Lecture 24

To begin, it may be helpful to make a few comments on the case of **regular singular points** before starting the next big topic of the course, namely the Laplace Transform.

If $x = x_0$ is a regular singular point of a linear, second order ODE

$$(*) \qquad y'' + p(x)y' + q(x) = 0,$$

then we look for series solutions just like in the case of an ordinary point, but with a difference, namely we try

$$y \;=\; x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where r is a constant to be determined, which r need not be an integer, or even a rational number. (When r is not rational, it will be a quadratic irrational.) Assuming that the radius of convergence R (around x_0) is positive, we differentiate the series expression term by term (for x satisfying $|x - x_0| < R$) to get series expressions for y' and y''. Plugging this information back in the differential equation, we get an identity of the form

$$x^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \;=\; 0,$$

where the coefficients c_n of the power series on the left are determined by the coefficients a_n of y (and hence of y', y'') as well as by $r, p(x).q(x)$. It follows that for such an identity to hold for all x close to x_0, we need to have

$$c_n \;=\; 0, \quad \forall \; n \geq 0.$$

A key difference with the ordinary case occurs when we look at the condition $c_0 = 0$, because this leads to an equation for r, called the *indicial equation* associated to $(*)$, which will be a quadratic equation in r. In this course we will only look at the case when this has two distinct real roots, and moreover assume that the two roots do not differ by a integer. (As usual, it is more subtle when there is a repeated root, and even when the roots are not repeated, if they differ by a positive integer, we will not be able to find easily a second series solution.) For an r satisfying the indicial equation, we may plug its value in and solve the equations $c_n = 0$ for all $n > 0$. This results, as in the ordinary case, a formula for the coefficients a_n, which can be used to find an explicit series solutions of $(*)$.

Let us illustrate this by solving the following ODE:

$$(i) \qquad x^2 y'' - (x+1)y = 0,$$

where $p(x) = 0$ and $q(x) = -(x+1)/x^2$. The only singular point of the ODE, given by looking at where the coefficient of y'' vanishes, is the point $x = 0$. Since $xp(x) = 0$ and $x^2 q(x) = -x - 1$, both of which are obviously analytic, $x = 0$ is a regular singular point. So we try a series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

which we assume has a positive radius of convergence R. Then for $|x| < R$, we may differentiate (again and again) the series expression for y term by term. We get

$$y' = x^{r-1} \sum_{n=0}^{\infty} (r+n) a_n x^n.$$

Differentiating this again, and rearranging, we obtain

$$y'' = x^{r-2} \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^n,$$

so that

$$(ii) \qquad x^2 y'' = x^r \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^n.$$

Moreover,

$$(iii) \qquad (x+1)y = x^r \sum_{n \geq 1} a_{n-1} x^n + \sum_{n \geq 0} a_n x^n.$$

Plugging (ii) and (iii) back into (i) and rearranging,

$$x^r \sum_{n=0}^{\infty} c_n x^n = 0,$$

where

$$c_0 \;=\; r(r-1) - 1,$$

and

$$(iv) \qquad c_n \;=\; ((r-1+n)(r+n)-1)\,a_n - a_{n-1}, \;\; \forall\, n \geq 1.$$

We need to have $c_n = 0$ for all $n \geq 0$. Just from the $n = 0$ case, and assuming $a_0 \neq 0$, we get the indicial equation:

$$(r-1)r - 1 \;=\; 0,$$

whose solutions are

$$r_\pm \;=\; \frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

Clearly, these are distinct roots, with $r_+ - r_- = \sqrt{5}$, which is not an integer.

Plugging in for r (and using (iv) in $c_n = 0$), we get for every $n \geq 1$,

$$a_n \;=\; \frac{a_{n-1}}{(n+r)(n+r-1)},$$

which makes sense because the denominator is non-zero. (This is where we would run into a problem if the two roots differ by an integer.) It follows that

$$a_n \;=\; \left(\prod_{j=1}^{n} \frac{1}{(n-j+r+1)(n-j+r)}\right) a_0.$$

This leads to two series solutions with $a_0 = 1$:

$$y_\pm \;=\; x^{r_\pm}\left(1 + \sum_{n=1}^{\infty} a_{n,\pm} x^n\right),$$

with

$$a_{n,\pm} \;=\; \prod_{j=1}^{n} \frac{1}{(n-j+r_\pm+1)(n-j+r_\pm)}, \;\; \forall\, n \geq 1.$$

(Check that these series have positive radius of convergence R. Is $R = \infty$?)

We claim that these two solutions are independent. Indeed, otherwise y_- would be a multiple of y_+ for all x near 0. Since they both have equal constant term (corresponding to $n = 0$ in the respective series), we must have an equality of the $n = 1$ term as well, which does not hold since $r_+ \neq r_-$. Hence the Claim.

Laplace Transform

Let f be a function on the half-open interval $[0, \infty) \subset \mathbb{R}$. Its *Laplace transform* is defined for $s > 0$ (when it makes sense) by

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt.$$

The integral on the right is an *improper integral* and hence is evaluated as the limit

$$\lim_{A\to\infty} \int_0^A f(t)e^{-st}dt.$$

Before looking at explicit examples, let us state the following without proof:

Theorem 1 *Suppose f is a function on $[0, \infty)$ satisfying the following:*

(i) f is, for each $A > 0$, piecewise continuous on $[0, A]$; and

(ii) f is a-nice for some $a \in \mathbb{R}$, i.e.,there exist real numbers C, a, T with $C, T > 0$, such that

$$|f(t)| \leq Ce^{at}, \ \forall\, t > T.$$

Then the Laplace transform $F(s)$ is defined for all positive $s > a$.

Recall that f is piecewise continuous on $[0, A]$ means we can partition the interval as

$$0 = b_0 < b_1 < \cdots < b_{n-1} < b_n = A,$$

for some $n > 0$, such that f is continuous on each subinterval $[b_{j-1}, b_j]$, for $j = 1, 2 \ldots, n$.

The simplest example of a piecewise continuous function is a step function.

Remarks:

(1) The reason for the hypothesis (i) (of Theorem 1) is this: *If f is a piecewise continuous function on a closed interval $[0, A]$, then we know (cf. Ma1a) that f is integrable there, and so is $f(t)e^{-st}$, which is also piecewise continuous.*Thus at least $\int_0^A f(t)e^{-st}$ makes sense for such a function.

(2) The reason for the hypothesis (ii) is this: *We can write (for f a-nice):*

$$\lim_{A\to\infty}\int_0^A f(t)e^{-st} = \int_0^T f(t)e^{-st}dt + \lim_{A\to\infty}\int_T^A f(t)e^{-st}dt,$$

and for $t > T$, $|f(t)e^{-st}|$ is less than or equal to $Ce^{(a-s)t}$, which goes to zero at $t \to \infty$ when $a - s < 0$, which makes $F(s)$ well defined.

(3) $\mathcal{L}$ is a *linear operator*, i.e., for all f, g as in Theorem, and for all α, β scalars,

$$\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g).$$

(4) $\mathcal{L}$ is a prime example of an *integral transform.* Other important examples are the *Fourier transform* and the *Mellin transform.* Also useful is the *discrete version* of the Laplace transform, called the *z-transform,* which associates to any sequence $\{f(n)\}$ the "transform": $\sum_n f(n)z^{-n}$, with z, resp. n, playing the role of e^s, resp. t.

Note that any bounded function on $[0, \infty)$, such as $\sin t$ or e^{-t}, satisfies (ii) with $a = 0$. Moreover, any polynomial, even a rational function, and $\ln x$, are a-nice for any $a > 0$, because any positive power of the exponential function dominates any such function as t goes to ∞.

Examples:

(1) (*Heaviside function*) For any $c > 0$, define the associated simple step function u_c on $[0, \infty)$ by setting it to be 1 if $t \geq c$ and 0 if $0 \leq t < c$. Then f is evidently piecewise continuous and 0-nice, so that for all $s > 0$, we have

$$F(s) = \lim_{A\to\infty}\int_c^A e^{-st}dt = \frac{1}{s}\lim_{A\to\infty}(e^{-cs} - e^{As}) = \frac{e^{-cs}}{s}.$$

(2) Put $f(t) = t/2$ if $t < \pi/2$, and $f(t) = \sin t$ if $t \geq \pi/2$. Note that f is then piecewise continuous and bounded (in fact with $|f(t)| \leq 1$), so its Laplace transform is well defined for all $s > 0$. Since $\int_0^{\pi/2}\frac{t}{2}dt = \pi^2/4$, we have

$$\mathcal{L}(f(t)) = \frac{\pi^2}{4} + \lim_{A\to\infty}\int_{\pi/2}^A \sin t\, e^{-st}dt.$$

One solves the integral of $\sin t\, e^{-st}$ by integration by parts (using the methods of Ma1a).

Here is the reason why we need to bother with $\mathcal{L}$:

The Laplace transform converts differentiation into multiplication, up to an additional term involving the initial value.

Theorem 2 *Let f be differentiable with f, f' piecewise continuous and a-nice. Then we have, for all $s > a$,*

$$\mathcal{L}(f'(t)) \;=\; s\mathcal{L}(f(t)) - f(0).$$

More generally, for any $n \geq 1$, if f is n-times differentiable with all the derivatives $f^{(j)}$, for $0 \leq j \leq n$, satisfying the hypotheses of Theorem 1, we have

$$\mathcal{L}(f^{(n)}(t)) \;=\; s^n\mathcal{L}(f(t)) - s^{n-1}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Here is the reason: Since $f'(t)dt = d(f(t))$, we have by the integration by parts

$$\int_0^A f'(t)e^{-st}dt \;=\; f(t)e^{-st}|_0^A - \int_0^A f(t)d(e^{-st})$$
$$= f(A)e^{-sA} + s\int_0^A f(t)e^{-st}dt,$$

and since $|f(A)| \leq Ce^{aA}$ for large A, and since $e^{(a-s)A}$ goes to 0 as A goes to ∞ (by virtue of s being $> a$), we see, by taking the limits as $A \to \infty$,

$$\mathcal{L}(f'(t)) \;=\; -f(0) + s\mathcal{L}(f(t)),$$

as asserted. The generalization for any n follows the same way by repeated differentiation, and will be left as an exercise to check.
QED.