Math 2a Prac
Lectures on Differential Equations*

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*Based on notes taken in class by Stephanie Laga, with a few added comments

Lecture 1

A differential equation, often shortened as diffEq or just DE, is an equation involving the derivatives of a function $y = f(t)$, where t is an *independent variable*. Here the *dependent variable* y could be just a single variable, which is what we will consider at the beginning, or a vector $(y_1, \ldots, y_n)$ in n-space; our main focus will be on $n = 2, 3$. The diffEq is called *ordinary*, and written as ODE, unless the equation contains partial derivatives, in which is case it is called a partial differential equation, abbreviated as PDE. In this course we will deal mostly with ODE's, though some partial derivatives will at times be used in their analysis. The *order* of an ODE is the highest order of the derivatives which appear.

Examples:

(1) In classical Mechanics, Newton's law (in 1 dimension) says

$$F = ma, \ a = \frac{dv}{dt}, \ t \geq 0.$$

It is a first order ODE in v, the velocity.

Moreover, $v = \frac{dx}{dt}$, where x is the position, so the force is $F = m\frac{d^2x}{dt^2}$, and we obtain a second order ODE in the independent variable x.

Suppose you want to solve $F = m\frac{dv}{dt}$, with fixed mass m. One can break up into three cases:

(i) $F = 0$ (no force):

$$\Rightarrow \frac{dv}{dt} = 0 \implies v = v_0, \text{ the initial value :}$$

This is the principle of inertia: If there is no force acting on a particle, it stays its course, i.e., remains at rest (for $v_0 = 0$), or it moves at constant non-zero speed (when $v_0 \neq 0$); the sign of v_0 gives the direction.

(ii) $F = c \neq 0$ (constant non-zero force)
e.g. $c = mg$: the case of a "freely falling particle"

$$F = m\frac{dv}{dt} = mg \implies v = gt + v_0$$

$$v = \frac{dx}{dt} \implies x = g\frac{t^2}{2} + v_0 t + x_0,$$

where x_0 is the initial position.

(iii) F non-constant:

$$\begin{aligned} F &= mg - \gamma v \\ m\frac{dv}{dt} &= mg - \gamma v \qquad \text{Two forces} : \downarrow mg \ \uparrow \text{drag} = -\gamma v, \quad \gamma : \text{constant} \\ &= -\gamma\left(v - \frac{mg}{\gamma}\right) \end{aligned}$$

$$\text{Stationary point: } v = \frac{mg}{\gamma} \Leftrightarrow \frac{dv}{dt} = 0$$

To get non-stationary solutions $v \neq \frac{mg}{\gamma}$, divide by $v - \frac{mg}{\gamma} \neq 0$ and obtain

$$\frac{1}{(v - \frac{mg}{\gamma})}\frac{dv}{dt} = -\frac{\gamma}{m}$$

$$\Rightarrow \int \frac{1}{v - \frac{mg}{\gamma}}\left(\frac{dv}{dt}\right) dt = -\frac{\gamma}{m}\int dt + C$$

Want to find a function $\varphi(v)$ such that

$$\frac{d\varphi}{dv} = \frac{1}{v - \frac{mg}{\gamma}}$$

By applying the chain rule,

$$\frac{d\varphi}{dt} = \frac{1}{v - \frac{mg}{\gamma}}\frac{dv}{dt},$$

implying

$$\varphi(v) = \ln\left|\left(v - \frac{mg}{\gamma}\right)\right|$$

up to a constant. We get $\ln|v - \frac{mg}{\gamma}| = -\frac{\gamma}{m}t + C$, which we exponentiate to get

$$\begin{aligned} \left|v - \frac{mg}{\gamma}\right| &= e^{\ln(v - \frac{mg}{\gamma})} = e^{-\frac{\gamma}{mt} + C} \\ &= Ae^{-\frac{\gamma}{m}t}, \text{ where } A = e^C > 0 \\ \Rightarrow v - \frac{mg}{\gamma} &= \pm Ae^{-\frac{\gamma}{m}t} \end{aligned}$$

So, including the stationary solution as well, we get

$$v = \tfrac{mg}{\gamma} + Be^{-\frac{\gamma}{m}t},$$

where B is any constant; $B = 0$ corresponds to the stationary state.

When $t = 0, v = \frac{mg}{\gamma} + B$, thus $B = v_0 - \frac{mg}{\gamma}$.

Since $v = \frac{dx}{dt}$, we may integrate the expression for v to get

$$x \;=\; x_0 + \frac{mg}{\gamma}t + \left(g - \frac{\gamma}{m}v_0\right) e^{\frac{\gamma}{m}t}.$$

(2) *Radioactive decay*:

$$\frac{dP}{dt} = -\lambda P, \quad \lambda > 0 : \text{ decay constant}$$

1st order ODE in $P(t)$:

$$\frac{1}{P}\frac{dP}{dt} = -\lambda$$

$$\int \left(\frac{1}{P} dPdt\right) dt = -\lambda \int dt + C$$

$$\ln|P| = -\lambda t + C$$

exponentiating $|P| = Ae^{-\lambda t}, A = e^C$

$$P = \pm Ae^{-\lambda t}$$

At $t = 0, P_0 = B$, so

$$P \;=\; P_0 e^{-\lambda t}.$$

Asymptotic: $\lim_{t\to\infty} e^{-\lambda t} = 0$, since $\lambda > 0$.

(3) *Population growth of rabbits*:

Idealized situation:

$$\frac{dP}{dt} = \lambda P, \quad \lambda > 0$$

A better model:

$$\frac{dP}{dt} = \lambda P - M,$$

where the correction factor M accounts for deaths in the population, and still $\lambda > 0$. Writing the ODE as $\frac{dP}{dt} = \lambda\left(P - \frac{M}{\lambda}\right)$, we see that there is a unique stationary point, also called the equilibrium solution:

$$P = \frac{M}{\lambda}.$$

If $P \neq \frac{M}{\lambda}$,

$$\frac{1}{P - M/\lambda}\frac{dP}{dt} = \lambda$$

$$\ln\left|P - \frac{M}{\lambda}\right| = \lambda t + C$$

$$P - \frac{M}{\lambda} = Be^{\lambda t} \Rightarrow P = \frac{M}{\lambda} + Be^{\lambda t}$$

All these examples led to the same type of equation, namely,

$$\tfrac{dy}{dx} = ay + b,$$

and the solutions involve exponentials.

Lecture 2

Recall that an ordinary *diffEQ* is an equation involving derivatives of a function $y = f(x)$ of an independent variable x. We write

$$y^{(m)} = \frac{d^m y}{dx^m}, \quad \text{the } m\text{-th derivative}$$

Convention: $y^{(1)} = \frac{dy}{dx}, y^{(0)} = y$.

Such an ODE is of *order* n if the highest derivative which occurs in the equation is

$$\frac{d^n y}{dx^n},$$

so an ODE (in one independent variable x) of *order* n looks like

$$F(y, y^{(1)}, \ldots, y^{(n)}) = g(x),$$

where $g(x)$ is independent of y.

$n = 1$: $F(y, y') = g(x)$.

Note that the ODE

$$\left(\frac{dy}{dx}\right)^2 - 2y = 3$$

has order 1, not 2. Some say the degree of this equation is 2, because of the appearance of the square of y' on the left hand side (LHS), but still the order is 1.

Other examples:

(i) Earlier we looked at $y' = ay + b$, with a, b constants, $F(y, y') = y' - ay$, and $g(x) = b$.

A simpler equation is $y' = ax + b$:

$$F(y, y') = y',\ g(x) = ax + b$$

$$\text{Solution} \ : \ y = \int (ax + b)dx + c$$

Get $y = a\frac{x^2}{2} + bx + c, \quad c = y_0 = y(0)$

(ii) $y' = ay + bx$, $F(y, y') = y' - ay, g(x) = bx$: Solution?

Definition: An ODE $F(y, y^{(1)}, \ldots, y^{(n)}) = g(x)$ is *linear* iff each term in F is linear. In particular, if we scale y to ay, then each term should multiply by a. This means F cannot have terms like y^2 or yy'. For the latter case, note that when $y \mapsto ay$, y' also gets scaled by a, and so yy' becomes multiplied by a^2. (We say yy' is a quadratic term.) Note that the coefficients of a linear equation can be functions of the independent variable x.

Examples of other non-linear equations:

(a) $yy'' = x + 2$

(b) $y'' + 3y' + 4y^3 = 0$

(c) $y'' = \sin y$

The last equation is non-linear as $\sin(y_1+y_2) \neq \sin(y_1)+\sin(y_2)$. Note also that the first order equation $(y')^2 - 2y = 3$ mentioned earlier is non-linear.

A *linear* ODE of order n looks like

$$(*) \qquad a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_{n-1}(x)y^{n-1} + a_n(x)y^n \;=\; g(x),$$

$$a_n(x) \neq 0$$

A simple but commonly occurring example of a linear ODE is

$$\frac{d^2y}{dt^2} \;=\; ct,$$

where c is a constant. This is the case, for instance, in Hooke's law for springs, where y is the displacement and $c = -k/m$, k: spring constant and m: mass.

A *linear* ODE has *constant coefficients* iff it is of the form

$$a_0y + a_1y' + a_2y'' + \cdots + a_ny^n = g(x)$$

where $a_1, a_2, a_3 \ldots a_n$ are constants, so independent of x.

As we will see later, such an equation can be solved with the help of matrices by converting it to a *linear system of first order equations*.

If given a linear ODE like $(*)$, one can ask if given any solution y_1, any scalar multiple cy_1 is also a solution. Consider for example,

$$y' \;=\; ay + b.$$

If y_1 is a solution, then $(cy_1)' = cy_1' = c(ay_1 + b) = a(cy_1) + bc$. Thus cy_1 won't be a solution (for $c \neq 1$), except when $b = 0$; RHS is 0.

Same argument for order m:

Scaling gives new solutions to $(*)$ iff $g(x)$ is identically 0.

Definition A linear ODE is *homogeneous* iff it's of the form

$$a_0(x)y + a_1(x)y' + \cdots + a_n(x)y^n = 0.$$

Suppose we have a linear ODE $(*)$. If y_1 and y_2 are two different solutions, then $y_1 - y_2$ is a solution of the homogeneous cousin of $(*)$, namely

$$(**) \qquad a_0(x)y + a_1(x)y' + \cdots + a_n(x)y^n = 0.$$

Moreover if we have a particular solution y_1 of $(*)$, and if y_2 is a solution of $(**)$, then $y_1 + y_2$ is a solution of $(*)$.

$n = 1$: A linear ODE of order 1 is of the form

$$a_0(x)y + a_1(x)y' = g(x), \qquad \text{with } a_1(x) \neq 0.$$

We may rewrite it as

$$y' = -b_0(x)y + h(x), \quad b_0 = \frac{a_0}{a_1}, \quad h = \frac{g}{a_1}.$$

Suppose $y_1 = u_1(x)$ and $y_2 = u_2(x)$ are both solutions of

$$y' = -b_0(x)y + h(x).$$

then $(y_1 - y_2)' = -b_0(x)(y_1 - y_2)$, showing that $y_1 - y_2$ is a solution of the homogeneous form $y' = -b_0(x)y$.

Let's now review how we solved (last time) the equation

$$y' = ay + b.$$

We wrote the right hand side (RHS) as $a\left(y + \frac{b}{a}\right)$ and first noted that there is a unique stationary point given by $y = -\frac{b}{a}$. Moreover, when $y \neq fracba$, we may divide both sides by $y + \frac{b}{a}$ and get

$$\frac{\frac{dy}{dx}}{y + \frac{b}{a}} = a.$$

Integrating both sides relative to x,

$$\int \frac{dy/dx}{y + \frac{b}{a}} dx = \int a dx + c.$$

If $u(x) = \ln |y + \frac{b}{a}|$, then $\frac{du}{dy} = \frac{1}{y + \frac{b}{a}}$, and by chain rule,

$$\frac{du}{dx} = \frac{1}{y + \frac{b}{a}} \cdot \frac{dy}{dx}.$$

Thus

$$\ln |y + \frac{b}{a}| = ax + c$$

$$\Rightarrow |y + \frac{b}{a}| = e^{ax+c} = Ae^{ax}$$

$$\Rightarrow y = -\frac{b}{a} + Ae^{ax}$$

When $x = 0$, $y = y_0 = -\frac{b}{a} \pm A$, and consequently, the final solution is

$$y = -\tfrac{b}{a} + (y_0 + \tfrac{b}{a})e^{ax}.$$

If we had started with the stationary solution, i.e., with $y_0 = -\frac{b}{a}$, y will forever remain the same as y_0. Otherwise, the term involving e^{ax} will dominate for large ax.

Lecture 3

First Order ODE:

$$\frac{dy}{dt} = f(t, y)$$

- Very hard in general to solve explicitly in closed form.

Things to do:

1) To get a qualitative picture, look at the **slope field** (or the *gradient field*), obtained by drawing short arrows in the direction of $y'(t)$ at a grid of points in the (t, y)-plane. One calculates the slope (of the arrows) as $f(t_i, y_i)$ at the selected points (t_i, y_i). Of importance are the **equilibrium points**, which are the points where $dy/dt = 0$; these are the stationary points.

2) If $f(t, y)$ is independent of y, say $\frac{dy}{dt} = g(t)$, with $g(t)$ integrable, then the solutions are given by the indefinite integral

$$y = \int g(t)dt,$$

which is determined only up to a constant c. If we know $y_0 = y(0)$, the **initial condition**, we can find c. Otherwise there will be infinitely many solutions.

One way to guarantee that g is integrable is for it to be continuous.

3) If $f(y, t)$ is independent of t, i.e.

$$\frac{dy}{dt} - \varphi(y) = 0 \qquad (*)$$

the equilibrium solution is given by $\varphi(y) = 0$. When $\varphi(y) \neq 0$, $(*)$ becomes

$$\tfrac{1}{\varphi(y)} \tfrac{dy}{dt} = 1.$$

Integrating,

$$\int \left(\frac{1}{\varphi(y)} \right) \frac{dy}{dt} dt = t + c$$

We can solve this if we can integrate $\frac{1}{\varphi(y)} dy/dt$.

For example, when $\varphi(y) = y + a$

$$\underbrace{\int \frac{dy/dt}{y+a} dt}_{\ln|y+a|} = t + c$$

By chain rule,

$$\frac{d}{dt} \ln|y+a| = \frac{dy/dt}{y+a},$$

and exponentiating both sides, we obtain $|y + a| = e^{t+c}$, an thus, for a non-stationary solution, $y = -a + (\pm e^c)e^t$. Clearly, $\pm e^c$ can take on any non-zero value B. The final solution is

$$y = -a + Be^t,$$

B is any constant; the solution is stationary iff $B = 0$.

4) **Separation of variables**:

Check to see if $f(y, t)$ factors as $\varphi(y)g(t)$. If so,

$$\frac{dy}{dt} = \varphi(y)g(t)$$

can be solved as in case 3): Firstly, the stationary solutions are given by $\varphi(y) = 0$ (if $g(t) \neq 0$). If $\varphi(y) \neq 0$ we can write

$$\int \left(\frac{1}{\varphi(y)} \frac{dy}{dt} \right) dt = \int g(t) dt + c,$$

and one can proceed further if $\frac{1}{\varphi(y)} dy/dt$ and $g(t)$ can be integrated.

Example:

$$\frac{dy}{dt} = yt.$$

The stationary solutions are when $y = 0$, and when $y \neq 0$,

$$\int \frac{\frac{dy}{dt}}{y} dt = \int t dt + c,$$

yielding

$$\ln y = \frac{1}{2} t^2 + c, \; c = y_0.$$

5) **The case of Leibniz**:

$$(*) \qquad y' + h(t)y = g(t), \quad \text{i.e.,} \quad f(y,t) = g(t) - h(t)y.$$

Idea: Multiply both sides by a suitable factor $u(t)$, called an *integrating factor*, satisfying

$$(\boxdot) \qquad \frac{d}{dt}(u(t)y) = u(t)(y' + h(t)y).$$

Why is it useful?

Suppose we have found some $u(t)$ so that $(\boxdot)$ holds. Then we can multiply the given ODE $(*)$ by $u(t)$ on both sides to get

$$\underbrace{u(t)[y' + h(t)y]}_{=\frac{d}{dt}(u(t)y)} = u(t)g(t)$$

Integrating both sides relative to t, we get

$$\int \left[\frac{d}{dt}(u(t)y)\right] dt = \int u(t)g(t)dt + c$$

By the Fundamental Theorem of Calculus, $u(t)y = \int u(t)g(t)dt + c$.

$$\implies y = \frac{1}{u(t)} \int u(t)g(t)dt + c$$

Can we find $u(t)$ st $(\boxdot)$ holds? *YES* if $h(t)$ is integrable. indeed, we have

$$\frac{d}{dt}(u(t)y) = \underbrace{u(t)y' + u'(t)}_{=u(t)[y'+h(t)y]} y.$$

So we want $u(t)$ to satisfy:

$$u'(t) = u(t)h(t),$$

which doesn't involve y. Then

$$\frac{d}{dt}(\ln|u(t)|) = \frac{u'(t)}{u(t)} = h(t),$$

implying that we may choose u such that

$$ln|u(t)| = \int h(t)dt + c,$$

so that

$$u(t) \ = \ Be^{\int h(t)dt},$$

for a constant B, which we may take to be 1. (We just need to find *some* factor u which works.) In other words,

$$y = e^{-\int h(t)dt} \int e^{h(t)dt} g(t)dt + c$$

An Example of Leibniz's method

$$(*) \qquad y' \ = \ t^2 - ty.$$

To find the stationary solutions, we solve $t^2 - ty = 0$, i.e., $t(t - y) = 0$, and so the stationary solution is $y = t$. To find all solutions, rewrite $(*)$ as

$$y' + ty = t^2,$$

so that

$$h(t) = t, \ \text{ and } g(t) = t^2$$

in the notation above. Then

$$u(t) = e^{\int h(t)dt} = e^{\int tdt} = e^{\frac{1}{2}t^2},$$

so that

$$y \ = \ e^{-\frac{1}{2}t^2} \int e^{\frac{1}{2}t^2} \cdot t^2 dt + c,$$

which can be evaluated.

Mathematical Models

Given a physical (chemical, biological, financial, ...) system, do the following:

I) Try to identify the independent and dependent variables, and formulate the problem as a mathematical equation, clearly noting the hypotheses one makes;

II) Try to solve the equation analytically, and many practical situations are described by differential equations;

III) Check if the solutions obtained from the model are compatible with experimental observation.

A basic example: "Money in the bank"

$$M(t + \Delta t) \;=\; M(t) + I(t)M(t)\Delta(t) + g(t)\Delta(t)$$

where $M(t)$ is the amount of money in your account at time t, I the interest rate (which could depend on t, but often taken to be fixed), Δt a small increment in time, and $g(t)$ the amount deposited minus the amount withdrawn between t and $t+\Delta t$. Subtracting $M(t)$ from both sides and dividing by Δt, one gets

$$\frac{M(t + \Delta t) - M(t)}{\Delta t} \;=\; I(t)M(t) + g(t).$$

If one imagines that Δt is very small, one can approximate this equation by the first order, linear ODE

$$\frac{dM}{dt} \;=\; I(t)M + g(t).$$

It has constant coefficients iff $I(t)$ and $g(t)$ are independent of t.

Lecture 4

Exact Equations

We have been looking at 1st order ODEs with one independent variable, say x, and one dependent variable, say y, given by:

$$\frac{dy}{dx} = f(x, y).$$

It is said to be a *linear ODE* if $f(x, y)$ is linear in y, i.e., of the form $a(x)y + b(x)$, and the ODE is linear with *constant coefficients* when a, b are independent of x. Note that we do not require f to be linear in both x and y for the ODE to be linear.

Often $f(x, y)$ is a quotient of functions such as

$$\frac{x^2 \sin y}{y}.$$

(Clearly, the ODE is not linear in this example.)

We can clear denominators, and rewrite the ODE in the form

$$M(x, y) + N(x, y)y' = 0 \qquad (*)$$

Often it is not easy to solve for y explicitly in terms of x, but it may be possible to express the solution as a relation $\psi(x, y) = 0$.

Example:

$$(x + 1) - yy' = 0, \quad y' = \frac{dy}{dx}.$$

Since $yy' = \frac{1}{2}d(y^2)/dx$, we can show that the solutions are given by the points on the hyperbola

$$(x + 1)^2 - y^2 = c,$$

for an undetermined constant c. (Strictly speaking, in the degenerate case $c = 0$, this describes the union of two lines with equations $y = x + 1$ and $y = -x - 1$; moreover, for $c \neq 0$, there is no (real) y if $(x + 1)^2 < c$.) The solutions can be written as

$$y = \sqrt{(x + 1)^2 - c} \quad \text{or} \quad y = -\sqrt{(x + 1)^2 - c}.$$

This is a simple example. In general, it is not so easy to solve for y precisely. We may only be able to express y *implicitly* by writing the solution as a relation between x and y.

Another Example:

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x + \sin y)\frac{dy}{dx} = 0$$

The general solution will turn out to be

$$\psi(x,y) \;=\; c, \quad \text{with} \quad \psi(x,y) = e^x \sin y + 2y\cos x - \cos y.$$

Here we are unable to give a formula for y expressing it explicitly as a function of x.

Let us consider again the general 1st order equation $(*)$. The situation is good if we can rewrite $(*)$ in the form

$$d(\psi(x,y)) = 0 \tag{**}$$

In this case, the general solution will simply be given by the relation

$$\psi(x,y) \;=\; c,$$

for a constant c, which we can evaluate if we know the initial value y_0 at $x = x_0$.

When $(*)$ can be converted to $(**)$, i.e., when we can find such a ψ, we call $(*)$ an *exact* differential equation of the 1st order.

Suppose $\psi(x,y)$ is differentiable (with continuous partial derivatives):

$$\psi_x = \frac{\partial \psi}{\partial x}, \quad \psi_y = \frac{\partial \psi}{\partial y}$$

If the dependence of y on x is given by a function $y = \varphi(x)$, then ψ becomes a function of x and it makes sense to consider the total derivative $d\psi/dx$.

For example, consider

$$\psi(x,y) = xy, \quad y = \varphi(x).$$

when $\psi_x = y$ and $\psi_y = x$. Moreover, since $y = \varphi(x)$, $\psi = x\varphi(x)$, which is totally a function of of x, and

$$\frac{d\psi}{dx} \;=\; \varphi(x) + x\varphi'(x).$$

In other words, since $\varphi(x) = y = \psi_x$ and $\varphi'(x) = dy/dx$,

$$\frac{d\psi}{dx} = \psi_x + \psi_y y'. \tag{1}$$

This holds in general for $\psi(x, \varphi(x))$, not just in this example.

Let's go back to $(*)$. We now realize that it can be rewritten as an exact equation $(**)$ **IFF** we have

$$M = \psi_x, \quad \text{and} \quad N = \psi_y,$$

for some function $\psi(x, y)$. We need even a better criterion for the equation $(*)$ to be exact, because in general we don't know how to guess $\psi(x, y)$. Some times one can guess ψ, however!
Example:

$$\underbrace{(x+y)}_{M(x,y)} + \underbrace{(x-y)}_{N(x,y)} y' \ = \ 0.$$

Put

$$\psi(x, y) = (x^2 + 2xy - y^2)/2,$$

which is an easy guess! Then

$$\psi_x = x + y = M, \quad \textit{and} \quad \psi_y = x - y = N.$$

So the ODE is exact and of the form $\frac{d\psi}{dx} = 0$, with solution $\psi(x, y) = 0$.

Warning! In general, as remarked above, it is not easy to guess ψ so easily! Also **the equation is not always exact, so there may be no** ψ.

Here's an idea on how to find ψ when it exists.

First Step: *Determine if it is exact*!

If there is a ψ, we may consider the second (mixed) partial derivatives

$$\psi_{xy} = \frac{\partial^2 \psi}{\partial x \partial y}, \qquad \psi_{yx} = \frac{\partial^2 \psi}{\partial y \partial x},$$

assuming ψ is twice differentiable.

Recall from Ma1a: If ψ is twice differentiable and has continuous 2nd partial derivatives, denoted $\psi \in \mathcal{C}^2$, then

$$\psi_{xy} = \psi_{yx}.$$

Given $(*)$, we want to know if $\exists\psi(x,y)$ and $\psi_x = M$ and $\psi_y = N$. Suppose M, N are continuously differentiable (i.e., $M, N \in \mathcal{C}^1$). Then $\psi_{yx} = M_y, \psi_{xy} = N_x$, so $M_y = N_x$.
Thus we have the following

Theorem *Suppose $M(x,y), N(x,y)$ are continuously differentiable in an open rectangular region $(a,b) \times (c,d)$ in $\mathbb{R}^2$. Then,*

$$M(x,y) + N(x,y)y' = 0, \tag{*}$$

with M, N continuously differentiable, is exact iff we have

$$(**) \qquad M_y = N_x.$$

Second Step: *Find ψ when $(*)$ is exact*!

What we want: $\psi(x,y)$, such that $\psi_x = M$ and $\psi_y = N$. What we know: $M(x,y) + N(x,y)y' = 0$, with $M_y = N_x$.

This forces

$$\psi(x,y) = \int M(x,y)dx + g(y),$$

where the integration over x is performed by keeping y constant.

Usually one writes $Q(x,y) = \int M(x,y)dx$, so that

$$\psi(x,y) = Q(x,y) + g(y). \tag{2}$$

Then $\psi_x = Q_x = M$

Equation (1) implies $\psi_y = Q_y + g'(y)$. This should be $N(x,y)$, so we need $g'(y) = N - Q_y$.

$$\Rightarrow g(y) = \int (N - Q_y)dy + c_1,$$

for a constant c_1 (which we may take to be zero, as we want just any ψ which works), and the integration over y is performed keeping x constant. Thus

$$\psi(x,y) = \int M(x,y)dx + \int (N(x,y) - Q_y)\, dy.$$

We may also write this as

$$\psi(x,y) \;=\; Q(x,y) + \int \left(N(x,y) - \frac{d}{dy}\int M(x,y)dx \right) dy.$$

Summary: *Given* $M(x,y)+N(x,y)\frac{dy}{dx}=0$ *with* M,N *continuously differentiable, we can express it as an exact equation* $\frac{d\psi}{dx}=0$ *iff we have* $M_y=N_x$*. In such a case, the general solution is given by* $\psi(x,y)=c$*. If we are given an initial condition, say* $y=b$ *when* $x=0$*, then we can solve for* c*.*

Lecture 5

Integrating Factors

Last time we looked at exact equations and obtained solutions of the form

$$\psi(x, y) = 0.$$

When we do this, we are writing y *implicitly* as a function of x, but not explicitly. An example of an implicit expression for y (depending on x) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Differentiating with respect to x,

$$(*) \qquad M(x, y) + N(x, y)y' = 0, \quad \text{with} \quad M = \frac{2x}{a^2}, \quad N = \frac{2y}{b^2}.$$

We also learned of a procedure to find ψ when the given equation $(*)$ is exact. The idea is to find a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. There are two stages in finding a candidate ψ:

(i) $\psi(x, y) = Q(x, y) + g(y)$,
with $Q(x, y) = \int M(x, y)dx$ (integration with y fixed);

(ii) $g(y) = \int (N(x, y) - Q_y)dy$ (integration with x fixed)

Remark:

1) The exactness condition on $(*)$ is not always satisfied. However, one has exactness if M is purely a function of x (i.e., independent of y) *and* (simultaneously) N is purely a function of y (i.e., independent of x), since in this case $M_y = 0 = N_x$; this is the case of *separation of variables.*

2) To check exactness, we need M and N to be continuously differentiable with $M_y = N_x$.

Example:

$$0 = \underbrace{(e^x \sin y - 2y \sin x)}_{M(x,y)} dx + \underbrace{(e^x \cos y + 2 \cos x + 3 \sin y)}_{N(x,y)} dy \qquad (*)$$

$$\implies M_y = e^x \cos y - 2\sin x \overset{**}{,} \; N_x = e^x \cos y - 2\sin y.$$

Hence $(*)$ is exact. (All this makes sense because M and N are continuously differentiable.) To find ψ, first solve for Q, then for g, and take $\psi(x,y) = Q(x,y) + g(y)$:

$$\begin{aligned} Q(x,y) &= \int M(x,y)dx \quad \text{keeping } y \text{ fixed in the integral} \\ &= e^x \sin y + 2y\cos x \\ \psi(x,y) &= Q(x,y) + g(y) \\ Q_y &= e^x \cos y + 2\cos x \\ N &= e^x \cos y + 2\cos x + 3\sin y \\ g(y) &= \int N(x,y) - Q(y)dy \text{ keeping } x \text{ fixed} \\ &= \int 3\sin y dy = -3\cos y + c \end{aligned}$$

Take

$$\begin{aligned} \psi(x,y) &= Q(x,y) + (y) \\ &= e^x \sin y + 2y\cos x - 3\cos y \end{aligned}$$

Solution of $(*)$: $e^x \sin y + 2y\cos x - 3\cos y = c.$

If we had the initial condition $y = 0$ when $x = 0$, then we can solve for c:

$$e^0 \sin(0) + 2(0)\cos(0) - 3\cos(0) = c \quad \text{so} \quad c = -3.$$

Many ODE's are not exact:
Example:

$$(x+2)\sin y dx + x\cos y dy = 0 \qquad (*)$$

$$\begin{aligned} M &= (x+2)\sin y, \text{ so } M_y = (x+2)\cos y \\ N &= x\cos y, \quad \text{so} \quad N_x = \cos y, \quad \text{and} \quad M_y \neq N_x. \end{aligned}$$

Sometimes, it may happen that the given ODE $(*)$ is not exact, but it becomes exact after multiplying it by an *integrating factor*. In other words, given $M(x,y) + N(x,y)\frac{dy}{dx} = 0$,

$$M_y \neq N_x, \text{ but for some } u(t) \text{ we may have}$$

$$(uM)_y = (uN)_x$$

In other words, the modified ODE

$$u(x,y)M(x,y) + u(x,y)(Nx,y)\frac{dy}{dx} = 0$$

is an exact equation. Note:

$$(uM)y = (uN)_x \;\Leftrightarrow\; u_y M + uM_y = u_x N + uN_x$$
$$\implies u_y M - M_x N = u(N_x - M_y)$$

Even when such a u exists for $(*)$, it's not easy to find. Here are a few tricks to try (which work sometimes):

1) Take u to be purely a function of x. Then we need

$$-u_x N = u(N_x - M_y)$$

If non zero,

$$\frac{-u'(x)}{u(x)} = \frac{N_x - M_y}{N}$$

For this it is necessary and sufficient that $\frac{N_x - M_y}{N}$ is independent of y

Suppose $\frac{N_x - M_y}{N} = -\varphi(x)$ then $\frac{u'(x)}{u(x)} = \varphi x$

$$\Rightarrow \ln|u(x)| = \int \varphi(x)dx$$

$$\Rightarrow u(x) = Be^{\int \varphi(x)dx}; \quad \text{can take} \quad B = 1.$$

Example:

$$\underbrace{(x+2)\sin y}_{M}\, dx + \underbrace{x\cos y dy}_{N} = 0$$

$$M_y = (x_2)\cos y + N_x = \cos y$$

$$\frac{N_x - M_y}{N} = \frac{\cos y - (x+2)\cos y}{x\cos y} = \frac{-(x+1)}{x}$$

So $\varphi(x) = (x+1)/x$ in this case. Try

$$u(x) = e^{\int \varphi(x)dx)} = e^{(x+\ln|x|)+c},$$

which suggests that we take $u(x) = xe^x$.

2) Same as 1) with x and y switched. Try to choose u to be just a function of y

$$u_y M = u(N_x - M_y),$$

i.e., $\frac{u'(y)}{u(y)} = \frac{N_x - M_y}{M}$ needs to be a function of y alone.

3) Given $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $M_y \neq N_x$, see if you can find a $u(x,y)$ st uN is independent of x. In this case, we only have to solve $(uM)_y = (uN)_x$, whence $(uM)_y = 0$.

Example:

$$(*) \qquad \underbrace{x^2y^3}_{M(x,y)} + \underbrace{x(1+y^2)}_{N(x,y)}\, y' = 0.$$

Hence $(*)$ is not exact.

$$M_y = 3x^2y^2, N_x = 1 + y^2 \implies M_y \neq N_x.$$

Choose $u(x,y)$ to be $\frac{1}{x}h(y)$.

Then $uN = h(y)(1+y^2)$, which is independent of x.

We need to check if we can choose $h(y)$ st $(uM)_y = 0$:

$$uM = xy^3h(y) \implies (uM)_y = 3xy^2h(y) + xy^3h'(y)$$

So we need, for $xy \neq 0$,

$$3h(y) + yh'(y) \;=\; 0,$$

which can be solved by taking

$$h(y) \;=\; y^{-3}.$$

Hence an integrating factor is given by

$$u(x,y) \;=\; y^{-3}(1+y^2).$$

This can be used to get all the non-equilibrium solutions. (The equilibrium solutions occur when $xy = 0$.)