8 & 0 \\ -3 & \lambda + 8 & 0 \\ 0 & 0 & \lambda - 3 \end{pmatrix} &= \begin{vmatrix} \lambda - 7 & 8 \\ -3 & \lambda + 8 \end{vmatrix} (\lambda - 3) \\ &= [(\lambda - 7)(\lambda + 8) + 24](\lambda - 3) \\ &= (\lambda^2 + \lambda - 56 + 24)(\lambda - 3) \\ &= (\lambda - 1)(\lambda + 2)](\lambda - 3) \end{aligned}$$

Thus $\lambda_1 = 1,\ \lambda_2 = -2,\ \lambda_3 = 3$.

Eigenvectors: Look for $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq 0$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$$\lambda_3 = 3 : \underbrace{\begin{pmatrix} 7 & -8 & 0 \\ 3 & -8 & 0 \\ 00 & 3 & \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\begin{pmatrix} 7a - 8b \\ 3a - 8b \\ 3c \end{pmatrix}}$$

We may take $\mathbf{v}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda_1 = 1$: Want $A(\mathbf{v}^{(1)} = \mathbf{v}^{(1)}$. Put $\mathbf{v}^{(1)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so that

$$\begin{aligned} 7a - 8b &= a \\ 3a - 8b &= b \\ 3c &= c \Rightarrow c = 0 \end{aligned}$$

We may take $\mathbf{v}^{(1)} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$.

$\lambda_2 = -2$: Check: $\mathbf{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ works!

Note: Put

$$\begin{aligned} M &= \begin{pmatrix} \mathbf{v}^{(1)} & \mathbf{v}^{(2)} & \mathbf{v}^{(3)} \end{pmatrix}, \text{matrix of eigenvectors} \\ &= \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \text{Check : } M^{-1} &= \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \text{using } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} &= \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ \Longrightarrow \underbrace{M^{-1}AM}_{\text{conjugation of } A \text{ by } M} &= \begin{pmatrix} 1 & -2 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 7 & -8 & 0 \\ 3 & -8 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \leftarrow \text{ the diagonal matrix of eigenvalues} \end{aligned}$$

Since the eigenvalues of A are all (real and) distinct, we know that a fundamental set (basis) of solutions of $A\mathbf{x} = \mathbf{x}'$ is given by:

$$\mathbf{x}^{(j)}, \mathbf{x}^{(j)}, \mathbf{x}^{(j)}, \text{ with } \mathbf{x}^{(j)} = \mathbf{v}^{(j)} e^{\lambda j t}, \text{ for each } j = \{1, 2, 3\}$$

Explicitly,

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} e^t,$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} e^{-2t},$$

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}.$$

The associated fundamental matrix

$$\Psi = \begin{pmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \mathbf{x}^{(3)} \end{pmatrix} = \begin{pmatrix} 3e^t & 2e^{-2t} & 0 \\ e^t & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix},$$

whose Wronskian is

$$W\begin{pmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \mathbf{x}^{(3)} \end{pmatrix} = (3e^{-t} - 2e^{-t})e^{3t} = e^{2t} \neq 0.$$

Asymptotics of the fundamental solutions

$\mathbf{x}^{(1)}(0) = \mathbf{v}^{(1)}$: starting point at $t = 0$

$\mathbf{x}^{(1)}(t) = \mathbf{v}^{(1)} e^t \to \begin{pmatrix} \infty \\ \infty \\ 0 \end{pmatrix}$ as $t \to \infty$; the first 2 coordinates go to $+\infty$ while third one stays at 0. Also, as $t \to -\infty$, $\mathbf{x}^{(1)}(t) \to \begin{pmatrix} \infty \\ \infty \\ 0 \end{pmatrix}$. Similarly,

$$\mathbf{x}^{(2)}(t) = \mathbf{v}^{(2)} e^{-2t} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} e^{-2t} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{as} \quad t \to \infty$$

$$\mathbf{x}^{(3)}(t) = \mathbf{v}^{(3)} e^{3t} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t} \to \begin{pmatrix} 0 \\ 0 \\ \infty \end{pmatrix}, \quad \text{as} \quad t \to \infty$$

Note: No non-zero linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_3(t)$ goes to the equilibrium solution $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as $t \to \infty$, while $\mathbf{x}^{(2)}$ does approach the equilibrium solution as $t \to \infty$.

$$\mathbf{x}^{(2)}(0) = \mathbf{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

General Solution:

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)},$$

where c_1, c_2, c_3 are scalars.

The given initial condition requires that $\mathbf{x}(0) = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix}$. This gives a system of equations for c_1, c_2, c_3:

$$c_1 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} \qquad (*)$$

Write $C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ as a vector of constants.

Consider the matrix of eigenvectors $M = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which diagonalizes the matrix A defining the linear system. It follows that

$$MC = \mathbf{x}(0)$$

We know that M is invertible, and so

$$C = M^{-1}\mathbf{x}(0) = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

So the *unique solution satisfying the Initial Condition* is:

$$\mathbf{x} = \mathbf{x}^{(1)} + 2\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} e^{-2t} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}.$$

Terminology

Let $\mathbf{x}' = A\mathbf{x}$ be a homogeneous, linear system, with A an $n \times n$-matrix with constant coefficients.

If we have n solutions, say $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots \mathbf{y}^{(n)}$ of this system of ODE's, then the *Wronskian* of $\{\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}\}$ is

$$W\left(\mathbf{y}^{(1)}, \mathbf{y}(2), \ldots, \mathbf{y}^{(n)}\right) \;=\; \det(\mathbf{y}^{(1)}, \mathbf{y}^{(2)} \ldots \mathbf{y}^{(n)}).$$

The $\mathbf{y}^{(j)}$ give a *basis of solutions* in an interval $(-a, a)$, for some $a > 0$, iff $W(\mathbf{y}^{(1)}, \ldots \mathbf{y}^{(n)}) \neq 0$ on $(-a, a)$. This is the same as saying: The $\mathbf{y}^{(j)}$ are linearly independent on the interval.

If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is a fundamental set of solutions, the associated *fundamental matrix* is

$$\Psi \;=\; (\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \ldots \mathbf{x}^{(n)}.$$

A simple example where A is real, but has non-real eigenvalues in the plane

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

As we saw earlier, the *eigenvalues of* A are $\lambda_1 = i$, $\lambda_2 = -i$, with associated eigenvectors

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \;\; \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

We get distinct solutions:

$$\mathbf{z}^{(1)} = \mathbf{v}^{(1)} e^{\lambda t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} e^{it} \\ ie^{it} \end{pmatrix}$$

$$\mathbf{z}^{(2)} = \mathbf{v}^{(2)} e^{\lambda_2 t} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it} = \begin{pmatrix} e^{-it} \\ -ie^{-it} \end{pmatrix}$$

The only catch is that the solutions are complex, not real!

If $\mathbf{z}$ is a complex solution, i.e., if $\mathbf{z}' = A\mathbf{z}$, then $\mathbf{x} = Re(\mathbf{z})$ and $\mathbf{y} = Im(\mathbf{z})$ are also solutions, since A is real, $\mathrm{Re}(\mathbf{z}') = \mathbf{x}'$, and $\mathrm{Im}(\mathbf{z}') = \mathbf{y}'$. The nice thing is that $\mathbf{x}$, $\mathbf{y}$ are real solutions. Since

$$e^{\pm it} = \cos t \pm i \sin t, \;\; \pm i e^{\pm it} = -\sin t \pm i \cos t,$$

the real solutions in the example above are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \;\; \mathbf{y}^{(1)}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

and

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \;\; \mathbf{y}^{(2)}(t) = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}.$$

Note that

$$\mathbf{x}^{(1)}(t) = \mathbf{x}^{(2)}(t), \;\; \mathbf{y}^{(1)}(t) = -\mathbf{y}^{(2)}(t).$$

So it suffices to consider just the solutions $\mathbf{x}^{(1)}$ and $\mathbf{y}^{(1)}$. Moreover, their Wronskian is

$$W\left(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}\right) \;=\; \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \;=\; \cos^2 t + \sin^2 t \;=\; 1 \;\neq\; 0.$$

So these two real solutions are linearly independent (over $\mathbb{R}$), and the corresponding fundamental matrix

$$\Psi \;=\; \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

is a *rotation matrix*, representing the rotation (with center $\mathbf{0}$) of the points in the plane through the *angle* t in the counterclockwise direction.

Finally, the general real solution of $\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{u}$ is given by

$$\mathbf{u}(t) \;=\; b_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + b_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

where b_1, b_2 are arbitrary real constants.