THE MATHEMATICAL METHOD VIA CALCULUS

DINAKAR RAMAKRISHNAN
Taussky-Todd-Lonergan Professor of Mathematics
California Institute of Technology

0. Proofs in Mathematics

It is not difficult to write mathematical proofs of statements, but one needs some experience, especially since the high school courses do not stress this aspect of Math. This is essential for the mathematical method, and so we will try to give a few selected ways of writing proofs. It is very helpful to understand this aspect now just using integers, rational numbers, and (later) real numbers, before trying out such arguments in Calculus proper.

First we need some preliminary notation. There is nothing complicated here, just that one has to attain a bit of familiarity. Throughout these Notes, we will mean by a set simply a collection X of objects having a common property. (This is only a rough definition, but it will do for our purposes in Ma 1a.) If x belongs to X, we will write $x \in X$. If $x_1, x_2, x_3, \ldots, x_n, \ldots$ are the elements of X, then we will write

$$X = \{x_1, x_2, \ldots, x_n, \ldots\}.$$

The symbol $\emptyset$ will denote the *empty set*, which is a set with no element, not to be confused with the set $\{0\}$ consisting of 0. The symbol $\forall$ will denote *for all* (or *for every*), and $\exists$ will mean *there exists*.

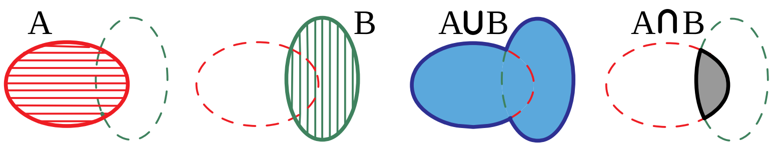
The union of two sets A, B is

$$A \cup B = \{z \,|\, z \in A \text{ or } z \in B\},$$

and their intersection is

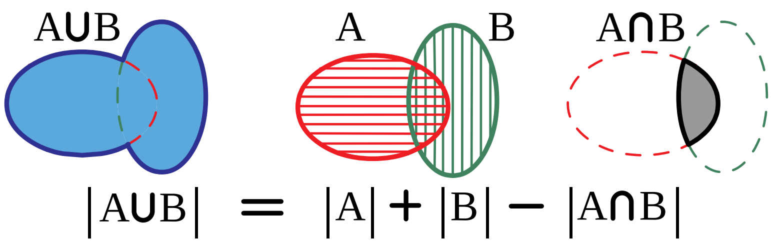
$$A \cap B = \{z \,|\, z \in A \text{ and } z \in B\}.$$

These definitions can be extended to the case when there are more than two sets.

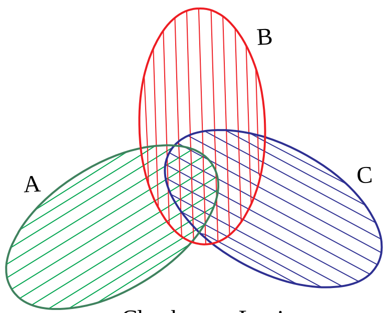


We say that a set C is a *subset* of another set A, denoted $C \subset A$, if every element of C is also an element of A. For example, $A \cap B$ is a subset of A (and of B), while A and B are subsets of $A \cup B$.

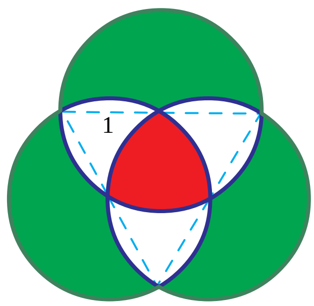
The *cardinality* of a set A, denoted $|A|$ (or $\#A$) is, when it is finite, the number of elements in it. If A is not finite, we will write $|A| = \infty$. There are different kinds of infinities, which we will discuss a bit later. Given two sets A, B, the cardinality of their union is seen to be the sum of the cardinalities of A and B minus the cardinality of their intersection. In other words, we include the cardinality of $A \cap B$ twice in $|A| + |B|$, so we have to exclude it once afterwards. This is a simple instance of the *inclusion-exclusion principle*. Here is a pictorial representation:



Can you derive the formula for the cardinality of the union of three sets by using this principle?



Check your Logic



Find the area of each Shape

This principle applies equally to computing the *area of a union of plane regions* or to the *probability of occurrence of one of the events*. (If X, Y are two events, their union symbolizes one of the events happening, while their intersection symbolizes both events occurring, all at some prescribed time.) The formula is the same in these cases, with $A(X)$, resp. $\mathbb{P}(X)$, replacing $|X|$.

We will define the set of natural numbers to be

$$\mathbb{N} = \{1, 2, 3, \ldots, \},$$

which encompasses all the positive integers. Note that 0 is not in $\mathbb{N}$. We put

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\},$$

which consists of all integers, positive, negative and zero. The set of *rational numbers* is

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\},$$

with the identification of $\frac{a}{b}$ with $\frac{c}{d}$ whenever $ad = bc$. Note that we can add, subtract or multiply any two numbers a, b in $\mathbb{Z}$, but we cannot divide 1 by b, unless $b = \pm 1$. This is the reason for considering $\mathbb{Q}$, where we can form any ratio x/y, unless $y = 0$.

The set $\mathbb{R}$ of real numbers, which we will discuss later, contains $\mathbb{Q}$ as a subset. Of course, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. We will write, as usual, $a \leq b$ (resp. $a \geq b$) to denote *a is less than or equal to b* (*a is greater than or equal to b*). Similarly for $a < b$ and $a > b$.

We write $A \implies B$ to signify A implies B, when A and B are statements asserting something. In plain English, this means that *if A is true, then B is true*, often denoted in shorthand by *if A, then B*. Note that if B is true, then one can say nothing about whether A is true.

If A implies B *and* B implies A, then we write

$$A \iff B,$$

and say *A is true iff B is true*, or just A iff B; here *iff* is shorthand for *if and only if*.

$\neg A$ will mean *Not* A, the negation of A. If A says, for example, that m is an even integer, then $\neg A$ says that m is not an even integer, i.e., m is an odd integer. It is useful to observe that $A \implies B$ is the same as $\neg B \implies \neg A$.

0.1. **Proof by direct verification.** This is the least subtle way of arriving at a proof of a statement, but nevertheless important to know. Often it involves nothing more than applying definitions.

Example 1: Let us prove the following proposition:

$$x^2 + y^2 \geq 2xy, \forall\, x, y \in \mathbb{Q}.$$

Recall that

$$(x - y)^2 = x^2 - 2xy + y^2,$$

and that the square of any number in $\mathbb{Q}$, e.g., $(x - y)^2$, is non-negative. Thus

$$x^2 - 2xy + y^2 \geq 0.$$

The assertion follows by adding $2xy$ to both sides.

Example 2: Let us now prove the following:

Proposition Any integer n has the same *parity* as its square, i.e., n and n^2 are both odd or both even.

Proof. Suppose first that n is even. Then as 2 divides it, we may write $n = 2m$, for some integer m. Then $n^2 = (2m)^2 = 4m^2$, which is even. Next consider when n is odd. Then $n - 1$ is even, and so

$n-1=2k$ for some integer k, which implies that $n=2k+1$. Then $n^2=(2k+1)^2=4k^2+4k+1$, which is odd. Done!

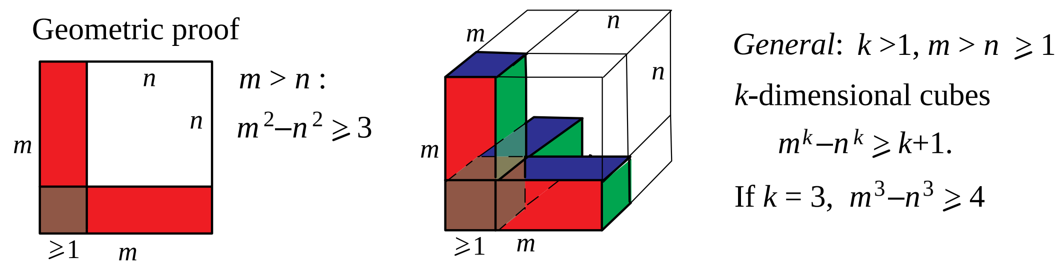
0.2. **Proof by contradiction.** If we want to prove $A \implies B$, then we can try to see if there will be a contradiction to something which is known to be true if we assume that A does not imply B.

Example 1 The equation $x^2-y^2=1$ cannot be solved with x, y being positive integers.

Proof. Suppose there are positive integers x, y such that x^2-y^2 equals 1. Then, factoring x^2-y^2 as $(x+y)(x-y)$ we see, since $x+y$ and $x-y$ are integers too, that there are exactly two possibilities for $(x+y, x-y)$, namely $(1,1),(-1,-1)$. The second solution is not really possible as $x+y$ is positive. So we must have $x+y=x-y=1$, implying $2y=0$, which is impossible as y is a positive integer.

In other words, if we assume that the Proposition is false, then we get a contradiction either to the positivity of x, y or to their integrality. QED!

One can also see geometrically why $m^2-n^2 \geq 3$ for positive integers m, n with $m>n$.; in fact, this can be jazzed up to show that $m^k-n^k \geq k+1$ for any $k>1$:



Example 2 There is no rational number x such that $x^2=2$.

Proof. Suppose not, and assume the existence of some $x=a/b$ in $\mathbb{Q}$ such that $x^2=2$. If (a, b) works, then so will $(-a, b),(a,-b)$, and $(-a,-b)$, as we are squaring x to get 2. So we may take a, b to be positive, Similarly, if (a, b) works and if d divides both a and b, then $(a/d, b/d)$ will also work, and so we may take a, b to have no common factor. (In other words, the *greatest common divisor* of a, b, denoted $\operatorname{gcd}(a, b)$ is 1.) Then we have

$$\frac{a^2}{b^2}=2, \text{ i.e., } a^2=2b^2.$$

Since $2b^2$ is even, a^2 must also be even, and this means a itself must be even (by Example 2 of section 0.1). So we may write $a = 2a_1$, with a_1 a positive integer. Now we get $(2a_1)^2 = 2b^2$, which yields, after dividing by 2,

$$2a_1^2 = b^2.$$

Then b^2, and hence b will be even. But this means 2 divides both a and b, contradicting the way we chose them, namely with them being relatively prime. This furnishes the desired contradiction, and so the assertion that there is no $\sqrt{2}$ in $\mathbb{Q}$ is true. QED.

It is this fact, that $\mathbb{Q}$ does not contain square roots of $2, 3,$, etc., which drives one to expand the number system further from $\mathbb{Q}$.

0.3. **Proof by Contrapositive.** Suppose we want to prove that $A \implies B$. In this method, we assume $\neg B$ and try to prove $\neg A$. Note that we are proving the contrapositive, namely $\neg B \implies \neg A$, which, as we saw earlier, is equivalent to $A \implies B$. Note that this method is not the same as proof by contradiction, where one assume that A holds but B doesn't, in order to arrive at a contradiction.

Example 1 Suppose x, y are rational numbers such that $x + y$ is not an integer. Then either x or y is not an integer.

Proof. Consider rational numbers x, y. The negation of *x or y is not an integer* is the statement: *x and y are both integers*. Let us assume this. Then we need to prove the negation of *$x + y$ is not an integer*, which is *$x + y$ is an integer*. In other words, we need only check that if x, y are in $\mathbb{Z}$, then $x + y$ is in $\mathbb{Z}$, which is true. QED.

0.4. **Proof by Induction.** Adding odd integers

$1 = 1$	$1 + 3 + 5 + 7 = 16$	
$1 + 3 = 4$	$1 + 3 + 5 + 7 + 9 = 25$	*Conjecture*: $\sum_{k=1}^{n}(2k-1) = n^2$
$1 + 3 + 5 = 9$		How can one prove this?

Note: if this is true for $n - 1$, what about for n? We have

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n-1}(2k-1) + (2n-1),$$

where the first term on the right is known to be $(n-1)^2$. So the right hand side becomes

$$(n-1)^2 + 2n - 1 = n^2 - 2n + 1 + 2n - 1\, n^2.$$

True for n!

This leads us to the following

The method of Induction: Let $P(n)$ be a property of natural numbers n. The *Principle of Induction*, or P.O.I. for short, allows us to conclude that $P(n)$ is true for every $n \geq n_1$ if we can perform the following steps for some integer $n_1 \in \mathbb{N}$:

(a) Prove that $P(n_1)$ is true
(b) Let n be an arbitrary fixed integer $> n_1$. Assume that $P(k)$ is true for all $k < n$ and prove that $P(n)$ is true.

$$P(n_1) \Rightarrow P(n_1+1) \Rightarrow P(n_1+2) \Rightarrow \dots$$

Usually, $n_1 = 1$.

Example 1: Prove by induction the following:

$$P(n) : \ 1^2 + 2^2 + \cdots + n^2 = \tfrac{n^3}{3} + \tfrac{n^2}{2} + \tfrac{n}{6}$$

First check (a) with $n_1 = 1$. $P(1)$ is true because

$$1^2 \ = \ 1 \ = \ \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{6}$$

Let's check (b) for $P(n)$, under the induction hypothesis that $P(k)$ is true for all $k < n$.

$$(*) \ \ 1^2 + \cdots + n^2 = \big(1^2 + \cdots + (n-1)^2\big) + n^2 = \tfrac{(n-1)^3}{3} + \tfrac{(n-1)^2}{2} + \tfrac{n-1}{6} + n^2.$$

Note that

$$(n-1)^3 \ = \ n^3 - 3n^2 + 3n - 1, \ \text{and} \ (n-1)^2 = n^2 - 2n + 1.$$

So the right hand side of $(*)$ is

$$\left(\tfrac{n^3}{3} - n^2 + n - \frac{1}{3}\right) + \left(\tfrac{n^2}{2} - n + \frac{1}{2}\right) + \left(\tfrac{n}{6} - \frac{1}{6}\right) + n^2 + \tfrac{n}{6},$$

which simplifies as

$$\tfrac{n^3}{3} + \tfrac{n^2}{2} + \tfrac{n}{6}.$$

Hence $P(n)$ holds.

It is easy to wrongly apply induction, and it may be worthwhile to consider the following:

False Example: *All cats have the same color eyes.*

Mathematically, this says, for any $n \geq 1$:

$P(n)$: For any collection of n cats, all of them have the same color eyes.

Here is the argument:

$P(1)$ is obviously true as we have only one cat to consider.

Now assume that the statement is true for all $k < n$. Let us try to prove $P(n)$. Consider $S := \{C_1, C_2, \dots, C_n\}$, where we have indexed the cats by C_1, C_2, etc. Consider the subsets

$$S_1 := \{C_1, C_2, \dots, C_{n-1}\},\ S_2 := \{C_{n-1}, C_n\}.$$

By induction, both these sets consist of cats with the same color eyes. Since C_{n-1} is in both sets, every cat in S_1 has the same color as either in S_2. So $P(n)$ holds!

Of course this argument is fallacious, because in the "proof" of $P(n)$, we have implicitly assumed that n is at least 3. The argument clearly fails when $n = 2$!

There is another Principle in Mathematics which is, for some people like me, more believable than induction, and that is called the **Principle of Well Ordering**, or P.W.O for short. It says the following:

Every non-empty set S of positive integers has a least element.

Explicitly this says that there is an integer m in S such that every k in S satisfies: $m \leq k$. Of course, such an m must be unique, since if m' is another such, then $m \leq m'$ and $m' \leq m$, implying that $m' = m$.

Proposition *Well Ordering implies Induction.*

Proof. Suppose $P(n)$ is an assertion about positive integers n such that (a) $P(1)$ holds, and (b) for every $n > 1$, if $P(k)$ holds for all $k < n$, then $P(n)$ holds. Let S denote the set of all positive integers n for which $P(n)$ is false. If S is empty, then the Principle of Induction holds for the property P, and there is nothing to do. So assume that S is non-empty. Then by Well Ordering, S has a least element; call it m. Since $P(k)$ holds for all k outside S (by the definition of S), and since m is the smallest number in S, $P(k)$ must be true for all $k < m$. Then by (b), $P(m)$ must be true as well, contradicting the fact that m is in S. So S must be empty, and the Proposition is proved.

It can be shown, see Apostol's Calculus, volume I, page 37, for example, that conversely, the Principle of Induction implies the Well Ordering Principle. So, despite the difference in their appearances, they are equivalent principles. Feel free to use either in your Homework solutions, midterm or Final exam.

0.5. **Disproof by counterexample.** Sometimes one has to deal with a property which may or may not be true. Before trying to prove it, it is always wise (in such situations) to see if one can disprove the assertion, and for this all one has to do is to produce one example, called a *counterexample*, where the assertion fails.

Example 1 Prove the following or disprove it by giving a counterexample:

If a, b, c are integers such that $a \mid bc$, i.e., if a divides the product bc, then a divides b or a divides c.

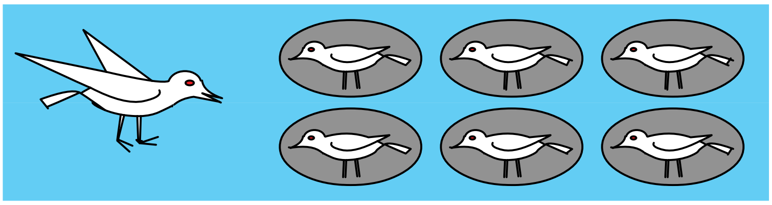
Counterexample: Consider $a = 6, b = 4$ and $c = 21$. Then a divides $bc = 4(21) = 84$, because $84 = 6(14)$, but a does not divide either b or c.

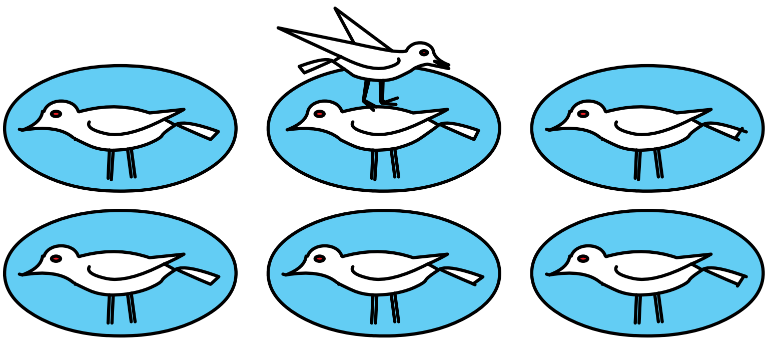
There is one situation in which this assertion is true, and that is when a is a *prime number*, which means a is an integer > 1 such that the only positive integers dividing a are 1 and a.

0.6. **The Pigeon Hole Principle.** Suppose we have n objects (or *pigeons*) and k boxes (or *holes/ cages*) to put them in. The *pigeon hole principle* then asserts the following:

If $n > k$, then at least one box must have two objects.

This simple principle is very useful and comes in handy in a variety of places.





For any rational (or real) number x, denote by $\lceil x \rceil$ the **ceiling of** x, i.e., the smallest integer greater than or equal to x. The **floor of** x, denoted $\lfloor x \rfloor$, is similarly defined as the largest integer less than or equal to x; this is also sometimes called the **integral part of** x, denoted by $[x]$. The **fractional part of** x, denoted $\{x\}$, is $x - [x]$, which is always between 0 and 1.

Here is a more precise version of this principle:

If we put n objects in k boxes with $n > k$, then one of the boxes must have at least $\lceil \frac{n}{k} \rceil$ objects.

This version is used in a lot of situations, some too difficult for this course. For example, it is used in the proof of *Kronecker's theorem* which asserts that given any fixed irrational number $\alpha > 0$, we can approximate well any real number β between 0 and 1 by a fractional part $\{n\alpha\}$ for a suitable positive integer n; the choice of n depends on the level of precision one wants in approximating β. As a fun exercise, you could try to (write a program and) compute the fractional parts $\{n\sqrt{2}\}$ for $n \leq 1000$.

1 The Real Number System

The rational numbers are beautiful, but are not big enough for various purposes, and the set $\mathbb{R}$ of real numbers was constructed in the late nineteenth century, as a kind of an envelope of $\mathbb{Q}$. (More later on this.) For a nonconstructive approach, one starts with a list of axioms, called the *Field axioms*, which $\mathbb{R}$ satisfies. This will not be sufficient to have $\mathbb{R}$, and we will also need *order axioms*, and the *completeness axiom*. In Math we like to build things based on axioms, much like in experimental sciences one starts from facts noticed in nature.

Field axioms: There exist two binary operations, called addition $+$ and multiplication $\cdot$, such that the following hold:

1) commutativity $x + y = y + x$, $xy = yx$
2) associativity $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$
3) distributivity $x(y + z) = xy + xz$
4) Existence of $0, 1$ such that $x + 0 = x$, $1 \cdot x = x$,
5) Existence of negatives: For every x there exists y such that $x + y = 0$.
6) Existence of reciprocals: For every $x \neq 0$ there exists y such that $xy = 1$.

We are, by abuse of notation, writing xy instead of $x \cdot y$.

A lot of properties can be derived from these axioms.

Example 1 $a + b = a + c \Rightarrow b = c$

Proof Ax. 5$\Rightarrow$ $\exists y$: $a + y = 0$. Take $y + (a + b) = y + (a + c)$ and use Ax. 2$\Rightarrow$ $(y + a) + b = (y + a) + c \Rightarrow 0 + b = 0 + c$ Use Ax. 4 $\Rightarrow b = c$.

Example 2 $(-1)(-1) = 1$.

Proof. (by direct verification) By definition of negatives, $(-1) + 1 = 0$. Hence

$$0 = ((-1) + 1)((-1) + 1) = (-1)(-1) + (-1)(1) + (1)(-1) + (1)(1),$$

where we have used the distributive property. By definition of the multiplicative unit 1, $(a)(1) = (1)(a) = a$ for any a. Taking a to be -1 and 1, we see that $(-1)(1) = (1)(-1) = -1$ and $(1)(1) = 1$. Thus we get from the equation above,

$$0 = (-1)(-1) - 1 - 1 + 1 = (-1)(-1) - 1,$$

which implies, as asserted, that $(-1)(-1)$ is 1. QED.

Note that it is not just $\mathbb{R}$ which satisfies the field axioms; $\mathbb{Q}$ does too. In fact, there are many subsets F of $\mathbb{R}$ besides $\mathbb{Q}$ which satisfy the field axioms. When such an F does, we call it a *field*. Here is a useful result:

Lemma *Let F be a subset of $\mathbb{R}$ containing $0, 1$ and such that for all x, y in F, the numbers $x + y$, $x - y$, xy, and x/y (if $y \neq 0$) lie in F. Then F is a field, i.e., satisfies all the field axioms.*

We are at this point inserting this Lemma without having built $\mathbb{R}$. You may take for granted that $\mathbb{R}$ can and will be constructed (soon).

Proof of Lemma. Given x, y, z in F, their sums and products of these lie again in F, by hypothesis. Since F is a subset of $\mathbb{R}$, the first three field axioms involving commutativity, associativity and distributivity hold for $x, y, z \in F$ because they hold in $\mathbb{R}$. Similarly, since 0 and 1 are assumed to be in F, for every x in F, the identities giving the remaining three axioms also hold in F because they hold in $\mathbb{R}$. Again, the point is that to check that the identities hold for a set of elements in F, it suffices to check that they hold in a larger set, namely $\mathbb{R}$. QED.

Example: *Consider the collection F of all real numbers of the form $x + y\sqrt{2}$, where x and y are rational numbers. Prove (by direct verification) that F satisfies all the field axioms (just like $\mathbb{R}$) under the usual addition and multiplication.*

We prove this using the Lemma above:

Since $0 = 0 + 0\sqrt{2}$ and $1 = 1 + 0\sqrt{2}$, it follows that F contains 0 and 1. Let x and y be two elements of F. By definition of F, there are a, b, c and d in $\mathbb{Q}$, such that $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. Now $x + y = (a + c) + (b + d)\sqrt{2}$ and $x - y = (a - c) + (b - d)\sqrt{2}$, thus these numbers are in F. Furthermore, $xy = (ac + 2bd) + (ad + bc)\sqrt{2}$ is in F as well. Assume now that $y \neq 0$, i.e.

that either c or d is non zero (or both). Thus also $c - d\sqrt{2} \neq 0$. Then

$$\begin{aligned}
\frac{1}{y} &= \frac{1}{c+d\sqrt{2}} = \frac{c-d\sqrt{2}}{(c-d\sqrt{2})(c+d\sqrt{2})} = \frac{c-d\sqrt{2}}{c^2-2d^2} \text{ and so} \\
\frac{x}{y} &= \frac{(a+b\sqrt{2})(c-d\sqrt{2})}{c^2-2d^2} = \frac{(ac-2bd)+(bc-ad)\sqrt{2}}{c^2-2d^2} \\
&= \frac{ac-2bd}{c^2-2d^2} + \frac{bc-ad}{c^2-2d^2}\sqrt{2}\ \text{, which is in } F.
\end{aligned}$$

The conditions of the lemma are satisfied and therefore F is a field.

Order axioms: Assume the existence of a subset $\mathbb{R}_+ \subset \mathbb{R}$, the set of positive numbers, which satisfies the following three order axioms:

7) If x and y are in $\mathbb{R}_+$, then $x+y, xy \in \mathbb{R}_+$

8) For every $x \neq 0$ either $x \in \mathbb{R}_+$ or $-x \in \mathbb{R}_+$, but not both.

9) $0 \notin \mathbb{R}_+$.

We now define

$$\begin{array}{lll}
x < y & \text{means} & y - x \in \mathbb{R}_+ \\
y > x & \text{means} & x < y \\
x \leq y & \text{means} & x < y \text{ or } x = y \\
x \geq y & \text{means} & x > y \text{ or } x = y
\end{array}$$

Examples:

1) (transitivity) $(a < b)$ and $(b < c) \Rightarrow (a < c)$

   Proof

   As $b - a > 0,\ c - b > 0$, by Axiom 7), $c - a = (b-a) + (c-b)$ is also > 0. Hence $a < c$. QED.

2) $(a < b)$ and $(c > 0) \Rightarrow ac < bc$.

   Proof As $b - a > 0,\ c > 0$, by Axiom 7), $(bc - ac) = (b-a)c$ is in $\mathbb{R}_+$. So $ac < bc$.

3) If $a \neq 0$ then $a^2 > 0$.

   Proof. If $a > 0$ then $a^2 > 0$ by Axiom 7. If $a^2 \not> 0$, then, as $a \neq 0$, $-a > 0$ and $(-a)(-a) > 0$. This means $a^2 > 0$, a contradiction. So we may not take a^2 to be negative..

Note: $(-a)(-b) = ab$ follows from the associativity of multiplication. $\square$

All axioms we have so far are satisfied by $\mathbb{Q}$.

A problem: Various geometric constructs, such as the length δ of the diagonal of the unit square, are not rational.

Pythagoras' theorem (from 6th Century BC) shows that $\delta^2 = 2$.

We have seen in the previous section that δ cannot be rational.

One would like to have a number system that includes numbers like $\sqrt{2}$.

Real numbers are commonly pictured as points of the line.

The *last remaining axiom* means that the **line has no holes**. Some say the real numbers form a "continuum."

Def. Suppose S is a nonempty set, and there exists B such that $x \leq B$ for any $x \in S$. Then B is called an **upper bound** for S. If $B \in S$ then B is called the **maximum element** of S.
Def. A number B is called a **least upper bound** of a nonempty set S if B has the following 2 properties:

(a) B is an upper bound for S.

(b) No number less than B is an upper bound for S.

Theorem 1.1 *The least upper bound for a set S is uniquely defined.*

Proof. If B_1 and B_2 are two least upper bounds then $B_1 \leq B_2$ and $B_2 \leq B_1 \Rightarrow B_1 = B_2$. $\square$

The least upper bound B of S, if it exists, is called the **supremum** of S.

Notation: $B = \sup S$.

Continuity axiom 10. Every nonempty set S of real numbers which is bounded above (has **an** upper bound) has a supremum: there exists $B = \sup S$.

Note that B may or may not belong to S.

Similarly one defines lower bounds, the greatest lower bound, which is called the **infimum**; notation

$$L = \inf S.$$

Theorem 1.2 *Every nonempty set S bounded from below has an infimum.*

Proof. Apply the continuity axiom to -S. $\square$

The Archimedean property of $\mathbb{R}$

Archimedes lived in the 3rd century BC.

Theorem 1.3 $\mathbb{N}$ *is unbounded above.*

Proof. Assume not. Then Axiom 10 $\Rightarrow \exists B = \sup \mathbb{N}$. $B - 1$ is not an upper bound $\Rightarrow \exists n \in \mathbb{N}\colon n > B - 1$. Adding 1 we get $n + 1 > B$. But $n + 1 \in \mathbb{N} \Rightarrow$ contradiction. □

Here is an important consequence:

Archimedean property If $x > 0$ and y is an arbitrary real number, there exists $n \in \mathbb{N}\colon nx > y$.

Proof. If not then y/x would have been an upper bound for $\mathbb{N}$. □

Geometrically: any line segment, no matter how long, may be covered by a finite number of line segments of a given positive length, no matter how small.

Existence of square roots

Theorem 1.4 *Every nonnegative real number a has a unique nonnegative square root.*

Proof. If $a = 0$ then 0 is the only square root of a. Assume $a > 0$. Let $S = \{x \in \mathbb{R}_+ | x^2 \leq a\}$. Since $(1 + a)^2 > a$, the number $1 + a$ is an upper bound for S. Also, $S \neq \emptyset$; $\frac{a}{1+a} \in S$ because $a^2 \leq a(1 + a)^2 \Leftrightarrow (\frac{a}{1+a})^2 \leq a$. By Ax. 10, there exists $B = \sup S$.

There are 3 possibilities: $B^2 > a$, $B^2 < a$, $B^2 = a$.

Assume $B^2 > a$. Let $C = \frac{1}{2}(B + \frac{a}{B})$. Then $C < B$ and $C^2 = \frac{1}{4}(B^2 + 2a + (\frac{a}{B})^2) = a + \frac{(B^2-a)^2}{4B^2} > a$. Hence, C is a smaller upper bound for $S \Rightarrow$ contradiction.

Assume $B^2 < a$. Choose C such that $(C < B)$ and $(C < \frac{a-B^2}{3B})$ and $(C > 0)$. Then

$$(B + C)^2 = B^2 + C(2B + C) < B^2 + 3BC < B^2 + a - B^2 = a.$$

Hence, $B + C \in S$ and $B + C > B$. Contradiction. □

We will prove later the existence, for any $n > 1$, of the nth roots of positive numbers using more powerful techniques arising from Calculus.

Representation of real numbers by decimals

A real number of the form

$$r = a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}$$

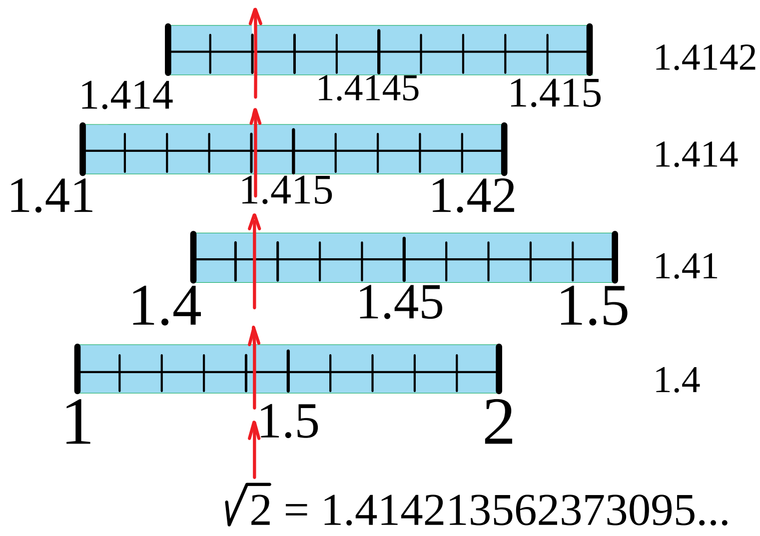
where a_0 is a nonnegative integer and $0 \leq a_i \leq 9, i = 1, 2, \ldots, n$, is usually written as

$$r = a_0 \cdot a_1 a_2 \ldots a_n$$

This is a **finite decimal representation** of r. Not any real number, and not even every rational number, can be represented in such a form. However, we can *approximate* an arbitrary real $x > 0$ to any desired degree of accuracy by finite decimals:

If $x \notin \mathbb{Z}$ then $\exists a_0$ such that $a_0 < x < a_0+1$. Divide the segment (a_0, a_0+1) into 10 equal parts. If x is not a subdivision point then $\exists a_1 \in \{0, \ldots, 9\}$ such that $a_0 + \frac{a_1}{10} < x < a_0 + \frac{a_1+1}{10}$. Continuing like that, we get at the nth stage

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} < x < a_0 + \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10}.$$



If x is never a subdivision point, we say that x has the infinite decimal representation

$$x = a_0 \cdot a_1 a_2 a_3 \ldots$$

For example, $1/3 = 0.333\ldots$

In what sense does the decimal representation define x?

Consider the set

$$S = \{a_0, a_0 \cdot a_1, a_0 \cdot a_1 a_2, a_0 \cdot a_1 a_2 a_3, \ldots\}$$

Then we can define $x = \sup S$.

Note that for a given x the decimal representation need not be uniquely defined! For example,

$$\frac{1}{2} = 0.500\ldots = 0.4999\ldots$$

This only happens when x is one of the subdivision points, that is, if it has a finite decimal representation.

If $x > 0$ has decimal expansion $a_0 \cdot a_1 \ldots a_n \ldots$, one says that $-a_0.a_1 \ldots a_n \ldots$ is the decimal expansion of $-x$.

Decimal expansions will not have much of a role in our course.

We will revisit the construction of real numbers in the next section, and learn how to think of them as limits of Cauchy sequences of rational numbers.

2 Sequences and series

We will first deal with sequences, and then study infinite series in terms of the associated sequence of partial sums.

2.1 Sequences

By a *sequence*, we will mean a collection of numbers

$$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots,$$

which is indexed by the set $\mathbb{N}$ of natural numbers. We will often denote it simply as $\{a_n\}$.

A simple example to keep in mind is given by $a_n = \frac{1}{n}$, which appears to decrease towards zero as n gets larger and larger. In this case we would like to have 0 declared as the *limit of the sequence*. A quick example of a sequence which does not tend to any limit is given by the sequence $\{1, -1, 1, -1, \ldots\}$, because it just oscillates between two values; for this sequence, $a_n = (-1)^{n+1}$, which is certainly bounded.

Definition 2.1 *A sequence $\{a_n\}$ is said to converge, i.e., have a limit A, iff for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ s.t. for all $n \geq N$ we have $|a_n - A| < \varepsilon$.*

Notation: $\lim_{n\to\infty} a_n = A$, or $a_n \to A$ as $n \to \infty$.

Two Remarks:

(i) It is immediate from the definition that for a sequence $\{a_n\}$ to converge, it is *necessary* that it be bounded. However, it is *not sufficient*, i.e., $\{a_n\}$ could be bounded without being convergent. Indeed, look at the example $a_n = (-1)^{n+1}$ considered above.

(ii) It is only the *tail* of the sequence which matters for convergence. Otherwise put, we can throw away any number of the terms of the sequence occurring at the beginning without upsetting whether or not the later terms bunch up near a limit point.

Lemma 2.2 *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*, then*

1) $\lim_{n\to\infty}(a_n + b_n) = A + B$

2) $\lim_{n\to\infty}(ca_n) = cA$, *for any* $c \in \mathbb{R}$

3) $\lim_{n\to\infty} a_n b_n = AB$

4) $\lim_{n\to\infty} a_n/b_n = A/B$, *if* $B \neq 0$.

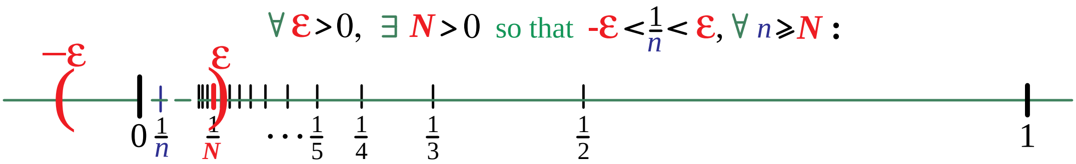
Proof of 1) For any $\varepsilon > 0$, choose N_1 and N_2 so that for $n_1 \geq N_1$, $n_2 \geq N_2$, $|a_{n_1} - A| < \varepsilon/2$, $|b_n - B| < \varepsilon/2$. Then for $n \geq \max\{N_1, N_2\}$, we have

$$|a_n + b_n - A - B| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Hence $A + B$ is the limit of $a_n + b_n$ as $n \to \infty$. □

Proofs of the remaining three assertions are similar. For the last one, note that since $\{b_n\}$ converges to a non-zero number B, eventually all the terms b_n will necessarily be non-zero, as they will be very close to B. So, in the sequence a_n/b_n, we will just throw away some of the initial terms when $b_n = 0$, which doesn't affect the limit as the b_n occurring in the tail will all be non-zero. □

Example: We have $\lim_{n\to\infty} \frac{1}{n} = 0$.



Indeed, given $\varepsilon > 0$, we may choose an $N \in \mathbb{N}$ such that $\exists N > \frac{1}{\varepsilon}$, because $\mathbb{N}$ is unbounded. This implies that for $n \geq N$, we have $|\frac{1}{n} - 0| \leq \frac{1}{N} < \varepsilon$. Hence $\{\frac{1}{n}\}$ converges to 0. □

Definition A sequence is said to be *monotone increasing*, denoted $a_n \nearrow$, if $a_{n+1} \geq a_n$ for all $n \geq 1$, and *monotone decreasing*, denoted $a_n \searrow$, if $a_{n+1} \leq a_n$ for $n \geq 1$. We say $\{a_n\}$ is monotone (or monotonic) if it is of one of these two types.

Theorem 2.3 *A bounded, monotonic sequence converges.*

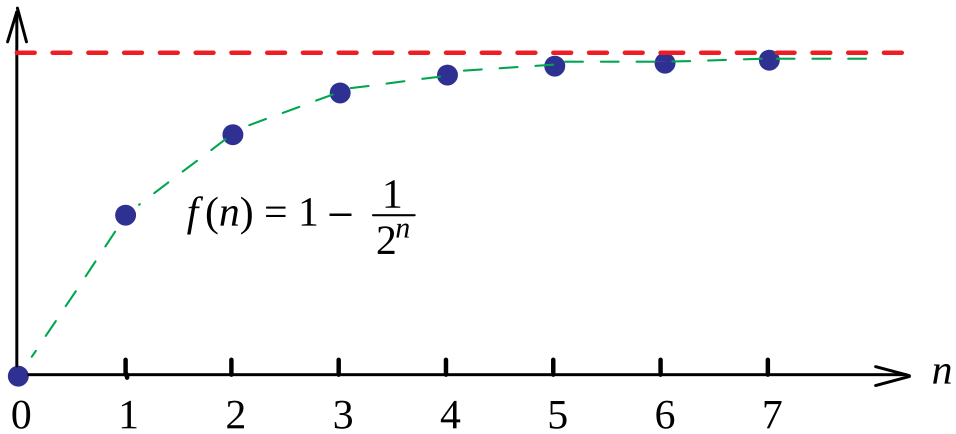
Proof. Assume bounded and $a_n \nearrow$. Let $A = \sup\{a_n\}$. (We write *sup* for *supremum*, which is the same as the least upper bound, and *inf* for *infimum*, which is the greatest lower bound.) For any $\varepsilon > 0$, $A - \varepsilon$ is not an upper bound $\Rightarrow \exists a_N > A - \varepsilon$. But $a_n \geq a_N$ for $n \geq N \Rightarrow -\varepsilon < a_n - A \leq 0$ for all $n \geq N$. Hence $\{a_n\}$ converges with limit A.

If $\{a_n\}$ is (bounded and) monotone decreasing, then look at $\{b_n\}$, with $b_n = -a_n$. Then this new sequence is monotone increasing and has a limit B, which is the sup of $\{b_n\}$. Then $A = -B$ is the inf of $\{a_n\}$, and $a_n \to A$ as $n \to \infty$.

$\square$

Example: $\lim_{n\to\infty} \frac{1}{2^n} = 0$. Indeed, the sequence is bounded and monotone decreasing, hence $\exists A$ such that $\lim_{n\to\infty} 1/2^n = A$. Note that if we multiply the terms of the sequence by 2, then this new sequence also converges to A. On the other hand, by Lemma 1.2, part 2), this limit should be $2A$. Then $2A = A$, implying that $A = 0$. $\square$

A monotone increasing sequence which is bounded can be easily constructed by taking the negative of the sequence above. If we want the values to also remain positive, we can consider the sequence $f(n) = 1 - \frac{1}{2^n}$:



The sequence $g(n) = \frac{1}{2^n}$ having the limit 0 as $n \to \infty$ is a special case of a more general phenomenon:

Lemma 2.4 *Let y be a positive real number. Then $\{y^n\}$ is unbounded if $y > 1$, while*

$$\lim_{n\to\infty} y^n = 0 \quad \text{if} \quad y < 1.$$

Proof of Lemma. Suppose $y > 1$. Write $y = 1 + t$ with $t > 0$. Then by the binomial theorem (which can be proved by induction),

$$y^n = (1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k,$$

which is $\geq 1 + nt$ (as $t > 0$). Since $1 + nt$ is unbounded, i.e, larger than any number for a big enough n, y^n is also unbounded.

Now let $y < 1$. Then y^{-1} is > 1 and hence $\{y^{-n}\}$ is unbounded. This implies that, for any $\epsilon > 0$, y^n is $< \epsilon$ for large enough n. Hence the sequence y^n converges to 0. $\square$

As an exercise, try to extend this Lemma and prove that for any $y \in \mathbb{R}$ with $|y| < 1$, the sequence $\{y^n\}$ converges to 0.

Here is an example. Define a sequence $\{s_n\}$ by putting

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}.$$

It is not hard to see that this sequence is bounded. Try to give a proof. (In fact one can show that it is bounded by 3, but we do not need the best possible bound at this point.) Clearly, $s_{n+1} > s_n$, so the sequence is also monotone increasing. So we may apply Theorem 1.3 above and conclude that it converges to a limit e, say, in $\mathbb{R}$. But it should be remarked that one can show with more work that e is irrational. So there is a valuable lesson to be learned here. Even though $\{a_n\}$ is a bounded, monotone sequence of rational numbers, there is no limit in $\mathbb{Q}$; one has to go to the enlarged number system $\mathbb{R}$.

2.2 The Squeeze Principle

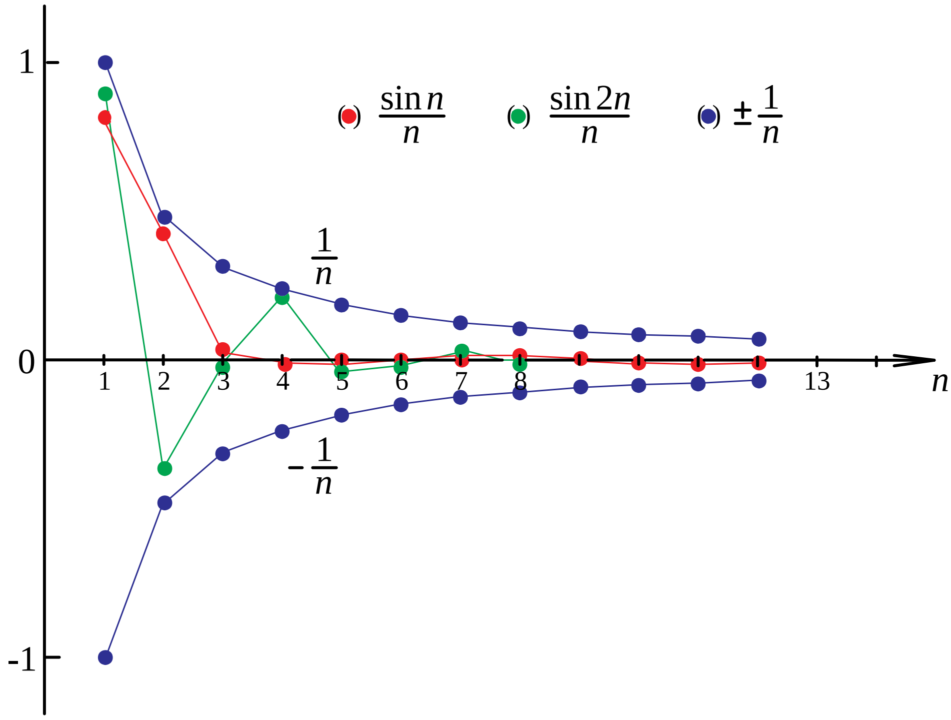
An efficient way to prove the convergence of a sequence is to see if it can be squeezed between two other sequences which converge to the same limit.

Proposition 1 *Suppose $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences of real numbers such that*

$$b_n \leq a_n \leq c_n, \ \forall\, n \geq 1.$$

Suppose moreover that the sequences $\{b_n\}$ and $\{c_n\}$ are convergent with the same limit L, say. Then the sequence $\{a_n\}$ converges as well, with

$$\lim_{n\to\infty} a_n = L.$$



Example: For any fixed real number x, consider the sequence $\{a_n\}$, with $a_n = \frac{\sin(nx)}{n}$. Then, since $\sin(nx)$ takes values between -1 and 1, the hypotheses of the Proposition are satisfied if we take $b_n = -1/n$ and $c_n = 1/n$, since a_n and b_n both converge to zero. We conclude that $\{\frac{\sin(nx)}{n}\}$ also converges to 0, regardless of what x is.

Before giving a proof of Proposition 1, let us note the following useful consequence for sequences $\{a_n\}$ with non-negative terms, by taking $b_n = 0$ for all n.

Corollary 2.5 *Suppose $\{a_n\}$, $\{c_n\}$ are sequences of* **non-negative** *real numbers, with $\{c_n\}$ convergent, such that*

$$a_n \le c_n, \ \forall\, n \ge 1,$$

and

$$\lim_{n\to\infty} c_n \,=\, 0.$$

Then the sequence $\{a_n\}$ converges as well, with

$$\lim_{n\to\infty} a_n \,=\, 0..$$

Proof of Proposition 1. Pick any $\varepsilon > 0$. Then, the convergence of $\{b_n\}$, resp. $\{c_n\}$, implies that we can find some $N_1 > 0$, resp. $N_2 > 0$, such that for all $n \geq N_1$, resp. $n \geq N_2$,

$$|L - b_n| < \varepsilon, \quad \text{resp.} \quad |L - c_n| < \varepsilon.$$

Put $N = \max(N_1, N_2)$. Then for all $n \geq N$, the fact that a_n is squeezed between b_n and c_n implies that

$$|L - a_n| \,<\, \varepsilon.$$

Since ε was arbitrary, this shows the sequence $\{a_n\}$ converges to the same limit L. □

2.3 Cauchy's criterion

The main problem with the definition of convergence of a sequence is that it is hard to verify it. For instance we need to have a candidate for the limit to verify the condition for convergence. One wants a better way to check for convergence. As seen above, boundedness is a necessary, but not sufficient, condition, unless the sequence is also monotone. Is there a necessary and sufficient condition? A nineteenth century French mathematician named Augustin-Louis Cauchy gave an affirmative answer. (His criterion also works for sequences of complex numbers.)

A sequence $\{a_n\}$ is said to be a *Cauchy sequence* iff we can find, for every positive ϵ, an $N > 0$ such that

$$(2.2.1) \qquad |a_m - a_n| \,<\, \epsilon \quad \text{whenever} \quad n, m > N.$$

This is nice because it does not mention any limit.

Lemma 2.6 *Every convergent sequence is Cauchy.*

Proof. Suppose $a_n \to L$. Pick any $\epsilon > 0$. Then by definition, we can find an $N > 0$ such that for all $n > N$, we have

$$|L - a_n| \;<\; \frac{\epsilon}{2}.$$

Then for $n, m > N$, the triangle inequality gives

$$|a_m - a_n| \;\leq\; |a_m - L| + |L - a_n| \;<\frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

□

Here comes the beautiful result of Cauchy, which we just state without proof:

Theorem 2.7 *Every Cauchy sequence of real (resp. complex) numbers converges in $\mathbb{R}$ (resp. $\mathbb{C}$).*

One can use Cauchy sequences to give a construction of $\mathbb{R}$. (A description of this, as well as a proof of Cauchy's theorem will be in the expanded Notes, which interested students could look up.)

2.4 Series: Basic Notions

Definition 2.8 *A series $\sum_{k=1}^{\infty} a_k$ converges if the sequence of partial sums $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots$ converges.*

$$\sum_{k=1}^{\infty} s_k \;=\; \lim_{n\to\infty} (a_1 + \cdots + a_n)$$

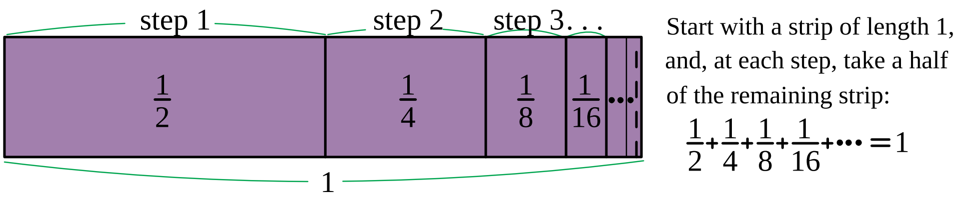
Note that, as seen with sequences, the convergence (or divergence) of a series does not depend on the first finitely many terms; it depends only on the "tail".

Example: (**Geometric Series**) We have

$$\sum_{k=0}^{\infty} x^k = \tfrac{1}{1-x}, \quad \text{for} \quad |x| < 1.$$

Indeed $\sum_{k=0}^{n} x^k \;=\; \frac{1-x^{n+1}}{1-x}$, and $x^{n+1} \to 0$ as $n \to \infty$ by the argument of the previous example.

Note that $\sum_{k=1}^{\infty} x^k \;=\; \frac{x}{1-x}$. Here is a visualization for $x = 1/2$:

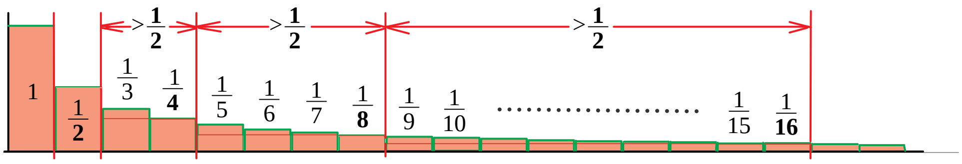


Lemma 2.9 *If $\sum a_n$ and $\sum b_n$ converge then $\sum(\alpha a_n + \beta b_n)$ also converges for any $\alpha, \beta \in \mathbb{R}$.*

This is a consequence of Lemma 2.2 and Definition 2.7.

Example: The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Indeed, note that $\sum_{k=n+1}^{2n} \frac{1}{k} > n \cdot \frac{1}{2n} = \frac{1}{2}$. Thus

$$\sum_{n=1}^{2^S} \frac{1}{n} = \sum_{1}^{2} + \sum_{3}^{4} + \sum_{5}^{8} + \cdots + \sum_{2^{S-1}}^{2^S} > \frac{S}{2} \Rightarrow$$ the sequence of partial sums is unbounded.



For the same reason $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges, since it is the same series without the first term.

What about $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?

Look at the partial sums

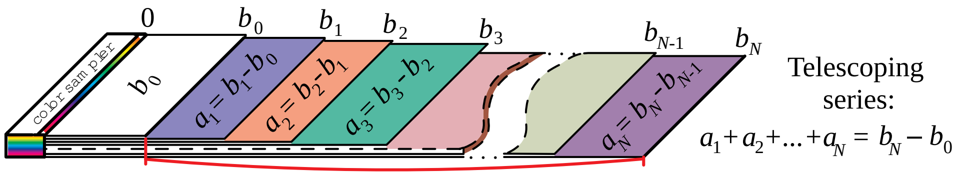
$$s_n = \sum_{n=1}^{n} \tfrac{1}{k(k+1)} = (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) = 1 - \tfrac{1}{n+1}.$$

This is an example of a **telescoping series**

$$\sum_{n=1}^{\infty} a_n \text{ with } a_n = b_n - b_{n+1} \quad n = 1, 2, \ldots,$$

with partial sums $\sum_{n=1}^{N} a_n = b_N - b_1$. If $\exists \lim_{n\to\infty} b_n = B$ then $\sum_{n\geq 1} a_n = B - b_1$.

Of course the series could start from $n = 0$ (when b_0 makes sense).



Question: Is the series $\sum_{n=1}^{\infty}(-1)^n$ convergent?
Hope you can figure this out yourself.

2.5 Tests for convergence of series

Proposition 2 *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Since $s_n = (a_1 + \cdots + a_n)$ goes to a limit L as $n \to \infty$, then the same is true of s_{n-1}, and since $a_n = s_n - s_{n-1}$, we must have $a_n \to L - L = 0$. □

This is a *necessary* condition.

Theorem 2.10 *Assume that* $a_n \geq 0$ *for any* $n \geq 1$. *Then the series* $\sum a_n$ *converges if and only if the sequence of its partial sums is bounded.*

Proof The sequence of partial sums s_n is monotone increasing because $a_n = s_n - s_{n-1}$ is non-negative. We know that convergence implies boundedness. Conversely, if the monotonic sequence $\{s_n\}$ is bounded, it is convergent by Theorem 2.3. □

Example: $\sum_{k=0}^{\infty} 1/k!$ converges. Indeed, $1/k! \leq 1/2^{k-1}$ for $k \geq 1$, and so

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} 1/2^{k-1} \leq 1 + \sum_{k=1}^{\infty} 1/2^{k-1} = 3.$$

This limit is denoted by e; we saw it earlier.

The Comparison test Assume $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq 1$. If there exists a positive constant C such that $a_n \leq cb_n$ for all $n \geq 1$ then convergence of $\sum b_n$ implies convergence of $\sum a_n$.

Proof. $\sum_{k=1}^{n} a_k \leq c \cdot \sum_{k=1}^{n} b_n$. □

Limit comparison test: Assume $a_n > 0$ and $b_n > 0$ for all $n \geq 1$, and suppose that $\lim_{n\to\infty} a_n/b_n = 1$. Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof. $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $1/2 < \frac{a_n}{b_n} < 3/2 \Rightarrow b_n < 2a_n$ and $a_n < \frac{2}{3}b_n$. Since the convergence does not depend on finitely many first terms, the result follows by applying the comparison test. $\square$

Example: $\sum \frac{1}{n^2}$ converges. Indeed, we showed before that $\sum \frac{1}{n(n+1)}$ converges, and $\lim_{n\to\infty} \frac{n(n+1)}{n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})}{1} = 1$. $\square$

Theorem 2.11 *(***Root test***). Let $\sum_{n=1}^{\infty} a_n$ be a series of nonnegative terms such that $a_n^{1/n} \to R$ as $n \to \infty$.*

(a) If $R < 1$, the series converges.

(b) If $R > 1$, the series diverges.

(c) If $R = 1$, the test is inconclusive.

Proof. Assume $R < 1$ and choose x: $R < x < 1$. Then $0 \leq a_n^{1/n} \leq x$ is satisfied for $n \geq N$. Hence, $a_n \leq x^n$, and $\sum a_n < \infty$ by the comparison test.

If $R > 1$, then $a_n > 1$ for infinitely many n, which implies $a_n \not\to 0$.Hence the series diverges. $\square$

For $R = 1$ we have two examples: $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

Example: $\sum n/3^n < \infty$.

Theorem 2.12 *(***Ratio test***) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that*

$$\frac{a_{n+1}}{a_n} \to L \text{ as } n \to \infty.$$

(a) If $L < 1$, the series converges.

(b) If $L > 1$, the series diverges,

(c) If $L = 1$, the test is inconclusive.

Proof. Assume $L < 1$ and choose x: $L < x < 1$. Then $\frac{a_{n+1}}{a_n} < x$ for all $n \geq N$. This implies

$$\frac{a_{n+1}}{x^{n+1}} < \frac{a_n}{x^n} \text{ for all } n \geq N.$$

Thus, the sequence $a_n/x^n \searrow$ for $n \geq N \Rightarrow \frac{a_n}{x^n} \leq$ const. Now the comparison test proves (a).

(b) follows from the fact that a_n is not decreasing.

(c) same 2 examples $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$. □

Example: $\sum \frac{2^n}{n!}$ converges. Indeed, $\frac{2^n}{n!} / \frac{2^{n+1}}{(n+1)!} = \frac{2}{(n+1)!} \to 0$.

2.6 Absolute and Conditional Convergence

Now let us consider series some of whose terms may be negative.

Proposition 3 *$\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

The converse is not true.

Proof. Let us prove that $\sum b_n$ with $b_n = a_n + |a_n|$ converges. We have $b_n = 0$ or $b_n = 2|a_n|$. Hence $b_n \leq 2|a_n|$, and by comparison test $\sum b_n < \infty$. Since $a_n = b_n - |a_n|$, we are done. □

Definition 2.13 *A series $\sum a_n$ is said to be* **absolutely convergent** *if $\sum |a_n|$ converges. If $\sum a_n$ converges and $\sum |a_n|$ diverges then $\sum a_n$ is said to be* **conditionally convergent**.

Theorem 2.14 *(Leibniz) If $\{a_n\}$ is a monotonic decreasing sequence with limit 0, then the* **alternating series** *$\sum(-1)^{n-1}a_n$ converges. If S is its sum and s_n is its nth partial sum then*

$$0 < |S - s_n| < a_{n+1} \text{ for all } n \geq 1.$$

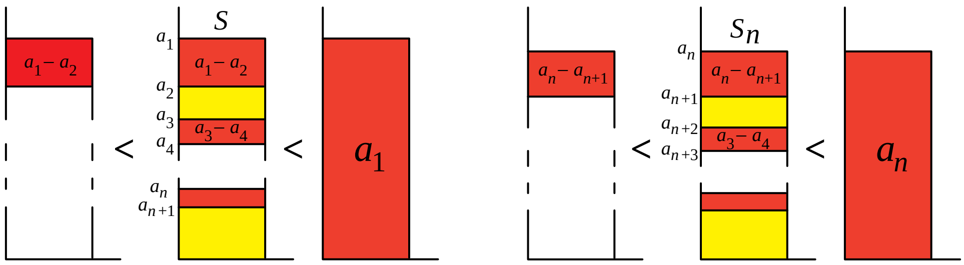
Proof The partial sums s_{2n} is monotone increasing because $s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} > 0$. By a similar argument, $s_{2n-1} \searrow$. Both sequences are bounded from below by s_2 and from above by s_1. Denote $S' = \lim_{n\to\infty} s_{2n}$, $S' = \lim_{n\to\infty} s_{2n-1}$. Then $S' - S'' = \lim(s_{2n} - s_{2n-1}) = \lim(-a_{2n}) = 0 \Rightarrow S' = S''$. Hence, the series $\sum a_n$ converges and its sum is $S = S' = S''$. Further,

$$0 < S - s_{2n} \leq s_{2n+1} - s_{2n} = a_{2n+1},$$

and

$$0 < s_{2n-1} - S \leq s_{2n-1} - s_{2n} = a_{2n}.$$

□



Example: $\sum(-1)^n/n$ is conditionally convergent.

Example: (James Gregory 1672, Gottfried Wilhelm Leibniz 1673)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} + \ldots.$$

This gives a very slowly convergent approximation for π by cutting off the tail at some large enough N.

Leibniz was a founder of **Optimism**: "Our universe is the best possible one God could have made." Some think incorrectly that he was mocked by Voltaire for this philosophy in *Candide*.

2.7 Power Series

Definition 2.15 *A* **power series** *is an infinite series of the form*

$$\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + (z-a)a_1 + (z-a)^2 a_2 + \ldots$$

Proposition 4 *Assume that the power series $\sum a_n z^n$ converges for a particular $z \neq 0$, say for $z = z_1$. Then the series converges absolutely for every z with $|z| < |z_1|$.*

Proof. Since $\sum a_n z_1^n$ converges, $a_n z_1^n \to 0 \Rightarrow |a_n z_1^n| < 1$ for $n \geq N$. If $|z| < |z_1|$ then

$$|a_n z^n| = |a_n z_1^n| \left|\frac{z^n}{z_1^n}\right| \leq \left|\frac{z}{z_1}\right|^n \quad \text{for } n \geq N$$

Since $\sum |\frac{z}{z_1}|^n < \infty$, by the comparison test, $\sum a_n z^n$ converges. □

Theorem 2.16 *Assume that the power series $\sum a_n z^n$ converges for at least one $z \neq 0$, say $z = z_1$, and that it diverges for at least one z, say $z = z_2$. Then there exists a positive real number ρ, called the* **radius of convergence***, such that the series converges absolutely if $|z| < r$ and diverges if $|z| > r$.*

Proof. Denote by A the set of positive z for which $\sum a_n z^n$ converge. We know that $A \neq \emptyset$ and $A \leq |z_2|$. Set $r = \sup A$. If $|z| < r$ then there exists $x \in (|z|, r)$ such that $\sum a_n x^n$ converges $\Rightarrow$ by the previous theorem $\sum a_n z^n$ absolutely converges. $\square$

Example: $S := \sum \frac{z^n}{n}$. Its radius of convergence is 1. Thus S converges whenever $|z| < 1$ and diverges when $|z| > 1$.

It is tricky understand what happens when $|z| = 1$. If $z \in \mathbb{R}$, then we are reduced to consider exactly two *boundary cases*, i.e., when $\rho = 1$, which are (i) the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges, and (ii) the *alternating series* $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, which converges by Leinbniz. So both situations can occur at the radius of convergence. The problem gets even more interesting if we allow z to be a complex number. Then $z = e^{i\theta}$ for some *angle* $\theta \in \mathbb{R}$ (which can be taken to be between 0 and 2π (with 0 included), and by de Moivre's theorem,

$$z^n \;=\; e^{in\theta} \;=\; \cos(n\theta) + i\sin(n\theta).$$

Consequently, the series splits up into a sum of two infinite series:

$$S = \sum_{n=1}^{\infty} \frac{z^n}{n} \;=\; \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}.$$

When $\theta = 0$, the first part, called the *real part* of S, is the harmonic series (which diverges), and the second part, said to be (i times) the *imaginary part* of S, is zero. It turns out that for any θ strictly between 0 and 2π, S converges.

3 Limits of functions, Continuity

After introducing the basic notions on functions, limits and continuity, we will go on to Bolzano's theorem, and the Intermediate Value Theorem (IVT) which follows from it, as well as the Extremal Value Theorem (EVT).

3.1 Functions

Def A function f is a set of ordered pairs (x, y), with $x, y \in \mathbb{R}$, such that no two (ordered pairs) have the same first member.

By definition, the second member y is determined by x, one may unambiguously write y as $f(x)$, where f denotes the assignment $x \mapsto y$. The set of all x (for which f is defined) is called the **domain** of f, and the set of the corresponding y is called the *image* (or *range*) of f.
Notation: $f : X \to Y$, where X is the domain, and Y contains the range.

One can plot the ordered pairs $\{(x, y = f(x))\}$ (defining a function f) in the Cartesian plane $\mathbb{R}^2 = \{(x, y) \,|\, x, y \in \mathbb{R}\}$, and the resulting *figure* is called the **graph** of f. It will be useful to become aware of the graphs of a number of standard functions, such as the ones below.

Examples:

(i) The *identity function*: $f(x) = x$;

(ii) *Constant function*: $f(x) = c$, for all $x \in \mathbb{R}$,
with c a fixed real number;

(iii) *Linear function*: $f = ax + b$, for constants a, b;

(iv) *Polynomial function of degree* $n \geq 0$: $f(x) = \sum_{j=0}^{n} a_j x^j$,
with $a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$;

(v) *Upper semicircle function*: $f(x) = \sqrt{r^2 - x^2}$,
$X = \{x \in \mathbb{R} \,|\, -r \leq x \leq r\}$, r: radius > 0;

(vi) *The integral part function*: $f(x) = [x]$, the largest integer not greater than x, $X = \mathbb{R}$, Image$(f) = \mathbb{Z}$.

We will call a function f *one-to-one*, or *injective*, iff for every y in the range, there is a unique x such that $f(x) = y$. The function $f : X \to Y$ is said to be *onto*, or *surjective*, iff Y is the image of f, i.e., for every y in Y, there is an $x \in X$ such that $y = f(x)$. Note that the linear function example above is injective on $X = \mathbb{R}$ iff $a \neq 0$, in which case it is onto all of $\mathbb{R}$. Sometimes f is not an injective function on its natural domain X, but becomes one when restricted to a (large enough) subset X_1 of X. (It is always injective if restricted to one point, but this is not interesting!) For example, the upper semicircle function is injective on $\{x \,|\, 0 \leq x \leq r\}$; the graph of this restriction is a quarter circle of radius r (in the first quadrant of $\mathbb{R}^2$).

3.2 Open, closed and compact subsets of $\mathbb{R}$

By an **interval**, we will mean a subset I of $\mathbb{R}$ such that if a, b are in I, then any number x between a and b is in I. Examples are, for $a < b \in \mathbb{R}$, the **open interval**

$$(a, b) \;=\; \{x \in \mathbb{R} \,|\, a < x < b\},$$

the **closed interval**

$$[a, b] \;=\; \{x \in \mathbb{R} \,|\, a \leq x \leq b\},$$

the **half-open intervals**

$$[a, b) \;=\; \{x \in \mathbb{R} \,|\, a \leq x < b\}, \quad (a, b] \;=\; \{x \in \mathbb{R} \,|\, a < x \leq b\},$$

the infinite open intervals

$$(a, \infty) \;=\; \{x \in \mathbb{R} \,|\, a < x < \infty\},$$

$$(-\infty, b) \;=\; \{x \in \mathbb{R} \,|\, -\infty < x < b\},$$

and so on.

Definition 3.1 *Any open interval containing a point a as its midpoint will be called a* **standard neighborhood** *of a.*

We will usually drop the adjective "standard."
Notation: $N(a, r) = \{x \in \mathbb{R} |\; |x - a| < r\} \;=\; (a - r, a + r)$.

By an **open set** in $\mathbb{R}$, we will mean a subset U which contains a neighborhood of every point in U. One can check that open intervals are open, as are arbitrary unions of open intervals and finite intersections of them. The empty set $\emptyset$ and $\mathbb{R}$ are open, too.

The **complement** of a set X in $\mathbb{R}$, denoted X^c, is the set $\mathbb{R} \setminus X := \{z \in \mathbb{R} \,|\, z \notin X\}$. Clearly, $\mathbb{R}$ and $\emptyset$ are complements of each other, while the complement of an open interval (a, b) is the union $(-\infty, a] \cup [b, \infty)$.

By a **closed set** in $\mathbb{R}$, we will mean a subset F whose complement F^c is open. Try to verify that closed intervals are closed, as are finite unions of closed intervals, and arbitrary intersections of them. The empty set and $\mathbb{R}$ are closed, too. Prove that these two sets are the only subsets of $\mathbb{R}$ which are both open and closed.

By a **compact set** in $\mathbb{R}$, we will mean a subset C of $\mathbb{R}$ which is both closed and bounded. (Again, when we say "bounded," we mean it is bounded from above and below.) The closed interval $[a, b]$, with $a < b$ in $\mathbb{R}$, is evidently compact. The complement of an open interval (a, b) is closed, but not bounded, so non-compact. And (a, b) itself is bounded but not closed, and so not compact. Compact sets play a very important role in Calculus of one and several variables, as well as in (higher) Mathematical Analysis and Geometry.

3.3 Limits

Let $a \in \mathbb{R}$. Assume that f is a function defined on some neighborhood of a except possibly at a.

Definition 3.2 *f has limit A as $x \to a$ iff for every neighborhood $N_1(A)$ there exists a neighborhood $N_2(a)$ such that $f(x) \in N_1(A)$ if $x \in N_2(a) - \{a\}$. Equivalently, $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \neq a$, $|x - a| < \delta$, we have $|f(x) - A| < \varepsilon$.*

Notation: $\lim_{x \to a} f(x) = A$, or $f(x) \to A$ as $x \to a$.

Theorem 3.3 *The following statements are equivalent:*

(i) $\lim_{x \to a} f(x) = A$

(ii) *For every sequence $\{a_n\}_{n=1}^{\infty} \subset$ Domain (f), $a_n \neq a$, such that* $\lim_{n \to \infty} a_n = a$, *we have* $\lim_{n \to \infty} f(a_n) = A$.

Proof (i) $\Rightarrow$ (ii): Given $\varepsilon > 0$, pick $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - A| < \varepsilon$, which is possible by (i). Let $\{a_n\}$ be a sequence with limit a, so that we may choose an $N > 0$ such that $a_n \in N(a, \delta)$ for $n \geq N$. Then $f(a_n) \in N(A, \varepsilon)$ for $n \geq N$, and so (i) holds.

(ii) $\Rightarrow$ (i): Let us prove this by contra-positive, i.e., assume $\neg(i)$ (= Not (i)) and deduce $\neg(ii)$. Pick ε for which δ does not exist, i.e., for every $\delta > 0$, $\exists x = x(\delta)$ such that $|x - a| < \delta$, but $|f(x) - A| \geq \varepsilon$. Then picking $\delta = \frac{1}{n}$ we can construct a sequence $a_n = x(\delta_n)$ in $N(a, \frac{1}{n})$ s. t. $|f(a_n) - A| \geq \varepsilon$. Then $a_n \to a$ but $f(a_n) \not\to A$. So (ii) does not hold when (i) doesn't. □

Right and Left limits: $\lim_{x\to a^+}$, resp. $\lim_{x\to a^-}$.

These are defined just as above except for requiring that $f(x) \in N_1(A)$ only for all $x \in N_2(a) \cap \{x > a\}$, resp. $x \in N_2(a) \cap \{x < a\}$. This is equivalent to taking (all possible) sequences $a_n > a$, resp. $a_n < a$. Clearly, the limit as $x \to a$ exists iff the right and left limits both exist and are equal.

Examples

(1) The limit of a constant function is the same constant.

(2) Limit of the identity function $\lim_{x\to a} x = a$. (In the proof, take $\delta = \varepsilon$.)

(3) $\lim_{x\to k^-}[x] = k - 1$, $\lim_{x\to k^+}[x] = k$, so $\lim_{x\to k}[x]$ does not exist.

Theorem 3.4 *Let* $A = \lim_{x\to a} f(x)$ *and* $B = \lim_{x\to a} g(x)$. *Then*

$$\lim_{x\to a} \begin{pmatrix} f(x) + g(x) \\ f(x)g(x) \\ f(x)/g(x) \end{pmatrix} = \begin{pmatrix} A + B \\ AB \\ A/B \quad if \quad B \neq 0 \end{pmatrix}$$

Proof: Follows from the corresponding statement for sequences.

Theorem 3.5 *(***The squeeze principle***) If* $f(x) \leq g(x) \leq h(x)$ *in some neighborhood of a (not including a), and if* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = A$, *then* $\lim_{x\to a} g(x) = A$.

Proof Follows from the corresponding statement for sequences. □

Lemma 3.6 *(a) We have*

$$\lim_{x \to 0} x \sin(1/x) \;=\; 0.$$

(b)

$$\lim_{x \to 0} \frac{\sin x}{x} \;=\; 1.$$

Proof: (a) To prove this, note that for any x, $-1 \leq \sin(1/x) \leq 1$, implying that $x \sin(1/x)$ is squeezed between $-x$ and x, both of which tend to 0 as $x \to 0$. Thus, by the squeeze principle above, $x \sin(1/x)$ tends to 0 as $x \to 0$.

(b) Typically, people tend to use a circular argument involving L'Hôpital's rule. Here is a (correct) geometric argument: Consider the unit circle C with center at the origin O. By definition, $\sin x$ is the y-coordinate of the point $P = P(x)$ on C at an angle x from the x-axis in the counterclockwise direction. (Note that $\sin x$ is periodic with period 2π, so we may take $x \in [0, 2\pi)$.) We are interested in x near zero. Suppose $x > 0$. Let Q denote the point on the x-axis cut by the vertical drawn downwards from P, so that the line segment OQ, resp. PQ, has length $\cos x$, resp. $\sin x$. Let B be the point $(1, 0)$, and A the point where the vertical drawn upwards from B meets the extension of the line OP, so that OP and OB have length 1. Evidently, the triangles OPQ and OAB are similar, implying that

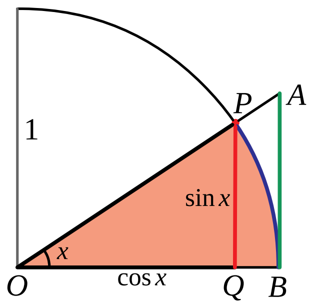
$$\frac{|AB|}{\sin x} \;=\; \frac{1}{\cos x}, \;\; i.e., \;\; |AB| = \frac{\sin x}{\cos x}.$$

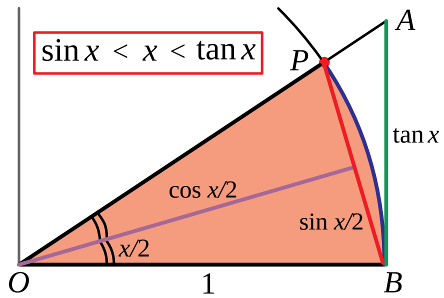
Now look at the angular sector S bounded between OP and OB, which contains the triangle OPQ and is contained in the triangle OAB. Hence the area of S, which is $x/2$, is squeezed between the areas of the triangular regions OPQ and OAB. We get

$$\frac{1}{2} \sin x \cos x \;<\; \frac{1}{2} x \;<\; \frac{\sin x}{2 \cos x}.$$

Dividing throughout by $\sin x/2$, which is positive as x is in the first quadrant (by virtue of being positive and near 0), we obtain

$$\cos x \;<\; \frac{x}{\sin x} \;<\; \frac{1}{\cos x}.$$





Since both $\cos x$ and its inverse go to 1 as $x \to 0$, we see that by the squeeze lemma, $\frac{x}{\sin x}$ goes to 1 as x goes to 0^+. So the right limit exists and equals 1. The left limit also follows the same argument (but with $x < 0$), which will be left as an exercise. $\square$

We end this section by looking also at limits of functions as $x \to \infty$ and $x \to -\infty$. The definition is analogous to the one for sequences. For instance, we say that $f(x)$ has a finite limit $A \in \mathbb{R}$ as $x \to \infty$ iff for every $\varepsilon > 0$, we can find a $T > 0$ such that whenever $x > T$, $|f(x) - A| < \varepsilon$. To compute such limits (for x large), it is useful to look for dominant terms. For example, if $f(x)$ is a polynomial function $a_0 + a_1x + \cdots + a_nx^n$, with $a_n \neq 0$, then the highest order term, namely a_nx^n, dominates. To give an example, consider the limit (of a rational function)

$$L := \lim_{x\to\infty} \frac{2x^2 - 35x + 16}{5x^2 + 12x + 1}.$$

We may rewrite the quotient inside the limit as

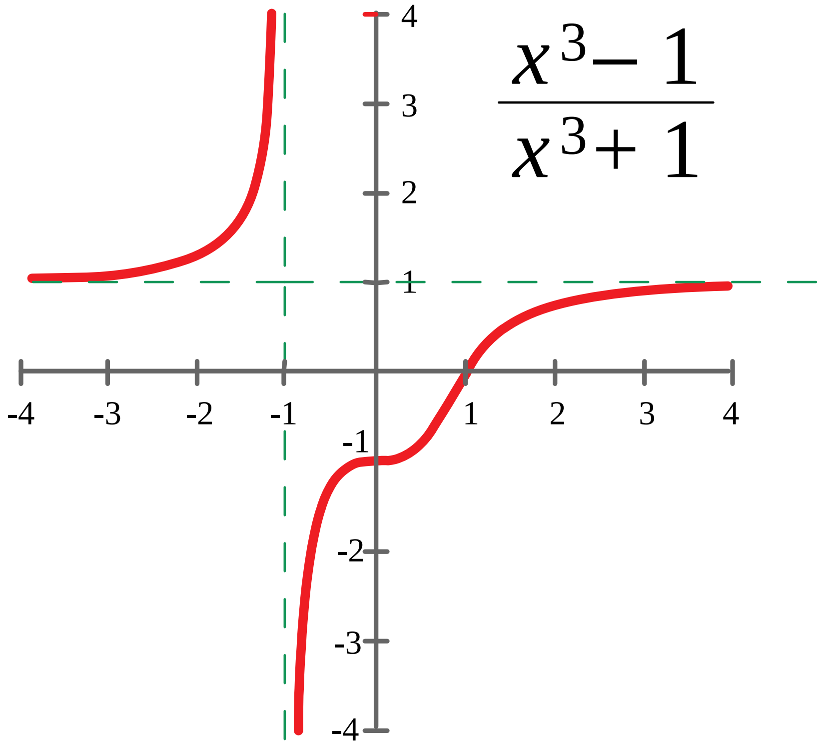
$$\frac{x^2(2 - 35/x + 16/x^2)}{x^2(5 + 12/x + 1/x^2)} = \frac{2 - 35/x + 16/x^2}{5 + 12/x + 1/x^2}.$$

Since $5/x$, $16/x^2$, $12/x$, and $1/x^2$ all go to zero as $x \to \infty$, and since $5 + 12/x + 1/x^2$ approaches $5 \neq 0$ as $x \to \infty$, we see, by applying Theorem 3.4 that

$$L = \frac{2}{5}.$$

Put another way, we may substitute $u = 1/x$, and the limit we want amounts to finding the limit of the quotient, now viewed as a function of u, as u goes to 0.

Similarly, if we look at the rational function $f(x) = \frac{x^3-1}{x^3+1}$, we see that it has the limit 1 as $x \to \infty$; same limit when $x \to -\infty$. However, this function is not defined at $x = -1$, and it goes to ∞ if we approach -1 from the left and goes to $-\infty$ if we approach it from the right, as seen in the graph:



This method applies to more than rational functions. For example, we may modify the example above to consider

$$L' := \lim_{x\to\infty} \frac{2x^2 - 35x\cos x + 16}{5x^2 + 12x + 1}.$$

Then the quotient inside the limit can be simplified (as above) to obtain

$$\frac{2 - 35\cos x/x + 16/x^2}{5 + 12/x + 1/x^2}.$$

To finish, note that (by the squeeze principle) the limit

$$\lim_{x\to\infty} \frac{\cos x}{x}$$

must be 0 as $1/x$ and $-1/x$ both goes to ∞ while $\cos x$ remains squeezed between -1 and 1. Consequently,

$$L' = \frac{2}{5} (= L).$$

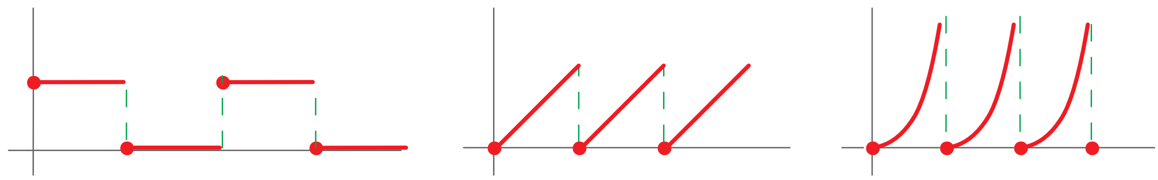
3.4 Continuity

Definition 3.7 *A function f is* **continuous** *at a point a if f is defined at a and $\lim_{x \to a} f(x) = f(a)$.*

Equivalently, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall x, |x - a| < \delta$ we have $|f(x) - f(a)| < \varepsilon$. In the examples above, (1) and (2) are continuous everywhere, while $[x]$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$. At the integral points we have **jump discontinuities**-when both left and right limits exist but not equal.

Removable discontinuity-left and right limits are the same but not equal to $f(a)$.

Here is a picture of some standard examples of discontinuous functions, where the discontinuity is not removable: In the last graph (on the right),



the function is even unbounded.
Exercise: Can you find a function f which is discontinuous at every point of $\mathbb{R}$, or just at every point of Q?

Lemma 3.8 *Polynomials are continuous functions, rational functions are continuous wherever defined.*

This follows from Theorem 3.4 and the definition of continuity.

Trigonometric functions such as $\sin x$, $\cos x$, and $\tan x$ *are continuous* where they are defined. (The proof of continuity of $\sin x$ was discussed in class.)

Proposition 1 *(composition) Assume f is continuous at a and g is continuous at $b = f(a)$. Then the composite function $g \circ f : x \mapsto g(f(x))$ is continuous at a.*

Proof. For any $N_1(g(b))$, there exist $N_2(b)$: $g(N_2(b)) \subset N_1(g(b))$ and $N_3(a)$: $f(N_3(a)) \subset N_2(b)$. Thus $g \circ f(N_3(a)) \subset N_1(g(q))$. $\square$

Example $f(x) = (x^3 + 1)^4$ is continuous everywhere.

The following Proposition follows from Theorem 3.3 (and the definition of continuity).

Proposition 2 *Suppose f is continuous at a, and that there is a sequence $\{x_n\}$ converging to a. Then the sequence $\{f(x_n)\}$ converges to $f(a)$.*

This result provides a useful way to prove that certain functions are discontinuous by considering its contra-positive. In other words, if $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} f(x_n) \neq f(a)$, then f cannot be continuous at a. Try to use this idea to prove that the function $f(x) = \sin(1/x)$ (for $x \neq 0$) cannot be extended in any way to be continuous at 0.

If f is a function on a set $X \subset \mathbb{R}$, the subset X_{dc} of X consisting of points a in X where f is discontinuous will be called the **set of discontinuities** of f.

We will denote by $\mathcal{C}(X)$ the collection of all continuous functions φ on X.

3.5 Bolzano and IVT

Bolzano's theorem: *Let f be continuous at each point of the closed interval, and assume that $f(a)$ and $f(b)$ have opposite signs. Then there is at least one c in the open interval (a, b) such that $f(c) = 0$.*

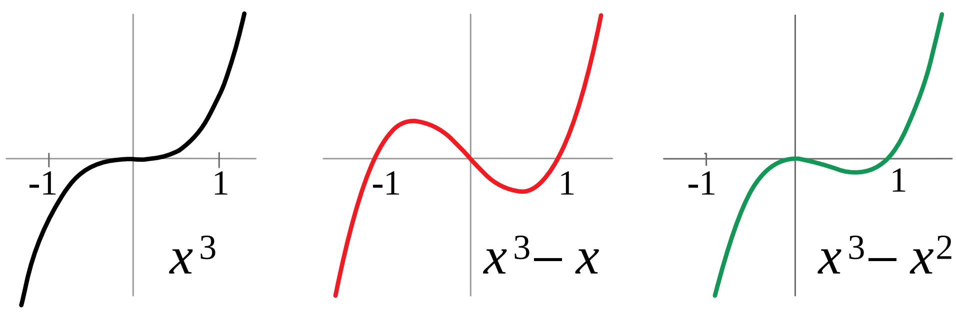
To prove this, we need the following

Lemma 3.9 *Let f be continuous at c, and suppose that $f(c) \neq 0$. Then there is an interval $(c - \delta), c + \delta)$ around c where f has the same sign as $f(c)$.*

Proof of Lemma. Pick $\varepsilon = |f(c)|/2$ in the definition of continuity to see that, for suitable $\delta > 0$, $|f(x) - f(c)| < |f(c)|/2$ whenever $|x - c| < \delta$. Consequently, for all $x \in (c - \delta, c + \delta)$, $f(x)$ has the same sign as $f(c)$. $\square$

Proof of Bolzano's theorem Assume $f(a) < 0$ and $f(b) > 0$. We will find the largest c for which $f(c) = 0$. Let B denote the set of all those points x in the interval $[a, b]$ for which $f(x) \leq 0$. It is nonempty because $f(a) < 0$, and it is evidently bounded. Let $c = \sup B$. We'll show $f(c) = 0$. If $f(c) > 0$ then it is positive in a neighborhood of c by the Lemma above, implying that some $c' < c$ is an upper bound for B, which contradicts the fact that c is the least upper bound. So $f(c) > 0$ is impossible. Similarly, if $f(c) < 0$, then f is negative in a neighborhood of c, implying that $\exists\, c'' > c$ which belongs to B. Then c is not an upper bound for B, giving us a contradiction, once again. Hence $f(c) = 0$. $\square$

A consequence of Bolzano's theorem is the fact that any odd degree polynomial $f(x)$ admites a real root, i.e., a real number a such that $f(a) = 0$. (Of course this is false for f of even degree, in fact already for the quadratic polynomial $x^2 + t$ if $t > 0$.) Indeed, if f has odd degree, then its sign is opposite for x large positive and x large negative; check this! In particular, if $f(x)$ is cubic, it has a root; in fact the number of roots is between 1 and 3:



Theorem 3.10 *(Intermediate Value Theorem or* **IVT***) Let $f \in \mathcal{C}([a, b])$. Choose two arbitrary points $x_1 < x_2$ in $[a, b]$ with $f(x_1) \neq f(x_2)$. Then f takes every value between $f(x_1)$ and $f(x_2)$ somewhere in the interval (x_1, x_2).*

Proof We may assume that $f(x_1) < f(x_2)$, as otherwise we may consider $-f$. If $t \in (f(x_1), f(x_2))$, consider the function $g(x) = f(x) - t$. Then $g(x_1) < 0$ and $g(x_2) > 0$, and we may apply Bolzano's theorem to $g(x)$ and find a point c in the interval such that $g(c) = 0$, which implies, by the definition of g, that $f(x) = t$. $\square$

Theorem 3.11 *If n is a positive integer and if $a > 0$, then there exists exactly one positive b, denoted $\sqrt[n]{a}$, such that $b^n = a$.*

Proof Choose $c > 1\colon 0 < a < c$, and consider $f(x) = x^n$ on $[0, c]$. Since $0 < a < c < c^n = f(c)$, by intermediate value thus $\exists\, b \in (0, c)$ such that $f(b) = a$. It is unique because f is strictly increasing. $\square$

Consider a function f with domain A and range R. For each x in A there is exactly one $y = f(x)$. Assume that for each $y \in B$ there exists only one x such that $y = f(x)$. Then we can construct a new function $g\colon B \to A$ such that $g(y) = x$ means $f(x) = y$. The new function g is called the **inverse function**, or just **inverse** of f. (It is important to note that g is *not* the multiplicative inverse $1/f$.)

Theorem 3.12 *Assume f is strictly increasing and continuous on $[a, b]$. Let $c = f(a)$, $d = f(b)$, and let g be the inverse of f. Then*

(a) g is strictly increasing on $[c, d]$.

(b) g is continuous on $[c, d]$.

Proof. (a) Since f is strictly increasing, we have

$$(*) \qquad x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2),$$

for all x_1, x_2 in $[a, b]$. If we put $y_j = f(x_j) \in [c, d]$, then by definition, $x_j = g(y_j)$, and hence $(*)$ is the same as the equivalence

$$(**) \qquad g(y_1) < g(y_2) \Leftrightarrow y_1 < y_2.$$

Since $c = f(a)$ and $d = f(c)$, every point in $[c, d]$ is in $f([a, b])$ by IVT, $(**)$ holds for all $y_1 < y_2$ in $[c, d]$. Thus g is strictly increasing on $[c, d]$.

(b) Choose a point $y_0 \in (c, d)$. We need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$g(y_0) - \varepsilon < g(y) < g(y_0) + \varepsilon \text{ whenever } y_0 - \delta < y < y_0 + \delta.$$

Set $x_0 = g(y_0)$ so that $f(x_0) = y_0$. Set

$$\delta = \min\{f(x_0) - f(x_0 - \varepsilon),\ f(x_0 + \varepsilon) - f(x_0)\}.$$

It works! $\square$

Corollary 3.13 *Let $n \in \mathbb{N}$. Then the n-th root function $f(x) = \sqrt[n]{x}$ is continuous on $\mathbb{R}_+$, and, if n is odd, it is also (defined and) continuous on $\mathbb{R}$.*

3.6 Max/Min and EVT

Definition 3.14 *The function f is said to have an* **absolute maximum** *(resp.* **absolute minimum***) on the set S if there is at least one point $c \in S$ such that $f(x) \le f(c)$ (resp. $f(x) \ge f(c)$), for all $x \in S$.*

If there is no confusion, we will abbreviate and just say **maximum** and **minimum**. (Later, we will introduce local maximum and local minimum.)

Example $f(x) = 1/x$ has no absolute maximum or minimum on $(0, 1)$.

By an **extremum** we will mean either a maximum or a minimum.

Theorem 3.15 *Let $f \in \mathcal{C}([a, b])$. Then f is bounded on $[a, b]$.*

Proof Assume f is unbounded. Let c be the midpoint of $[a, b]$. Then f is unbounded either on $[a, c]$ or $[c, b]$. Denote the corresponding interval by $[a_1, b_1]$. Continuing the procedure, we get, for any $n \ge 1$, a nested sequence of subintervals

$$[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset \cdots \subset [a_1, b_1] \subset [a, b], \quad \text{with} \quad b_n - a_n = \tfrac{b-a}{2^n}.$$

Let A be the sequence $\{a_n\}$ of real number, which is bounded since each $a_n \in [a, b]$. Put $\alpha = \sup A \in [a, b]$. By the continuity of f at α, there is an open interval $I = (\alpha - \delta, \alpha + \delta)$ such that $|f(x) - f(\alpha)| < 1 \Rightarrow |f(x)| < 1 + |f(\alpha)|$, which means f is bounded on this interval I. But $[a_n, b_n]$ is contained in $(\alpha - \delta, \alpha + \delta)$ for n large enough. To be precise, this happens when $a_n \in [\alpha - \delta, \alpha]$ and $(b - a)/2^n < \delta$. Contradiction, because f was unbounded on $[a_n, b_n]$ by construction! So f must be bounded on $[a, b]$. $\square$

Theorem 3.16 *(Extremal Value Theorem for continuous functions or* **EVT***) Assume f is continuous on a closed interval $[a, b]$. Then there exists points c and d in $[a, b]$ such that*

$$f(c) = \sup f = \max f, \quad \text{and} \quad f(d) = \inf f = \min f.$$

Proof It suffices to prove the first one, since then we may replace f by $-f$, reversing the roles of sup and inf (and max and min). Set $M = \sup f$ and $g(x) = M - f(x)$. Suppose M is not attained at any point. Then $g(x) > 0$ on $[a, b]$. Hence, $1/g(x)$ is well defined and continuous on $[a, b]$, which implies, by Theorem 3.15, that it is bounded on $[a, b]$. Consequently, for some $C > 0$, $\frac{1}{M-f(x)} < C$, or equivalently, $f(x) < M - \frac{1}{C}$, which contradicts the fact that $M = \sup f$. Hence M must be realized at some point, call it c, in $[a, b]$. Then, evidently, f attain its maximum on $[a, b]$ at c. □

Here is a very powerful general result, which we will note without proof:

Theorem 3.17 *If f is a continuous function on a compact set C in $\mathbb{R}$, then the image $f(C)$ is also compact.*

Recall that a subset of $\mathbb{R}$ is compact iff it is closed and bounded. In particular, as noted earlier, any closed interval $[a, b]$, with $a < b$ in $\mathbb{R}$, is compact. Thus Theorem 3.17 asserts that $f([a, b])$ is bounded (which is Theorem 3.15) and is closed (which implies Theorem 3.16).

4 Differential Calculus

4.1 Basic Notions

Definition 4.1 *The function f defined on a neighborhood of $a \in \mathbb{R}$ is called* **differentiable** *at a if the limit*

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists. It is called the **derivative** *of f at a and is denoted as $f'(a)$ (or as $\frac{df}{dx}(a)$.)*

Recall that a neighborhood of a point a contains some open interval $I = (a-t, a+t)$, and the limit above makes sense because for h sufficiently close to 0, $a+h$ lies in I. Note that h can be positive and negative. If one restricts to $h > 0$, then the corresponding limit is called the *right derivative* of f at a, and similarly for the *left derivative* (where $h < 0$). The derivative exits at a iff both the right and left derivatives exist *and* are equal.

Examples:

1) Derivative of a constant function exists at any a and equals 0.

2) $f(x) = mx + b \;\Rightarrow\; f'(a) = m$ for any $a \in \mathbb{R}$.

3) $f(x) = x^n$. Recall the *binomial formula*:

$$(a+h)^n \;=\; \sum_{j=0}^{n} \binom{n}{j} a^{n-j}h^j \;=\; a^n + na^{n-1}h + O(h^2),$$

where $O(h^2)$ denotes the sum of terms of order at least h^2. Hence

$$f'(a) = \lim_{h\to 0} \frac{(a+h)^n - a^n}{h} = \lim_{h\to 0} \{na^{n-1} + O(h)\} \;=\; na^{n-1}.$$

4) $f(x) = \sin x$. Since $\sin y - \sin x = 2\sin\frac{y-x}{2}\cos\frac{y+x}{2}$, we have, for any $a \in \mathbb{R}$,

$$f'(a) = \lim_{h\to 0} \frac{\sin(a+h)-\sin(a)}{h} = \lim_{h\to 0} \frac{\sin(h/2)}{h/2}\cos\left(a+\frac{h}{2}\right) \;=\; \cos a.$$

Similarly, $(\cos x)' = -\sin x$.

5) $f(x) = x^{\frac{1}{n}}$, for $n \in \mathbb{N}$. Set $u = (a+h)^{\frac{1}{n}}$ and $v = a^{\frac{1}{n}}$. Then

$$\frac{f(a+h) - f(a)}{h} = \frac{u - v}{u^n - v^n} = \frac{1}{u^{n-1} + u^{n-2}v + \cdots + v^{n-1}},$$

which goes to $\frac{1}{n}a^{\frac{1}{n}-1}$ as $h \to 0$.

We say that f is **differentiable on an open interval** I iff f is differentiable at every point in I. In this case, we may treat f' as a function on I. In general f' may not be continuous, and if it is so at a (resp. on I), we will say that f is $\mathcal{C}^1$, meaning it is **continuously differentiable** at a (resp. on I).

We will say that f is **twice differentiable** at a iff f and f' are both differentiable at a. We put $f'' = (f')'$ and call it the **second derivative**. This way we can define the third and fourth derivatives, and in fact, for any $n \in \mathbb{N}$, the n**-th derivative** of f, denoted by $f^{(n)}$, as the derivative of $f^{(n-1)}$.

f is said to be $\mathcal{C}^n$ at a (or on I) iff all the derivatives $f^{(j)}$ exist for $j \leq n$ and are continuous at a (resp. on I). Finally, one says that f is $\mathcal{C}^\infty$, or **infinitely differentiable**, iff $f^{(n)}$ exists for every n. The simplest examples of infinitely differentiable functions on all of $\mathbb{R}$ are polynomials and the sine and cosine functions. The same holds for rational functions and other trigonometric functions at the set of points where they are defined. For example, $f(x) = \frac{x^2+x-3}{x-1}$ is defined at every point $x \neq 1$ in $\mathbb{R}$, and it is $\mathcal{C}^\infty$ there, while $\tan x$ is defined and $\mathcal{C}^\infty$ at every point x where $\cos x$ is not 0, i.e., when x is not of the form $(2k+1)\pi/2$, for $k \in \mathbb{Z}$.

4.2 Geometric Interpretation of the derivative

Let f be any function defined around a. Given any h close to 0, one may consider the line, called the **secant line**, joining the points $(a, f(a))$ and $(a+h, f(a+h))$, whose slope m_h is given by

$$m_h = \frac{f(a+h) - f(a)}{h},$$

and as we let h approach zero, either from the right or from the left, the two points coalesce in the limit, and the existence of the derivative guarantees a uniquely defined line T passing through $(a, f(a))$, which touches the graph at this point without cutting across it in a small neighborhood. This line, called the **tangent line** to the graph passing through $(a, f(a))$, has slope m given by

$$m = \lim_{h \to 0} m_h.$$

Consequently, existence of the derivative at a is equivalent to the existence of a well defined tangent line T at $(a, f(a))$, and its slope m is none other than the derivative $f'(a)$.

We can in fact write down the **equation of the tangent line** T as

$$y - f(a) = f'(a)(x - a).$$

You may easily check that this is correct, as the equation is forced by the two facts that (i) the slope is $f'(a)$, and (ii) T passes through $(a, f(a))$. Of course this equation could be rearranged as

$$y = f'(a)x + (f(a) - f'(a)a).$$

The **normal** N to the graph of $y = f(x)$ at $P = (a, f(a))$ is the line perpendicular to T and passing through P. When $f'(a) = 0$, T is horizontal, and the equation of N is given by $x = a$. And when $f'(a) \neq 0$, N is not vertical and its the slope is given by $-1/f'(a)$. (So what is the equation of N?)

4.3 Basic Properties of the Derivative

Theorem 4.2 *If f is differentiable at a, then f is continuous at a.*

Proof $f(a + h) = f(a) + h \cdot (\frac{f(a+h)-f(a)}{h})$, and the existence $f'(a)$ implies that $\lim_{h \to 0} f(a + h) = f(a)$, proving the continuity of f at a. □

Remark: Note that $f(x) = |x|$ is continuous at 0 but not differentiable there; the right derivative is 1 (which is the slope of the line $y = x$), and the left derivative is -1 (=slope of the line $y = -x$). Clearly, the function is differentiable, even $\mathcal{C}^\infty$, at every $a \neq 0$.

Theorem 4.3 *Suppose f, g are differentiable at a. Then*

1) $(f + g)'(a) = f'(a) + g'(a)$

2) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

3) $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$, *if* $g(a) \neq 0$.

Proof 1) is immediate from the definition. Let us prove 2). We have

$$\frac{f(a+h)g(a+h) - f(a)g(a)}{h} = g(a)\frac{f(a+h) - f(a)}{h} + f(a+h)\frac{g(a+h) - g(a)}{h},$$

from which the assertion follows by letting $h \to 0$. □

3) Thanks to 2), it is enough to prove $(1/g)' = -g'/g^2$.

$$\frac{1/g(a+h) - 1/g(a)}{h} = -\frac{g(a+h) - g(a)}{h} \cdot \frac{1}{g(a)g(a+h)} \to -\frac{g'(a)}{g(a)^2}.$$

□

Example: $f(x) = x^r$, $r \in \mathbb{Q}$. Then, at any x in $\mathbb{R}$, $f'(x) = rx^{r-1}$. For $r = \frac{1}{n}$, this has been proved. Then extend to $\frac{m}{n}$ by induction on m. For $\mathbb{Q}_-$, the assertion follows from $\mathbb{Q}_+$.

Theorem 4.4 *(Chain rule) Let $\varphi = g \circ f$ ("the composite function"), with f differentiable at a and g differentiable at $b := f(a)$. Then φ is differentiable at a, and $\varphi'(a) = g'(b) \cdot f'(a)$.*

Proof Set $k = f(a+h) - f(a)$. Then $k \to 0$ as $h \to 0$, because f is continuous at a. Also, since $b = f(a)$, we have

$$f(a+h) = b + k, \quad \text{and} \quad \varphi(a+h) = g(b+k).$$

We are using here the fact: $\varphi(x) = g(f(x))$. Thus

$$\frac{\varphi(a+h) - \varphi(a)}{h} = \frac{g(b+k) - g(b)}{h}.$$

We may rewrite the right hand side, using the definition of k, as

$$\frac{g(b+k) - g(b)}{k} \cdot \frac{f(a+h) - f(a)}{h}$$

Here is a reasonable looking, but false, proof of the Chain rule. Since k goes to 0 when h does, we can try to argue that

$$\lim_{h\to 0} \frac{\varphi(a+h) - \varphi(a)}{h} = \left(\lim_{k\to 0} \frac{g(b+k) - g(b)}{k}\right) \cdot \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right),$$

which equals $g'(b)f'(a)$, as desired. The problem is that it may happen that $k = 0$ for infinitely many h near zero, preventing us from taking the limit of the first quotient as $h \to 0$ since then we would be dividing by zero. For example, look at the function $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, which is differentiable (see below) at $a = 0$, but $f(h)$ becomes zero infinitely often as h gets close to 0.

Here is a way we can overcome this problem by modifying the argument as follows. Put

$$G(t) = \frac{g(b+t) - g(b)}{t} - g'(b), \quad \text{if} \;\; t \neq 0$$

Then

$$g(b+t) - g(b) \; = \; t(G(t) + g'(b)), \;\; \forall\, t \neq 0.$$

Now extend G to all of $\mathbb{R}$ by setting $G(0) = 0$. Then, since g is differentiable at a, G is continuous at 0. We then have (by setting $t = k$),

$$\frac{\varphi(a+h) - \varphi(a)}{h} \; = \; \frac{k}{h}(G(k) + g'(b)).$$

(This way we are not dividing by k, and so we can take the limit we want.) When $h \to 0$, we have $\frac{k}{h} = \frac{f(a+h)-f(a)}{h} \to f'(a)$. Also, and $G(k) \to 0$, since $k = k(h) \to 0$ and G is continuous at 0. Hence, the right hand side (of the last displayed equation) tends to $g'(b)f'(a)$. □

Examples:

1) $(\cos(x^3))' = -3x^2 \sin(x^3)$.

2) $((g(x)^n)'g(x)^{n-1} \cdot g'(x)$.

3) $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y} = \mp \frac{x}{\sqrt{r^2-x^2}}$

   Hence, $(\pm\sqrt{r^2 - x^2})' = \mp \frac{x}{\sqrt{r^2-x^2}}$

4.4 An interesting class of Examples

Let us consider, for each integer $m \geq 0$, the function f_m on $D := \mathbb{R} \setminus \{0\}$ defined by

$$f_m(x) \; = \; x^m \sin(1/x),$$

which, by applying the product and quotient rules, is seen to be differentiable, even $\mathcal{C}^\infty$ (by using induction in addition), on D. It is natural to extend f_m to all of $\mathbb{R}$ by putting

$$f_m(0) \; = \; 0.$$

It was remarked earlier (in class) that f_0 is not continuous at 0, as $\sin(1/x)$ wildly fluctuates near zero and does not approach 0 as x goes to 0. (It's useful to look at the graph!) But f_1 is continuous at 0, since $\lim\limits_{x\to 0} x\sin(1/x) = 0 = f_1(0)$ (as seen in the previous chapter when we dealt with limits of functions).

On the other hand, the ratio $\frac{h\sin(1/h)}{h}$, being just $\sin(1/h)$, has no limit as $h \to 0$, proving that f_1 is not differentiable at 0. However, the limit $\lim\limits_{h\to 0} \frac{f_2(h)}{h}$ does exist, since $\frac{f_2(h)}{h} = h\sin(1/h)$, proving that f_2 is differentiable at 0. It is a quick thing to check that f_2' is not continuous at 0, i.e., f_2 is differentiable but not $\mathcal{C}^1$ at 0. It is also not hard to do one more step and show that f_3 is $\mathcal{C}^1$ at 0, but is not twice differentiable.

This process can be continued *ad infinitum*, and it will be left as an exercise to prove (try induction!) the following (for every non-negative integer m):

f_{2m} is m-times differentiable, but is not $\mathcal{C}^m$, at 0, while f_{2m+1} is $\mathcal{C}^m$, but is not $(m+1)$-times differentiable, at 0.

4.5 Local Extrema

Definition 4.5 *A function f defined on a set S is said to have a local maximum (minimum) at a point $c \in S$ if there exists an interval $I \ni c$ such that $f(x) \le f(c)$ for all $x \in I \cap S$.*

A local maximum or minimum is called an **extreme value**, *or an* **extremum**, *of f.*

Some call local extrema *relative extrema.*

This definition makes sense even if f is not continuous around a. However, if f is continuous and a is a local maximum, we see that the function increases to the left of a and decreases to the right. Similarly, if a is a local minimum, f decreases to the left and increases to the right of a.

Iff is moreover differentiable around a, if a is a local maximum, $f'(x)$ will necessarily be positive for $x < a$ and x close to a, while $f'(x)$ is negative for $x > a$ and x near a. We have a similar criterion for a local minimum.

Theorem 4.6 *Let f be defined on an open interval I, and assume that f has a local extremum at $a \in I$. If f is differentiable at a, then $f'(a) = 0$.*

Proof Set $Q(x) = \frac{f(x)-f(a)}{x-a}$ if $x \neq a$, and $Q(a) = f'(a)$. The assumption that f is differentiable at a then implies that Q is continuous at a. We'll show that $Q(a) = 0$. If $Q(a) > 0$ then by sign-preserving property of continuous functions, $Q > 0$ on some neighborhood of a. Then $f(x) > f(a)$ if $x > a$ and $f(x) < f(a)$ if $x < a$. This contradicts the fact that a is a local extremum. Similarly, $Q(a) < 0$ leads to a contradiction again, leaving us no choice ut to have $Q(a) = f'(a) = 0$. $\square$

Examples: 1) $f(x) = x^3$. $f'(0) = 0$, but this point is not a local extremum!

2) $f(x) = |x|$. $f(0)$ is an extremum but f is not differentiable at 0.

Definition 4.7 *A real number a is a* **critical point** *of a differentiable function f (around a) iff $f'(a) = 0$. It is called an inflection point if $f'(a) = 0$, but a is not a local extremum.*

Proposition 1 *Assume f is continuous on $[a, b]$ and differentiable on (a, b). Then*

(a) If $f'(x) > 0$ on $(, a, b)$ then f is strictly increasing on $[a, b]$;

(b) If $f'(x) < 0$ on (a, b) then f is strictly decreasing on $[a, b]$;

(c) If $f'(x) \equiv 0$ then $f \equiv$ constant.

Proof Use $f(y) - f(x) = f'(c)(y - x)$. $\square$

Corollary 4.8 *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x)$ changes sign at $c \in (a, b)$ the c is a local extremum.*

Theorem 4.9 *(***The second derivative test***) Let c be a critical point of f in an open interval (a, b). Assume that f is twice differentiable on (a, b). Then* *(a) If f'' is negative in (a, b), f has a local max at c.*

(b) If f'' is positive in (a, b), f has a local min at c.

Proof $f'(c) = 0$ and f' decreases or increases $\Rightarrow$ it changes sign at c. $\square$

Note that if $f(x) = x^k$, with $k = 2, 3, 4$, it has a critical point at $x = 0$, which is a local minimum for $k = 2$ and $k = 4$, but is an inflection point for $k = 3$, since the second derivative changes sign there.

4.6 A simple example

Consider the function $f(x) = 3x^3 - x + 1$ on the interval $[-2, 3]$. Being a polynomial, it is clearly continuous on $[-2, 3]$ and twice differentiable on the open interval $(-2, 3)$, even on all of $\mathbb{R}$. Let us find all the local extrema and also the absolute extrema of f on $[-2, 3]$.

Since $f'(x) = 9x^2 - 1$, the critical points are given by $(3x)^2 = 1$, i.e., $3x = \pm 1$. In other words, $x = 1/3$ and $x = -1/3$ are the sole critical points, and they both lie in $(-2, 3)$. We have

$$f(1/3) = 3(1/3)^3 - (1/3) + 1 = 7/9, \ f(-1/3) = 3(-1/3)^3 - (-1/3) + 1 = 11/9.$$

Moreover, since $f''(x) = 18x$, we have

$$f''(1/3) = 6 > 0, \quad \text{and} \quad f''(-1/3) = -6 < 0.$$

Hence $x = 1/3$ is a local minimum and $x = -1/3$ is a local maximum. f has no inflection point, even though it seems close to $g(x) = 9x^3$ which has an inflection point at $x = 0$.

It remains to find the absolute extrema of f on $[-2, 3]$. We see that f increases to the left of $x = -1/3$ decreases on $[-1/3, 1/3]$, and increases again to the right of $x = 1/3$. (Just look at the sign of $f'(x)$, for example.) So the absolute extrema will either occur at the critical points or at the boundary points $x = -2$ and $x = 3$. Recall that f, being continuous on the closed interval $I = [-2, 3]$, attains its (absolute) extrema on I. Direct checking gives

$$f(-2) = 3(-2)^3 - (-2) + 1 = -21, \quad \text{and} \quad f(3) = 3(3)^3 - (3) + 1 = 79.$$

Hence f has absolute minimum at $x = -2$ (with value -21), and it has absolute maximum at $x = 3$ (with value 79).

Neither local extremum is an absolute extremum in this example. But if we look at the (even simpler) example $f(x) = x^2 - x + 1$ on $[-2, 4]$, there is, since $f'(x) = 2x - 1$, a unique critical point at $x = 1/2$. And since $f''(x) = 2 > 0$ at every x, this critical point is a local minimum. Check that $f(1/2) = 3/4$. Now, at the end points, we have $f(-2) = 7$ and $f(4) = 13$. Thus the absolute maximum of f occurs at the right boundary point $x = 4$ (with value 13), while the absolute minimum occurs at the critical point $x = 1/2$ (with value $3/4$). Hence, in this example, the unique local minimum is also the absolute minimum.

4.7 The Mean Value Theorem

Let $\mathcal{C}[a, b] \cap \mathcal{D}(a, b)$ denote the collecion of functions f which are continuous on $[a, b]$ and differentiable on (a, b).

Rolle's theorem (1690) Let f be in $\mathcal{C}[a, b] \cap \mathcal{D}(a, b)$. Assume that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof Assume $f' \neq 0$ on (a, b). Then f must take its absolute maximum and minimum at a and b. Since $f(a) = f(b)$, we have $\min f = \max f \Rightarrow f \equiv$ const $\Rightarrow f' \equiv 0$. Contradiction. □

Example $f(x) = |x|$ on $(-1, +1)$. There are no points where $f'(x) = 0$!

Corollary 4.10 *(Mean-value theorem for derivatives) Assume that $f \in \mathcal{C}[a, b] \cap \mathcal{D}(a, b)$. Then $\exists c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$*

In the applications to Mechanics, this gives the attainment (at a particular time) of the average speed.

Proof Set $h(x) = f(x)(b - a) - x(f(b) - f(a))$ and apply Rolle's theorem. □

Cauchy's mean-value formula: Let $f, g \in \mathcal{C}[a, b] \cap \mathcal{D}(a, b)$. Then $\exists C \in (a, b) \cdot f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$.

Proof Set $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$ and apply Rolle's theorem □

Derivative test for convexity: Assume f is continuous on $[a, b]$ and differentiable on (a, b). If f' is increasing on (a, b) (e.g., if $f'' \geq 0$), then f is convex on $[a, b]$. (Similarly, if decreasing, then concave.)

Proof Take $x < y$ in $[a, b]$ and let $z = \alpha y + (1 - \alpha)x$, $0 < \alpha < 1$. We want to prove that $f(z) \leq \alpha f(y) + (1 + \alpha)f(x)$, which is equivalent to

$$(1 - \alpha)(f(z) - f(x)) \leq \alpha(f(y) - f(z))$$

By the mean-value theorem, $\exists c \in (x, z)$ and $d \in (z, y)$:

$$f(z) - f(x) = f'(c)(z - x) \text{ and } f(y) - f(z) = f'(d)(y - z)$$

We have $f'(c) \leq f'(d)$ by the hypothesis. Also,

$$(1 - \alpha)(z - x) = \alpha(y - z)$$

so we have

$$(1 - \alpha)(f(z) - f(x)) = (1 - \alpha)f'(c)(z - x) \leq \alpha f'(d)(y - z) = \alpha(f(y) - f(z))$$

$\square$

5 Integration

We will first discuss the question of integrability of bounded functions on closed intervals, followed by the integrability of continuous functions (which are nicer), and then move on to bounded functions with negligible discontinuities.

The main tool will be to approximate the integral from above by the **upper sum** and from below by the **lower sum**, relative to various partitions. This method was introduced by the famous nineteenth century German mathematician Riemann, and it is customary to call these sums **Riemann sums**.

5.1 Basic Notions

Definition 5.1 *If f is a bounded function on a closed interval $[a, b]$, then the* **span** *of f on $[a, b]$ is given by*

$$\mathrm{span}_f([a, b]) \;=\; \sup_f([a, b]) - \inf_f([a, b]),$$

where $\sup_f([a, b])$ (resp. $\inf_f([a, b])$) denotes the supremum (resp. infimum) of the values of f on $[a.b]$.

If f is continuous on $[a, b]$, then we know that it is bounded, and moreover, $\sup = \max$ and $\inf = \min$ (of $f([a, b])$).

Definition 5.2 *A* **partition** *of a closed interval $[a, b]$ is a collection of points $t_0, t_1, t_2, \ldots, t_n$ such that*

$$t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < b = t_n.$$

Definition 5.3 *A function S defined on $[a, b]$ is called a* **step function** *if there is a partition $\underline{P} = \{t_0, \ldots, t_n\}$ of $[a, b]$, and constants $c_1, c_2, \ldots, c_n$ such that such that*

$$S(x) \;=\; c_j \;\; \textit{if} \;\; x \in [t_{j-1}, t_j),$$

and $S(b) = c_n$.

A proper definition of integration must allow such a (step) function to be integrable, with its integral over $[a, b]$, denoted $\int_a^b S$, being the sum $\sum_{j=1}^n c_j(t_j - t_{j-1})$.

Definition 5.4 *If P, P' are partitions of $[a, b]$, we will say that P' is a* **refinement** *of P iff the set of points in P is contained in the set of points of P'.*

For example, $P : t_0 = 0 < t_1 = \frac{1}{2} < t_2 == 1$ and $P' : t'_0 = 0 < t'_1 = \frac{1}{4} < t'_2 = \frac{1}{2} < t'_3 = \frac{3}{4} < t'_4 = 1$ are both partitions of $[0, 1]$, with P' a refinement of P.

It is clear from the definition that given any two partitions P, P' of $[a, b]$, we can find a third partition P'' which is simultaneously a refinement of P and of P'. Such a P'' is called a **common refinement** of P, P'.

Remark: The sum and the product of two step functions are also seen to be step functions, by considering suitable refinements.

Here is a quick definition of integrability:

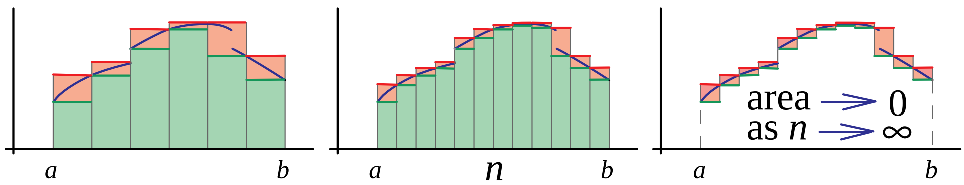
Definition 5.5 *A bounded function f on $[a, b]$ is* **integrable** *iff for every $\varepsilon > 0$, we can find a partition $P = t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < b = t_n$ such that the sum*

$$\Delta_f(P) := \sum_{j=1}^{n} (t_j - t_{j-1}) \mathrm{span}_f([t_{j-1}, t_j])$$

is less than ε.

Note that, for each $\varepsilon > 0$, the choice of P may depend on ε.

Clearly, $\Delta_f(P)$ is the area caught between the upper and lower Riemann sums. We want this area to be as small as possible for f to be integrable. Equivalently, we want to choose a sequence $\{P_n\}$ of partitions such that $\Delta_f(P_n) \to 0$ as $n \to \infty$.



The obvious question now is to ask if there are integrable functions. One such example is given by the **constant function** $f(x) = c$, for all $x \in [a, b]$. Then for any partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$, the sup and inf of f on each subinterval $[t_{j-1}, t_j]$ coincide, making $\Delta_f(P)$ zero.

Lemma 5.6 *Every step function S is integrable on $[a, b]$.*

Proof. Let S be the step functions associated to a partition $P := t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < b = t_n$ and constants c_j, so that $S(x) = c_j$ if $x \in [t_{j-1}, t_j)$ and $S(b) = c_n$. There is a jump in the value of S at each t_j for $j \in \{1, 2, \ldots, n-1\}$. This is harmless and can be taken care of as follows. For any $\varepsilon > 0$, consider a refinement of P given by

$$P' : a = t'_0 < t'_1 = t_1 - \delta < t'_2 = t_1 < t'_3 = t_2 - \delta < t'_4 = t_2 < \cdots < t'_{2n-2} = t_{n-1} < t'_{2n-1} = t_n = b,$$

where

$$\delta = \varepsilon/(n-1)\mu, \quad \text{with} \quad \mu = \max\{c_1, \ldots, c_n\} - \min\{c_1, \ldots, c_n\}.$$

Then, since the span of f is zero on each $[t'_{2i} - t'_{2i+1}]$, we get

$$\Delta_f(P) < \sum_{j=1}^{n-1} \delta(c_j - c_{j-1}) < \varepsilon,$$

as each $c_j - c_{j-1} < \mu$. □

5.2 Upper and Lower Sums

One can interpret $\Delta_f(P)$ for each partition P of $[a, b]$ as the difference between certain upper and lower sums of Riemann.

Definition 5.7 *The* **upper**, *resp.* **lower**, **sum** *of f over $[a, b]$ relative to the partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ is given by*

$$U(f, P) = \sum_{j=1}^{r} (t_j - t_{j-1}) \mathrm{sup}_f([t_{j-1}, t_j])$$

resp.

$$L(f, P) = \sum_{j=1}^{r} (t_j - t_{j-1}) \mathrm{inf}_f([t_{j-1}, t_j]),$$

Of course we have, for all P,

$$L(f,P) \leq U(f,P), \quad \text{and} \quad \Delta_f(P) = U(f,P) - L(f,P).$$

More importantly, it is clear from the definition that if P' is a refinement of P, then

$$L(f,P) \leq L(f,P') \quad \text{and} \quad U(f,P') \leq U(f,P).$$

Put

$$\mathcal{L}(f) = \{L(f,P) \mid P \text{ partition of } [a,b]\} \subseteq \mathbb{R}$$

and

$$\mathcal{U}(f) = \{U(f,P) \mid P \text{ partition of } [a,b]\} \subseteq \mathbb{R}.$$

Lemma 5.8 *$\mathcal{L}(f)$ admits a* **sup***, denoted $\underline{I}(f)$, called the* **lower integral** *of f over $[a,b]$. Similarly, $\mathcal{U}(f)$ admits an* **inf***, denoted $\overline{I}(f)$, called the* **upper integral**.

Proof. Thanks to the discussion in Chapter 1, all we have to do is show that $\mathcal{L}(f)$ (resp. $\mathcal{U}(f)$) is bounded from above (resp. below). So we will be done if we show that given any two partitions P, P' of $[a,b]$, we have $L(f,P) \leq U(f,P')$, as then $\mathcal{L}(f)$ will have $U(f,P')$ as an upper bound and $\mathcal{U}(f)$ will have $L(f,P)$ as a lower bound. Choose a third partition P'' which refines P and P' simultaneously. Then we have $L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P')$. Done.

□

We always have

$$\underline{I}(f) \leq \overline{I}(f).$$

Lemma 5.9 *A bounded function f is integrable over $[a,b]$ iff $\underline{I}(f) = \overline{I}(f)$.*

Proof Suppose f is integrable. Then, by definition, $\Delta_f(P) = U(f,P) - L(f,P)$ becomes arbitrarily small as P goes through a sequence of refinements. This forces the equality $\underline{I}(f) = \overline{I}(f)$ in the limit. Conversely, given this equality, the difference $U(f,P) - L(f,P)$ must become less than any given $\varepsilon > 0$, for a suitably refined P. □

When such an equality holds, we will simply write $I(f)$ for $\underline{I}(f)$ $(= \overline{I}(f))$, and call it the **integral of f over** $[a, b]$.

Quite often we will also write

$$I(f) = \int_a^b f \quad \text{or} \quad \int_a^b f(x)\, dx.$$

In practice one is loathe to consider *all* partitions P of $[a, b]$. The following lemma tells us something useful in this regard.

Lemma 5.10 *Let f be a bounded function on $[a, b]$. Suppose $\{P_n\}$ is an infinite sequence of partitions, with each P_n being a refinement of P_{n-1}, such that the corresponding sequences $\{U(f, P_n)\}$ and $\{L(f, P_n)\}$ both converge to a common limit λ in $\mathbb{R}$. Then f is integrable with $\lambda = \int\limits_a^b f(x)dx$.*

Note that for such a step function f defined by $(P, \{c_j\})$, we have an explicit formula for the integral, namely

$$\int_a^b f(x)dx = \sum_{j=1}^{n} c_j\,(t_j - t_{j-1}).$$

5.3 Integrability of monotone functions

Let $f : A \to \mathbb{R}$ be a function, with A a subset of $\mathbb{R}$. Recall that f is **monotone increasing** (resp. **monotone decreasing**), iff we have

$$x_1, x_2 \in A,\; x_1 < x_2 \implies f(x_1) \leq f(x_2) \quad (\text{resp.} \quad f(x_1) \geq f(x_2)).$$

A **monotone** function f on a closed interval $[a, b]$ is bounded on $[a, b]$. This is clear because f is bounded by $f(a)$ on one side and by $f(b)$ on the other.

Theorem 5.11 *Let f be a monotone function on $[a, b]$. Then f is integrable.*

Proof. Suppose f is monotone increasing on $[a, b]$. For $n \geq 1$, let $P_n = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be the n-th **standard partition** (or *equal partition*) with

$$t_{j+1} - t_j \;=\; \frac{b-a}{n} \quad \forall\, j \leq n-1.$$

Since f is increasing, $f(t_j)$ is, for each j, the *inf* of the values of f on the subinterval $[t_j, t_{j+1}]$, and $f(t_{j+1})$ is the *sup*. Hence

$$L(f, P_n) \;=\; \frac{b-a}{n}\left(f(t_0) + f(t_1) + \ldots + f(t_{n-1})\right)$$

and

$$U(f, P_n) \;=\; \frac{b-a}{n}\left(f(t_1) + f(t_2) + \ldots + f(t_n)\right).$$

It follows by telescoping that

$$\Delta_f(P_n) = U(f, P_n) - L(f, P_n) \;=\; \frac{b-a}{n}\left(f(b) - f(a)\right).$$

As n goes to infinity, this difference goes to zero. Hence f is integrable by Lemma 5.6.

$\square$

5.4 Computation of $\int_a^b x^s dx$

The function $f(x) = x^m$ is a monotone increasing function for any $m > 0$. So by Theorem 5.11, it is integrable on any closed interval $[a, b]$. There is a well known formula for the value of the integral, which can be derived in a myriad of ways. We will give two proofs in this chapter, and here is the first one – due to Riemann, which uses partitions $P = a = t_0 < t_1 < \ldots < t_n = b$ where the subintervals $[t_i, t_{i+1}]$ are not of equal length, but where the ratios t_{i+1}/t_i are kept constant.

Riemann's method, unlike the one we will describe later in section 5.7, is very general, and works also for x^s for *any* real exponent $s \neq -1$, as long as the limits a, b are positive. We have encountered x^s before for rational s, and also e^y for any real y (via the infinite series defining it). F.32or those who know about logarithms and exponentials, x^s is defined for any real s and positive x as $e^{s \log x}$; here $\log x$ denotes the *natural logarithm* of x, which some denote by $ln(x)$. It is problematic to define x^s for general s and negative x, except when s is an integer, because numbers like $(-1)^{1/2}$ are not in $\mathbb{R}$.

Proposition 1 *Let s, a, b be real numbers with $s \neq -1$ and $a < b$. If s is not an integer, assume that a is positive. Then*

$$\int_a^b x^s dx = \frac{b^{s+1} - a^{s+1}}{s+1}.$$

We will prove this result below only for positive integer values of s, but we will make a remark after the proof about what one needs for the extension to general s.

When $s = -1$, one cannot divide by $s+1$ and the Proposition cannot hold as stated. It may be useful to note that for any $b > 1$,

$$\int_1^b \frac{1}{x} dx = \log b.$$

If you are not familiar with the logarithm, you may take this as the definition of $\log b$.

Proof of Proposition for positive integral exponents:

We will take s to be a positive integer m. (The assertion is obvious for $m = 0$.) We will also assume, for simplicity of exposition, that $a > 0$, even though the asserted formula holds equally well when $a \leq 0$. Write

$$f(x) = x^m. \tag{5.4.1}$$

Put

$$u = b/a > 1, \tag{5.4.2}$$

and define, for each $n \geq 1$, a partition

$$P_n : a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b \tag{5.4.3}$$

such that for each $j \leq n$,

$$t_j = au^{j/n}.$$

Then we have, for all $j < n$,

$$\frac{t_{j+1}}{t_j} = u^{(j+1)/n - j/n} = u^{1/n}, \tag{5.4.4}$$

which is independent of j.

The lower sum is given, for each n, by

$$L(f, P_n) = \sum_{j=0}^{n-1} (t_{j+1} - t_j) t_j^m = (u^{1/n} - 1) \sum_{j=0}^{n-1} t_j^{m+1}, \tag{5.4.5}$$

where we have used (5.4.4). By the definition of t_j and the fact that $\sum_{j=0}^{n-1} z^j = \frac{1-z^n}{1-z}$, the expression on the right of (5.4.5) becomes

$$a^{m+1}(u^{1/n} - 1) \sum_{j=0}^{n-1} u^{j(m+1)/n} = -a^{m+1} \frac{(1 - u^{1/n})(1 - u^{m+1})}{1 - u^{(m+1)/n}}.$$

Since $-a^{m+1}(1 - u^{m+1})$ equals $b^{m+1} - a^{m+1}$, we get

$$L(f, P_n) = (b^{m+1} - a^{m+1}) \frac{1 - u^{1/n}}{1 - u^{(m+1)/n}} = (b^{m+1} - a^{m+1}) \frac{1}{1 + u^{1/n} + u^{2/n} + \ldots + u^{m/n}}. \tag{5.4.6}$$

We know that as $n \to \infty$, $u^{j/n}$ goes to 1 for any fixed j. This shows that

$$\lim_{n\to\infty} \frac{1}{1 + u^{1/n} + u^{2/n} + \ldots + u^{m/n}} = \frac{1}{m+1}. \tag{5.4.7}$$

Consequently, by (5.4.6),

$$\lim_{n\to\infty} L(f, P_n) = \frac{b^{m+1} - a^{m+1}}{m+1}. \tag{5.4.8}$$

On the other hand, since $t_{j+1}^m = u^{m/n} t_j^m$, the corresponding upper sum is

$$U(f, P_n) = \sum_{j=0}^{n-1} (t_{j+1} - t_j)(u^{m/n} t_j^m) = u^{m/n} L(f, P_n). \tag{5.4.9}$$

And since $u^{m/n} \to 1$ as $n \to \infty$, (5.4.8) and (5.4.9) imply that we have

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

It follows then (see Lemma 5.2.8) that the (definite) integral of x^m over $[a, b]$ equals this common limit. Incidentally, this computation shows explicitly that $f(x) = x^m$ is integrable and we don't really need to refer to Theorem 5.11 (from the previous section).

$\square$

Now suppose we want to prove the full force of the Proposition, i.e., treat the case of an arbitrary real exponent $s \neq -1$, by this method. Proceeding as above, we will get

$$L(f, P_n) = (b^{s+1} - a^{s+1})\phi_s(n), \tag{5.4.6}$$

where

$$\phi_s(n) = \frac{1 - u^{1/n}}{1 - u^{(s+1)/n}},$$

and

$$U(f, P_n) = u^{s/n} L(f, P_n).$$

As before, $u^{s/n}$ goes to 1 as $n \to \infty$. So the whole argument will go through if we can establish the following limit:

$$\lim_{n\to\infty} \phi_s(n) = \frac{1}{s+1}.$$

This can be done, but we will not do it here. In any case, you should feel free to use the Proposition for all $s \neq -1$.

5.5 Example of a non-integrable, bounded function

Define a function

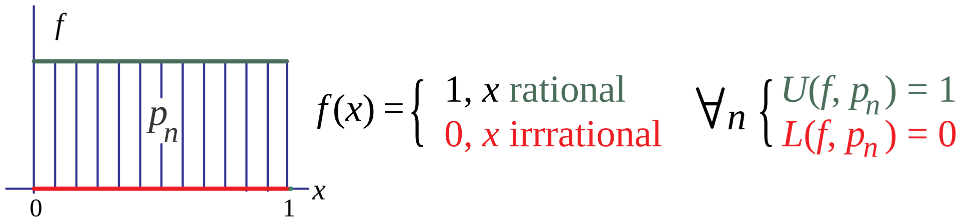
$$f : [0, 1] \to \mathbb{R}$$

by the following recipe. If x is irrational, set $f(x) = 0$, and if x is rational, put $f(x) = 1$.

This certainly defines a bounded function on [0.1], and one is led to wonder about the integrability of f.

Proposition 5.5.2 *This f is not integrable.*

Proof. Let P be any partition of $[0, 1]$ given by $0 = t_0 < t_1 < \ldots < t_n = 1$. By a basic property of $\mathbb{R}$, we know that there is a rational number



between any two real numbers. So in every subinterval $[t_j, t_{j+1}]$ there will be some rational number q_j (in fact infinitely many), with $f(q_j) = 1$ by definition. Consequently,

$$U(f, P) \;=\; \sum_{j=0}^{n-1} 1 \cdot (t_{j+1} - t_j) \;=\; (t_0 - t_1) + (t_1 - t_2) + \ldots + (t_{n-1} - t_n) \;=\; 1,$$

because $t_0 = 0$ and $t_n = 1$.

On the other hand, every interval $[c, d]$ in $\mathbb{R}$ must contain an irrational number y. Let us give a proof. If c or d is irrational, then we may take y to be that number, so we can assume that c and d are rational. Then the number $y = c + (d - c)\sqrt{2}/2$ is irrational and lies in $[c, d]$. Consequently, for every j, there is an irrational y_j in $[t_j, t_{j+1}]$, which implies that

$$L(f, P) \;=\; 0.$$

Hence

$$U(f, P) - L(f, P) \;=\; 1,$$

and this is independent of the partition P. So f is not integrable.

□

It should be noted, however, that there are non-zero integrable, bounded functions f on $[0, 1]$ which are supported on $\mathbb{Q}$, i.e., $f(x)$ is zero for irrational numbers x. But they are not constant on the rational numbers.

5.6 Properties of integrals

The integral $\int_a^b f(x)dx$ is often called a **definite integral** because it has a definite value, assuming that f is integrable. One calls f the **integrand**, a

the **lower limit** and b the **upper limit**. It is customary to use the convention

$$\int_b^a f(x)dx = -\int_a^b f(x)dx. \tag{5.6.1}$$

The definite integral has many nice properties as f or $[a, b]$ varies, making our life very pleasant, which we want to discuss in this section.

Proposition 2 *(Linearity in the integrand) If f, g are integrable over $[a, b]$, so is any linear combination $\alpha f + \beta g$, with $\alpha, \beta \in \mathbb{R}$, and moreover,*

$$\int_a^b \{\alpha f(x) + \beta g(x)\}dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Proof. If P is any partition, it is immediate from the definition that

$$L(\alpha f + \beta g, P) = \alpha L(f, P) + \beta L(g, P)$$

and

$$U(\alpha f + \beta g, P) = \alpha U(f, P) + \beta U(g, P).$$

It follows then that

$$\underline{I}(\alpha f + \beta g) = \alpha \underline{I}(f) + \beta \underline{I}(g)$$

and

$$\overline{I}(\alpha f + \beta g) = \alpha \overline{I}(f) + \beta \overline{I}(g).$$

Since f, g are integrable, $\underline{I}(f) = \overline{I}(f)$ and $\underline{I}(g) = \overline{I}(g)$. So the lower and upper integrals of $\alpha f + \beta g$ coincide, proving the assertion. $\square$

Proposition 3 *(Additivity in the limits) Let a, b, c are real numbers with $a < b < c$, and let f be integrable on $[a, c]$. Then f is integrable on $[a, b]$ and $[b, c]$, and moreover,*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

Proof. Given partitions P, P' of $[a, b], [c, d]$ respectively, $P \cup P'$ defines a partition of $[a, c]$. And if P'' is a partition of $[a, c]$, then we can refine it by adding b to get a partition of the type $P \cup P'$, with P (resp. P') being a partition of $[a, b]$ (resp. $[b, c]$). It follows easily that the lower (resp. upper) integral of f over $[a, c]$ is the sum of the lower (resp. upper) integrals of f over $[a, b]$ and $[b, c]$; whence the assertion.

□

If c is any real number , the function $x \to x + c$ is called **translation** by c. The following Proposition describes the **translation invariance** of the definite integral.

Proposition 4 *Suppose f is integrable on $[a, b]$ and $c \in \mathbb{R}$. Then the c-***translate** *of f, given by $x \to f(x + c)$, is integrable on $[a - c, b - c]$, and*

$$\int_{a-c}^{b-c} f(x + c)dx = \int_{a}^{b} f(x)dx.$$

For any $c \in \mathbb{R}$, the function $x \to cx$ is called the **homothety** (or **stretching**) by c. Some also use the terms **expansion** and **contraction** when $c > 1$ and $0 < c < 1$. The following Proposition describes the **behavior under homothety**.

Proposition 5 *Suppose f integrable on $[ac, bc]$. Then the function $x \to f(cx)$ is integrable on $[a, b]$ and*

$$\int_{ac}^{bc} f(x)dx = c\int_{a}^{b} f(cx)dx.$$

5.7 Even and odd functions, and the integral of x^m revisited

In section 5.4 we established the following identity for any $m \geq 0$ and any $[a, b]$:

$$(*) \qquad \int_{a}^{b} x^m dx = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

In view of the linearity property of the integral (see Proposition 2), we see that if we have any **polynomial**

$$f(x) = c_0 + c_1 x + \ldots + c_n x^n,$$

then f is integrable over any $[a, b]$. Furthermore, we have the **explicit formula**

$$(**) \qquad \int_a^b f(x)dx = c_0 x + c_1 \frac{x^2}{2} + \ldots + c_n \frac{x^{n+1}}{n+1}.$$

We will now give an **alternate proof** of $(*)$ for $0 < a < b$.

We first need a Lemma, which is of independent interest. We will call a function $f(x)$ **even**, resp. **odd**, iff $f(-x) = f(x)$, resp. $f(-x) = -f(x)$, for all x. Note that x^j is even if j is even and it is odd if j is odd. Also, $\cos x$ is even while $\sin x$ is odd.

Lemma 5.12 *Let $f(x)$ be an integrable function of $[-a, a]$, for some $a > 0$. Then*

$$f \text{ even} \quad \Longrightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx,$$

$$f \text{ odd} \quad \Longrightarrow \int_{-a}^{a} f(x)dx = 0.$$

Proof. By Proposition 5 (applied with $c = -1$) and the convention (5.6.1),

$$\int_{-a}^{0} f(x)dx = \int_0^a f(-x)dx,$$

which equals

$$(5.12) \qquad (-1)^r \int_0^a f(x)dx,$$

with r being 1, resp. -1, when f is even, resp. odd. The lemma now follows by Proposition 3, which implies that

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$

$\square$

Now let us begin the alternate proof of $(*)$. Put

$$I_m = \int_0^1 x^m dx.$$

The key is to show that

$$I_m = \frac{1}{m+1}. \tag{5.7.2}$$

Indeed, we can use Proposition 5 to deduce that

$$\int_0^b x^m dx = b\int_0^1 (bx)^m dx = b^{m+1} I_m, \tag{5.7.3}$$

and combining this with Proposition 3, we get

$$\int_a^b x^m dx = \int_0^b x^m dx - \int_0^a x^m dx = (b^{m+1} - a^{m+1}) I_m, \tag{5.7.4}$$

as desired. So it suffices to just prove (5.7.2).

By the translation invariance (Proposition 4) of definite integrals, we get

$$\int_{-b}^{b} (x+b)^m dx = \int_0^{2b} x^m dx = 2^{m+1} b^{m+1} I_m. \tag{5.7.5}$$

By the binomial theorem,

$$(x+b)^m = \sum_{j=0}^{m} \binom{m}{j} x^j b^{m-j}.$$

and so by (5.7.5),

$$2^{m+1}b^{m+1}I_m = \sum_{j=0}^{m}\binom{m}{j}b^{m-j}\int_{-b}^{b} x^j dx. \tag{5.7.6}$$

By Lemma 5.12, $\int_{-b}^{b} x^j dx$ equals 0 if j is odd, and twice $\int_{0}^{b} x^j dx = 2b^{j+1}I_j$ (by 5.7.3) when j is even.

Since $(*)$ is true for $m = 0$, we can take $m > 0$ and assume, by induction, that $(*)$ holds for all $j < m$. Then we get from the above, the following:

$$2^m I_m = \sum_{j=0,\, j\,\mathrm{even}}^{m}\binom{m}{j}I_j = \epsilon_m I_m + \sum_{j=0,\, j\,\mathrm{even}}^{m-1}\binom{m}{j}\frac{1}{j+1}, \tag{5.7.7}$$

where ϵ_m is 1 if m is even and 0 if m is odd. On the other hand,

$$\binom{m}{j}\frac{1}{j+1} = \frac{m!}{(j+1)!(m-j)!} = \binom{m+1}{j+1}\frac{1}{m+1},$$

so that

$$\sum_{j=0,\, j\,\mathrm{even}}^{m-1}\binom{m}{j}\frac{1}{j+1} = \frac{1}{m+1}\sum_{k=0,\, k\,\mathrm{odd}}^{m}\binom{m+1}{k}. \tag{5.7.8}$$

Next we note that for any integer $r \geq 1$,

$$\sum_{k=0,\, k\,\mathrm{odd}}^{r}\binom{r}{k} = \frac{1}{2}(1-(-1)^r) = 2^{r-1}.$$

(Check this!) Consequently,

$$\sum_{k=0,\, k\,\mathrm{odd}}^{m}\binom{m+1}{k} = 2^m - \epsilon_m. \tag{5.7.9}$$

Combining (5.7.7.), (5.7.8) and (5.7.9), we get

$$2^m I_m = \epsilon_m I_m + \frac{2^m}{m+1} - \epsilon_m\frac{1}{m+1}.$$

When m is odd, $\epsilon_m = 0$ and hence

$$2^m I_m = \frac{2^m}{m+1}.$$

When m is even, $\epsilon_m = 1$ and we get

$$2^m I_m = I_m + \frac{2^m}{m+1} - \frac{1}{m+1}.$$

In either case,

$$I_m = \frac{1}{m+1},$$

as asserted.

$\square$

5.8 Trigonometric functions

The following result is basic.

Proposition 6 *Let $[a, b]$ be any closed interval. Then the functions $\sin x$ and $\cos x$ are integrable on $[a, b]$. Explicitly,*

$$\int_a^b \sin x dx = \cos a - \cos b$$

and

$$\int_a^b \cos x dx = \sin b - \sin a.$$

Proof. By the periodicity and additivity of the integral (see section 5.6), we may assume that $0 \leq a < b \leq 2\pi$. Moreover, for $0 \leq x \leq \pi$, the *oddness* of $\sin x$ implies, when used in conjunction with its periodicity, that

(5.8.1) $$\sin(\pi + x) = -\sin(-\pi - x) = -\sin(\pi - x),$$

while the *evenness* of $\cos x$ implies

$$\cos(\pi + x) = \cos(-\pi + x) = \cos(\pi - x).$$

Moreover, the trigonometric definition of $\sin x$ and $\cos x$ gives immediately the following identities:

$$\sin(\pi - x) \;=\; \sin x \tag{5.8.2}$$

and

$$\cos(\pi - x) \;=\; -\cos x.$$

Thanks to (5.8.1) and (5.8.2) it suffices to prove the assertion of the Theorem when $0 \leq a < b \leq \pi/2$. Furthermore, since

$$\int_a^b f(x)dx \;=\; \int_0^b f(x)dx - \int_0^a f(x)dx$$

for any function $f(x)$, it suffices to prove for any a in $(0, \pi/2]$ that the functions $\sin x$ and $\cos x$ are integrable on $[0, a]$, and that

$$\int_0^a \sin x\, dx \;=\; 1 - \cos a \quad \text{and} \quad \int_0^a \cos x\, dx \;=\; \sin a. \tag{5.8.3}$$

We will prove the formula for the integral of $\sin x$ over $[0, a]$ and leave the proof of the corresponding one for $\cos x$ to the reader.

For every $n \geq 1$, define a partition P_n of $[0, a]$ to be given by

$$0 = t_0 \;<\; t_1 = \frac{a}{n} \;< t_2 = \frac{2a}{n} \;<\; \ldots < t_n = a, \tag{5.8.4}$$

so that $t_{j+1} - t_j = \frac{a}{n}$ for all $j \leq n - 1$.

Since $\sin x$ is a monotone increasing function in $[0, \pi/2]$, the upper and lower sums are given by

$$U(\sin x, P_n) \;=\; \frac{a}{n} \sum_{j=0}^{n-1} \sin(\frac{(j+1)a}{n}) \tag{5.8.5}$$

and

$$L(\sin x, P_n) \;=\; \frac{a}{n} \sum_{j=0}^{n-1} \sin(\frac{ja}{n}).$$

By the addition theorem for $\cos x$ (part (d) of Theorem 5.8.1), we have

$$2\sin x \sin y \;=\; \cos(x-y) - \cos(x+y). \tag{5.8.6}$$

In particular, when $y = \frac{a}{2n}$ and $x = \frac{(j+1)a}{n}$, we get

$$2\sin(\frac{a}{n})\sin(\frac{(j+1)a}{n}) \;=\; \cos(\frac{(2j+1)a}{2n}) - \cos(\frac{(2j+3)a}{2n}). \tag{5.8.7}$$

Applying this in conjunction with (5.8.5), we get

$$U(\sin x, P_n) \;=\; \frac{a/2n}{\sin(\frac{a}{2n})}\left((\cos(\frac{a}{2n}) - \cos(\frac{3a}{2n})) + \ldots + (\cos(\frac{(2n-1)a}{2n}) - \cos(\frac{(2n+1)a}{2n}))\right),$$

which simplifies to

$$U(\sin x, P_n) \;=\; \frac{a}{2n\sin(\frac{a}{2n})}\left(\cos(\frac{a}{2n}) - \cos(\frac{(2n+1)a}{2n})\right). \tag{5.8.8}$$

By Proposition 5.8.2,

$$\lim_{n\to\infty} \frac{\sin(\frac{a}{2n})}{\frac{a}{2n}} \;=\; 1. \tag{5.8.9}$$

Also, since $\cos 0 = 1$,

$$\lim_{n\to\infty} \cos(\frac{a}{2n}) - \cos(\frac{(2n+1)a}{2n}) \;=\; 1 - \cos a. \tag{5.8.10}$$

Combining (5.8.8) through (5.8.10), we get

$$\lim_{n\to\infty} U(\sin x, P_n) \;=\; 1 - \cos a. \tag{5.8.11}$$

The analog of (5.8.8) for the n-th lower sum is

$$L(\sin x, P_n) \;=\; \frac{a}{2n\sin(\frac{a}{2n})}\left(1) - \cos(\frac{(2n-1)a}{2n})\right). \tag{5.8.12}$$

We get the same limit, namely

$$\lim_{n\to\infty} L(\sin x, P_n) \;=\; 1 - \cos a. \tag{5.8.13}$$

In view of (5.8.11), (5.8.12), the desired identity (5.8.3) follows for the sine integral. The argument is entirely analogous for the cosine integral.

□

5.9 Functions with discontinuities

One is very often interested in being able to integrate bounded functions over $[a,b]$ which are continuous except on a subset which is very small, for example outside a **finite set**. To be precise, we say that a subset Y of $\mathbb{R}$ is **negligible**, or that it has **measure zero**, iff for every $\varepsilon > 0$, we can find a **countable number of closed intervals** $J_1, J_2, \ldots$ such that

(i) $Y \subset \cup_{i=1}^{\infty} J_i$, and

(ii) $\sum_{i=1}^{\infty} \ell(J_i) < \varepsilon$,

where $\ell(J_i)$ denotes the length of J_i.

If we can do this with just a **finite** number of closed intervals $\{J_i\}$ (for each ε), then we will say that Y has **content zero**. Of course, being of content zero is stronger than having measure zero.

Examples:

(1) *Any finite set of points in* $\mathbb{R}$ *has content zero.* (Proof is obvious!)

(2) *Any subset* Y *of* $\mathbb{R}$ *which contains a non-empty open interval* (a,b) *is* **not** *negligible.*

Proof of (2). It suffices to show that (a,b) has non-zero measure for $a < b$ in $\mathbb{R}$. Suppose (a,b) is covered by a finite union of a countable collection of closed intervals $J_1, J_2, \ldots$ in $\mathbb{R}$. Then clearly,

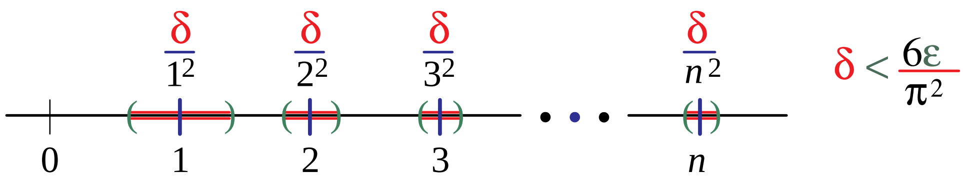
$$S := \sum_{i=1}^{m} \ell(J_i) \ \geq \ b - a.$$

So we can never make S less than $b - a$. $\square$

(3) *The set* $\mathbb{N}$ *of natural numbers is negligible in* $\mathbb{R}$.

Indeed, given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta < 6\varepsilon/\pi^2$, and choose the intervals $J_n = (-\frac{\delta}{2n^2}, \frac{\delta}{2n^2})$ surrounding n. Then $\mathbb{N} \subset \cup_n J_n$ and

$$\sum_n \ell(J_n) \ = \ \delta \sum_{n \geq 1} \frac{1}{n^2} \ < \ \varepsilon.$$



Theorem 5.13 *Let f be a bounded function on $[a, b]$ which is continuous except on a subset Y of measure zero. Then f is integrable on $[a, b]$.*

Proof when Y has content zero. Let $M > 0$ be such that $|f(x)| \leq M$, for all $x \in [a, b]$. Since Y has content zero, we can find a finite number of closed subintervals $J_1, J_2, \ldots, J_n$ of $[a, b]$ such that

(i) $Y \subseteq \cup_{m=1}^{n} J_m$, and

(ii) $\sum_{m=1}^{n} \ell(J_m) < \frac{\varepsilon}{4M}$.

We may assume that the closed intervals J_m are mutually disjoint except possibly at the endpoints. Extend $\{J_1, \ldots, J_n\}$ to a partition $P : a = t_0 < t_1 < \cdots < t_r = b$ of $[a, b]$, meaning that each J_m is some $[t_{i-1}, t_i]$. Applying the small span theorem, we may suppose that if some $[t_{j-1}, t_j]$ is not some J_m, then $\mathrm{span}_f([t_{j-1}, t_j]) < \frac{\varepsilon}{2(b-a)}$. (We can apply this theorem because f is continuous outside the union of $J_1, \ldots, J_n$.) Let $J_{n+1}, \ldots, J_r$ be the intervals $[t_{j-1}, t_j]$ which are not one of the J_m. So we have

$$U(f, P) - L(f, P) \;\leq\; 2M \sum_{i=1}^{n} \ell(J_i) + \sum_{i=n+1}^{r} \mathrm{span}_f(J_i)\ell(J_i),$$

which is

$$< \;(2M)\left(\frac{\varepsilon}{2M}\right) + \frac{\varepsilon}{2\ell([a, b])} \sum_{i=n+1}^{r} \ell(J_i) \;\leq\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

because $\sum_{i=n+1}^{r} \ell(J_i) \leq b - a$.

□

Even though we proved this result only for Y of content zero, it is correct for any Y of measure zero as well, and you should feel free to use it in that generality.

Remark: We can use this theorem to define the integral of a **continuous function** f **on any compact set** B in $\mathbb{R}$ **if the boundary** of B is **negligible**. Indeed, in such a case, we may enclose B in a closed interval $[a, b]$ and define a function $\tilde{f}$ on $[a, b]$ by making it equal f on B and 0 on $[a, b] - B$. Then $\tilde{f}$ will be continuous on all of $[a, b]$ except on the boundary of B, which has content zero. So $\tilde{f}$ is integrable on $[a, b]$. Since $\tilde{f}$ is 0 outside B, it is reasonable to set

$$\int_B f = \int_a^b \tilde{f}.$$

6 Fundamental Theorems, Substitution, Integration by Parts, and Polar Coordinates

So far we have separately learnt the basics of integration and differentiation. But they are not unrelated. In fact, they are **inverse operations**. This is what we will try to explore in the first section, via the two fundamental theorems of Calculus. After that we will discuss the two main methods one uses for integrating somewhat complicated functions, namely **integration by substitution** and **integration by parts**. The final section will discuss **integration in polar coordinates**, which comes up when there is radial symmetry.

6.1 The fundamental theorems

Suppose f is an integrable function on a closed interval $[a, b]$. Then we can consider the **signed area function A** on $[a, b]$ (relative to f) defined by the definite integral of f from a to x, i.e.,

$$A(x) = \int_a^x f(t)dt. \tag{6.1.1}$$

The reason for the **signed area** terminology is that f is not assumed to be ≥ 0, so a priori $A(x)$ could be negative.

It is extremely interesting to know how $A(x)$ varies with x. What conditions does one need to put on f to make sure that A is continuous, or even differentiable? The continuity part of the question is easy to answer.

Lemma 6.1 *Let f, A be as above. Then A is a continuous function on $[a, b]$.*

Proof. Let c be any point in $[a, b]$. Then f is continuous at c iff we have

$$\lim_{h \to 0} A(c + h) = A(c).$$

Of course, in taking the limit, we consider all small enough h for which $c + h$ lies in $[a, b]$, and then let h go to zero. By the *additivity* of the integral, we have (using (6.1.1)),

$$A(c+h) - A(c) \;=\; \int\limits_{I(c,h)} f(t)dt,$$

where $I(c,h)$ denotes the closed interval between c and $c+h$. Clearly, $I(c,h)$ is $[c, c+h]$, resp. $[c+h, c]$, if h is positive, resp. negative. When h goes to zero, $I(c,h)$ shrinks to the point $\{c\}$, and so

$$\lim_{h\to 0} A(c+h) - A(c) \;=\; 0,$$

which is what we needed to show.

$\square$

Remark 6.1.2: The general moral to remember is that, just as in real life,

Integration is good and Differentiation is bad !

Indeed, as seen in this Lemma, *integration* makes functions better; here it takes an integrable, but not necessarily a continuous, function f, and from it obtains a continuous function. The Theorem below says that if f is in addition continuous, then its integral is even differentiable. Differentiation, on the other hand, makes functions worse. The derivative of a differentiable function f is often not differentiable (think of $f(x) = \mathrm{sign}(x)x^2$ at $x = 0$); in fact, f' need not even be continuous (think of $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $= 0$ when $x = 0$).

A satisfactory answer to the question of differentiability of the integral is given by the following important result, which comes with an appropriately honorific title:

Theorem 6.2 *(***The first fundamental theorem of Calculus***) Let f be an integrable function on $[a,b]$, and let A be the function defined by (6.1.1). Pick any point c in (a,b), and suppose that f is continuous at c. Then A is differentiable there and moreover,*

$$A'(c) \;=\; f(c).$$

Some would write this symbolically as

(6.1.3) $$\frac{d}{dx}\int_a^x f(t)dt \;=\; f(x).$$

In plain words, this says that *differentiating the integral gives back the original* as long as the original function is continuous at the point in question.

Proof. To know if $A(x)$ is differentiable at c, we need to evaluate the limit

$$L = \lim_{h \to 0} \frac{A(c+h) - A(c)}{h}. \tag{6.1.4}$$

By the *additivity* of the definite integral, we have

$$A(c+h) - A(c) = \int\limits_{I(c,h)} f(x)dx, \tag{6.1.5}$$

where $I(c,h)$ is as in the proof of Lemma 6.1.

Denote by $M(c,h)$, resp. $m(c,h)$, the *supremum*, resp. *infimum*, of the values of f over $I(c,h)$. Then the following bounds evidently hold:

$$hm(c,h) \leq \int\limits_{I(c,h)} f(x)dx \leq hM(c,h). \tag{6.1.6}$$

Combining (6.1.4), (6.1.5) and (6.1.6), we get for all small h,

$$\lim_{h \to 0} m(c,h) \leq L \leq \lim_{h \to 0} M(c,h). \tag{6.1.7}$$

But by hypothesis, f is continuous at c. Then both $m(c,h)$ and $M(c,h)$ will tend to $f(c)$ as h goes to 0, which proves the Theorem in view of (6.1.7) and the squeeze theorem.

(Draw pictures to convince yourselves that if f is not continuous, then these limits, even if they exist, need not equal $f(c)$.)

$\square$

Let f be any function on an open interval I. Suppose there is a differentiable function ϕ on I such that $\phi'(x) = f(x)$ for all x in I. Then we will call ϕ a **primitive** of f on I. Note that the primitive is not unique. Indeed, for any constant α, the function $\phi + \alpha$ will have the same derivative as ϕ. Intuitively, one feels immediately that the notion of a primitive should be tied up with the notion of an integral. The following very important and oft-used result makes this expected relationship precise.

Theorem 6.3 *(***The second fundamental theorem of Calculus***) Suppose f, ϕ are functions on $[a,b]$, with f integrable on $[a,b]$ and ϕ a primitive of f on (a,b), with ϕ defined and continuous at the endpoints a, b. Then*

$$\phi(b) - \phi(a) \;=\; \int_a^b f(x)dx.$$

One can rewrite this, perhaps more expressively, as

$$\phi(b) - \phi(a) \;=\; \int_a^b \frac{d}{dx}\phi dx.$$

Proof. Choose any partition

$$P:\; a = t_0 < t_1 < \ldots < t_n = b,$$

and set, for each $j \in \{1, 2, \ldots, n\}$,

$$M_j \;=\; \sup(f([t_{j-1}, t_j])) \quad \text{and} \quad m_j \;=\; \inf(f([t_{j-1}, t_j])). \tag{6.1.8}$$

By definition,

$$\sum_{j=1}^{n} (t_j - t_{j-1})m_j \;\leq\; \int_a^b f(x)dx \;\leq\; \sum_{j=1}^{n} (t_j - t_{j-1})m_j. \tag{6.1.9}$$

On the other hand, the **Mean Value Theorem** gives us, for each j, a number c_j in $[t_{j-1}, t_j]$ such that

$$\phi'(c_j) \;=\; \frac{\phi(t_j) - \phi(t_{j-1})}{t_j - t_{j-1}}. \tag{6.1.10}$$

Since ϕ is by hypothesis the *primitive* of f on (a,b), $f(c_j) = \phi'(c_j)$ for each j. Moreover,

$$m_j \;\leq\; f(c_j) \;\leq\; M_j, \tag{6.1.11}$$

and

$$\sum_{j=1}^{n} \phi(t_j) - \phi(t_{j-1}) \;=\; \phi(b) - \phi(a). \tag{6.1.12}$$

Combining (6.1.10), (6.1.11) and (6.1.12), we obtain

$$\sum_{j=1}^{n}(t_j - t_{j-1})m_j \;\le\; \phi(b) - \phi(a) \;\le\; \sum_{j=1}^{n}(t_j - t_{j-1})m_j. \tag{6.1.13}$$

Since (6.1.9) and (6.1.13) hold for *every* partition P, and since f is integrable on $[a, b]$, the assertion of the Theorem follows.

□

6.2 The indefinite integral

Suppose ϕ is a **primitive** of a function f on an open interval I, i.e., which yields f back upon differentiation. It is not unusual to set, following Leibnitz,

$$\int f(x)dx \;=\; \phi(x). \tag{6.2.1}$$

This is called an **indefinite integral** because there are no limits and ϕ is non-unique. So one can think of such an indefinite integral as a function of x which is unique only up to addition of an arbitrary constant. One has, in other words, an equality for all scalars C

$$\int f(x)dx \;=\; \int f(x)dx + C.$$

It could be a bit unsettling to work with such an indefinite, nebulous function at first, but one learns soon enough that it is a useful concept to be aware of.

In many Calculus texts one finds formulas like

$$\int \cos x dx \;=\; \sin x + C$$

and

$$\int \frac{1}{x}dx \;=\; \log x + C.$$

All they mean is that $\sin x$ and $\log x$ are the primitives of $\cos x$ and $\frac{1}{x}$, i.e.,

$$\frac{d}{dx}\sin x \;=\; \cos x$$

and

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

Of course the situation is completely different in the case of definite integrals.

6.3 Integration by substitution

There are a host of techniques which are useful in evaluating various definite integrals. We will single out two of them in this chapter and analyze them. The first one is the **method of substitution**, which one should always try first before trying others.

Theorem 6.4 *Let $[a,b]$ be a closed interval and g a function differentiable on an open interval containing $[a,b]$, with g' continuous on $[a,b]$. Also let f be a continuous function on $g([a,b])$. Then we have the identity*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. Let ϕ denote a primitive of f, which exists because the continuity assumption on f makes it integrable on $[a,b]$. Then we have, by the second fundamental theorem of Calculus,

$$\int_{g(a)}^{g(b)} f(u)du = \phi(g(b)) - \phi(g(a)) = (\phi\circ g)(b) - (\phi\circ g)(a). \tag{6.3.1}$$

On the other hand, by the *chain rule* applied to the composite function $\phi\circ g$, we have

$$(\phi\circ g)'(x) = (\phi'\circ g)(x)\cdot g'(x) = (f\circ g)(x)\cdot g'(x). \tag{6.3.2}$$

Consequently,

$$\int_a^b f(g(x))g'(x)dx = \int_a^b (\phi\circ g)'(x)dx. \tag{6.3.3}$$

Applying the second fundamental theorem of Calculus again, the right hand side of (6.3.3) is the same as

$$(\phi \circ g)(b) - (\phi \circ g)(a). \tag{6.3.4}$$

The Theorem now follows by combining (6.3.1), (6.3.3) and (6.3.4).

□

Before giving some examples let us note that powers of $\sin x$ and $\cos x$, as well as polynomials, are differentiable on $\mathbb{R}$ with continuous derivatives. In fact we can differentiate them any number of times; one says they are *infinitely differentiable*. The same holds for ratios of such functions or their combinations, as long as the denominator is non-zero in the interval of interest.

Examples: (**1**) Let

$$I = \int_0^{\pi/2} sin^3 x \cos x dx.$$

Thanks to the remark above on the *infinite differentiability* of the functions in the *integrand*, we are allowed to apply Theorem 6.4 here, with

$$g(x) = \sin x \quad \text{and} \quad f(u) = u^3.$$

Then, since $g'(x) = \cos x$ (as proved earlier, $g(0) = 0$ and $g(\pi/2) = 1$, we obtain

$$I = \int_0^1 u^3 du = \frac{1}{4}.$$

(**2**) Put

$$I = \int_0^{\pi/4} \cos^2 x dx.$$

Recall that

$$\cos^2 x - \sin^2 x = \cos 2x.$$

Since $\cos^2 x + \sin^2 x = 1$, we get

$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

Using this and the easy integral $\int_0^{\pi/4} dx = \pi/4$, we get

$$I = \frac{\pi}{8} + J, \quad \text{with} \quad J = \frac{1}{2}\int \cos 2x dx.$$

Put $g(x) = 2x$ and $f(u) = \cos u$, which are both infinitely differentiable on all of $\mathbb{R}$, and use Theorem 6.4 to conclude that, since $g'(x) = 2$, $g(0) = 0$ and $g(\pi/4) = \pi/2$,

$$J = \frac{1}{4}\int_0^{\pi/4} \cos u du = \frac{\sin(\pi/4) - \sin 0)}{4} = \frac{1}{4\sqrt{2}},$$

This implies that

$$I = \frac{\pi}{8} + \frac{1}{4\sqrt{2}} = \frac{\pi + \sqrt{2}}{8}.$$

(**3**) Evaluate

$$I = \int_0^1 \sqrt{1 - x^2} dx.$$

Here we use the substitution theorem in the reverse direction. The basic idea is that $\sqrt{1 - x^2}$ would simplify if x were $\sin t$ or $\cos t$. Put

$$g(t) = \sin t \quad \text{and} \quad f(u) = \sqrt{u}.$$

Then g is differentiable everywhere with $g'(t) = \cos t$ being continuous on $[0, 1]$. We chose the interval $[0, 1]$ because $g(0) = 0$ and $g(\pi/2) = 1$, giving us the limits of integration of I. Also, f is continuous on $g([0, \pi/2]) = [0, 1]$. (At the end point 0, the continuity of f means it is right continuous there. This is good, because f is not defined to the left of 0.) So we have satisfied all the hypotheses of Theorem 6.4 and we may apply it to get

$$I = \int_0^{\pi/2} f(g(t))g'(t)dt = \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\cos t dt.$$

But

$$\sqrt{1 - \sin^2 t} = \sqrt{\cos^2 t} = |\cos t|,$$

which is just $\cos t$, because the cosine function is non-negative in the interval $[0, \pi/2]$. Hence

$$I = \int_0^{\pi/2} \cos^2 t dt.$$

We just evaluated the integral of $\cos^2 t$ in the previous example, albeit with different limits. In any case, proceeding as in that example, we get

$$I = \frac{\pi}{4} + \frac{\sin(\pi/2) - \sin 0}{4} = \frac{\pi - 1}{4}.$$

6.4 Integration by parts

Some consider this the most important technique in Calculus. Its use is pervasive.

Theorem 6.5 *Let $[a, b]$ be a closed interval and let f, g be differentiable functions in an open interval around $[a, b]$ such that f', g' continuous on $[a, b]$. Then we have*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

The notation used above signifies

$$f(x)g(x)|_a^b := f(b)g(b) - f(a)g(a). \tag{6.4.1}$$

Proof. By the *product rule*,

$$(fg)'(x) = f(x)g'(x) + f'(x)g'(x) \tag{6.4.2}$$

for all x where f and g are both differentiable. Subtracting $f'(x)g(x)$ from both sides and integrating over $[a, b]$ we get

$$\int_a^b f(x)g'(x)dx = \int_a^b (fg)'(x)dx - \int_a^b f'(x)g(x)dx. \tag{6.4.3}$$

But by the second fundamental theorem of Calculus,

$$\int_a^b (fg)'(x)dx \;=\; (fg)(b) - (fg)(a). \tag{6.4.4}$$

The assertion now follows by combining (6.4.3) and (6.4.4).

$\square$

An Example: Evaluate, for *any* integer $n \geq 0$,

$$I_n \;=\; \int_0^{\pi/2} \cos^n x dx.$$

First note that

$$I_0 \;=\; \int_0^{\pi/2} 1 \cdot dx \;=\; \frac{\pi}{2}$$

and

$$I_1 \;=\; \int_0^{\pi/2} \cos x dx \;=\; \sin(\pi/2) - \sin 0 \;=\; 1,$$

because $\sin(\pi/2) = 1$ and $\sin 0 = 0$.

We have already solved the $n = 2$ case in the previous section using substitution, but we will not use it here. Integration by parts is more powerful!

So we may suppose that $n > 1$. Put

$$f(x) \;=\; \cos^{n-1} x \quad \text{and} \quad g(x) \;=\; \sin x.$$

Then, as noted earlier, f and g are infinitely differentiable on all of $\mathbb{R}$, with

$$f'(x) \;=\; (n-1)\cos^{n-2} x \cdot (-\sin x) \quad \text{and} \quad g'(x) \;=\; \cos x,$$

where the first formula comes from the chain rule. Now we may apply Theorem 6.5 and obtain

$$I_n \;=\; \left(\cos^{n-1}(x)\sin x\right)\big|_0^{\pi/2} - (n-1)\int_0^{\pi/2} cos^{n-2}x(-\sin x)\cdot \sin x dx.$$

Note that $n - 1 \neq 0$ as $n > 1$ and $\cos\theta \sin\theta$ is 0 if θ is 0 or $\pi/2$. Therefore the first term on the right is zero, and we get

$$I_n = (n-1) \int_0^{\pi/2} \cos^{n-2} x \sin^2 x dx.$$

Since $\sin^2 x = 1 - \cos^2 x$, we get

$$I_n = (n-1)I_{n-2} + (n-1)I_n,$$

which translates into the neat **recursive relation**

$$I_n = \frac{n-1}{n} I_{n-2}.$$

In particular,

$$I_2 = \frac{1}{2} I_0 = \frac{\pi}{4},\ I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}, \ldots$$

and

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3},\ I_5 = \frac{4}{5} I_3 = \frac{8}{15}.$$

It will be left as a nice exercise for the reader to find closed expressions for I_{2n} and I_{2n-1}.

6.5 Polar coordinates

So far we have confined ourselves to **rectilinear coordinates** on the plane, which are often called **Cartesian coordinates** to honor René Descartes who introduced them. Simply put, we identify each point P on the plane by the pair (x, y), where x (resp. y) is the distance between the origin O and the point where the x-axis (resp. y-axis) meets the perpendicular to it from P. Instead one can look at the pair (r, θ), where r is the distance from P to the origin, measured on the line L connecting O to P, and the angle between the x-axis and L, measured in the counterclockwise direction. By definition $r \geq 0$, and we will take θ to lie in $[0, 2\pi)$. In particular, the angle is taken to be zero, and not 2π or 4π or -2π, for any point lying on the positive x-axis. It should also be noted that as defined, the angle does not make much sense for the origin; we take (r, θ) to be $(0, 0)$ for it.

The quantities r, θ are called the **polar coordinates** of P. It is easy to see that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{6.5.1}$$

In the reverse direction, one can almost recover (r,θ) from (x,y) by the easily verified formulae

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}. \tag{6.5.2}$$

We said *almost*, because we do not know at the moment to what extent $\tan\theta$ determines θ. We will come back to this question after studying inverse functions.

Sometimes the equation defining a curve, or a region, in the plane becomes simpler when we use polar coordinates, and this is the reason for studying them. For example, the circle centered at O defined by the equation $x^2+y^2 = a^2$ can be easily described as the graph of $r = a$. The region inside the circle is simply $r \leq a$. The rule of thumb is that whenever there is **circular symmetry** in a given situation, it is better to use polar coordinates.

Suppose S is an **angular sector**, i.e., the region bounded by $\theta = a, \theta = b$ and $r = \rho$, with $b - a \in [0, 2\pi]$. ρ is called the **radius**, and $b - a$ the **angle**, of s. Using the definition of π as the area of the region inside the unit circle, one can show

$$A(S) = \frac{1}{2}(b-a)\rho^2. \tag{6.5.3}$$

We will accept this as a basic fact.

The main problem here will be to understand the **radial sets**, and to know when they are measurable. Such a set is given as the region bounded by $\theta = a, \theta = b$ and $r = f(\theta)$, where f is a function of $[a,b]$ and $b-a \in [0, 2\pi]$. One can use Calculus to find the area of R under a hypothesis on f.

Let us call a function f on $[a,b]$ **square-integrable** iff f^2 is integrable on $[a,b]$. Note that every continuous function on $[a,b]$ is square-integrable.

Proposition 1 *Let R be a radial set bounded by $\theta = a, \theta = b$ and $r = f(\theta)$, where f is a square-integrable function of $[a,b]$ and $b - a \in [0, 2\pi]$. Then*

$$A(R) = \frac{1}{2}\int_a^b f(\theta)^2 d\theta.$$

Proof. Pick any partition

$$P : a = t_0 < t_1 < \ldots < t_n = b.$$

For every integer j with $1 \leq j \leq n$, write

$$m_j = \inf f([t_{j-1}, t_j]) \quad \text{and} \quad M_j = \sup f([t_{j-1}, t_j]), \tag{6.5.4}$$

and denote by S_j, resp. S_j', the angular sector of radius m_j, resp. M_j, between $\theta = t_{j-1}$ and $\theta = t_j$. Then

$$\frac{1}{2}L(f^2, P) = \frac{1}{2}\sum_{j=1}^{n}(T_j - t_{j-1})m_j^2 = \sum_{j=1}^{n} A(S_j) \tag{6.5.5}$$

and

$$\frac{1}{2}U(f^2, P) = \frac{1}{2}\sum_{j=1}^{n}(T_j - t_{j-1})M_j^2 = \sum_{j=1}^{n} A(S_j').$$

It follows that

$$\frac{1}{2}L(f^2, P) \leq A(R) \leq \frac{1}{2}U(f^2, P). \tag{6.5.6}$$

Since f^2 is integrable on $[a, b]$ by hypothesis, the upper sums and lower sums (of f^2) converge to a common limit, which is the integral of f^2 over $[a, b]$. Now the assertion of the Proposition 1 follows by virtue of (6.5.6).

□

As an **example**, let us look at the region R bounded by the **spiral of Archimedes**:

$$0 \leq \theta \leq 2\pi, \; r = \theta.$$

Since $f(\theta) = \theta$ is square-integrable we may apply Proposition 1 and deduce that

$$A(R) = \frac{1}{2}\int_0^{2\pi} \theta^2 d\theta = \frac{1}{2}\frac{(2\pi)^3}{3} = \frac{4\pi^3}{3}.$$

Remark: It is more subtle to find the **arc length** of curves in the plane, and this will be treated in Ma1c.

7 Improper Integrals, Exp, Log, Arcsin, and the Integral Test for Series

We have now attained a good level of understanding of integration of nice functions f over closed intervals $[a, b]$. In practice one often wants to extend the domain of integration and consider unbounded intervals such as $[a, \infty)$ and $(-\infty, b]$. The simplest non-trivial examples are the **infinite trumpets** defined by the areas under the graphs of x^t for $t > 0$, i.e., the *improper integrals*

$$A_t = \int_1^\infty \frac{1}{x^t} dx.$$

We will see below that A_t has a well defined meaning if $t > 1$, but becomes unbounded for $t \leq 1$.

One is also interested in integrals of functions f over finite intervals with f being unbounded. The natural examples are given (for $t > 0$) by

$$B_t = \int_0^1 \frac{1}{x^t} dx.$$

Here it turns out that B_t is well defined, i.e., has a finite value, if and only if $t < 1$. In particular, neither A_1 nor B_1 makes sense. Moreover,

$$A_t + B_t = \int_0^\infty \frac{1}{x^t} dx$$

is unbounded for every $t > 0$.

7.1 Improper Integrals

Let f be a function defined on the interior of a possibly infinite interval J such that either its upper endpoint – call it b, is ∞ or f becomes unbounded as one approaches b. But suppose that the lower endpoint – call it a, is finite and that $f(a)$ is defined. In the former case the interval is unbounded, while in the latter case the interval is bounded, but the function is unbounded. We

will say that the **integral of f over J** exists iff the following two conditions hold:
(7.1.1)

(i) For every $u \in (a,b)$, f is integrable on $[a,u]$; and

(ii) the limit

$$\lim_{u \to b,\, u<b} \int\limits_a^u f(x)dx$$

exists.

When this limit exists, we will call it the **integral of f over J** and write it symbolically as

$$\int\limits_a^b f(x)dx. \tag{7.1.2}$$

Sometimes we will also say that the integral (7.1.2) converges when it makes sense.

Similarly, if a is either $-\infty$ or is a finite point where f becomes unbounded, but with b a finite point where f is defined, one sets

$$\int\limits_a^b f(x)dx = \lim_{u \to a,\, u>a} \int\limits_u^b f(x)dx \tag{7.1.3}$$

when the limit on the right makes sense.

Lemma 7.1 *We have*

$$\int\limits_a^\infty f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^\infty f(x)dx \quad \forall\, c \in (a,\infty)$$

and

$$\int\limits_{-\infty}^b f(x)dx = \int\limits_{-\infty}^c f(x)dx + \int\limits_c^b f(x)dx \quad \forall\, c \in (-\infty,b),$$

whenever the integrals make sense.

Proof. We will prove the first **additivity formula** and leave the other as an exercise for the reader. Pick any $c \in (a, \infty)$. or all real numbers $u > c$, we have by the usual additivity formula,

$$\int_a^u f(x)dx = \int_a^c f(x)dx + \int_c^u f(x)dx.$$

Thus

$$\int_a^\infty f(x)dx = \lim_{u\to\infty} \left(\int_a^c f(x)dx + \int_c^u f(x)dx \right),$$

which equals

$$\int_a^c f(x)dx + \lim_{u\to\infty} \int_c^u f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx.$$

$\square$

If J is an interval with both of its endpoints being problematic, we will choose a point c in (a, b) and put

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{7.1.4}$$

if both the improper integrals on the right make sense. One can check using Lemma 7.1 above that this definition is independent of the choice of c.

When an improper integral does not make sense, we will call it **divergent**. Otherwise it is **convergent**.

Proposition 1 *Let t be a positive real number. Then for $t > 1$,*

$$\int_1^\infty \frac{1}{x^t}dx = \frac{1}{t-1}, \tag{A}$$

with the improper integral on the left being divergent for $t \leq 1$.

On the other hand, if $t \in (0,1)$*,*

$$(B) \qquad \int_0^1 \frac{1}{x^t} dx \;=\; \frac{1}{1-t},$$

with the improper integral on the left being divergent for $t \geq 1$*.*

Proof. We learnt in section 5.4 that for all $a, b \in \mathbb{R}$ with $a < b$, and for all $t \neq 1$:

$$\int_a^b \frac{1}{x^t} dx \;=\; \frac{b^{1-t} - a^{1-t}}{1-t}.$$

Hence for $t > 1$,

$$\lim_{u\to\infty} \int_1^u \frac{1}{x^t} dx \;=\; \lim_{u\to\infty} \frac{1}{(1-t)u^{t-1}} + \frac{1}{t-1} \;=\; \frac{1}{t-1},$$

because the term $\frac{1}{u^{t-1}}$ goes to zero as u goes to ∞. If $t < 1$, this term goes to ∞ as u goes to ∞, and so the integral is divergent. Finally let $t = 1$. Then we cannot use the above formula. But for each $N \geq 1$, we have the inequality

$$\sum_{n=1}^{N} \frac{1}{n} \;\leq\; \int_1^N \frac{1}{x} dx.$$

The reason is that the sum on the left is a lower Riemann sum for the function $f(x) = \frac{1}{x}$ over the interval $[1, N]$ relative to the partition $P:\ 1 < 2 < \ldots < N$. So, if the improper integral of this function exists over $[1, \infty)$, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \;=\; \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n}$$

must converge. But we have seen in Chapter 2 that this series diverges. So the integral is divergent for $t = 1$, and (A) is proved in all cases.

The proof of (B) is very similar and will be left as an exercise.

□

7.2 Inverse functions

Suppose f is a function with domain X and image (or range) Y. By definition, given any x in X, there is a **unique** y in Y such that $f(x) = y$. But this definition of a function is not egalitarian, because it does not require that a *unique* number x be sent to y by f; so y is special but x is not. A really nice kind of a function is what one calls a **one-to-one** (or **injective**) function. By definition, f is such a function iff

$$f(x) = f(x') \implies x = x'. \tag{7.2.1}$$

In such a case, we can define an **inverse function** g with domain Y and range X, given by

$$g(y) = x \quad \text{iff} \quad f(x) = y. \tag{7.2.2}$$

Clearly, when such an inverse function g exists, one has

$$g \circ f = 1_X \quad \text{and} \quad f \circ g = 1_Y, \tag{7.2.3}$$

where 1_X, resp. 1_Y, denotes the **identity function** on X, resp. Y.

We will be concerned in this chapter with X, Y which are subsets of the real numbers.

Proposition 2 *Let f be a one-to-one function with domain $X \subset \mathbb{R}$ and range Y, with inverse g. Suppose in addition that f is differentiable at x with $f'(x) \neq 0$. Then g is differentiable at $y = f(x)$ and we have*

$$g'(y) = \frac{1}{f'(x)}$$

for all x in X with $y = f(x)$.

Note that if we know *a priori* that f and g are both differentiable, then this is easy to prove. Indeed, in that case their composite function $g \circ f$, which is the identity on X, would be differentiable. By differentiating the identity

$$g(f(x)) = x$$

with respect to x, and applying the *chain rule*, we get

$$g'(f(x)) \cdot f'(x) \;=\; 1,$$

because the derivative of x is 1. Done.

Proof. We have to compute the limit

$$L \;=\; \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}.$$

Since f is differentiable, it is in particular continuous, which implies that

$$\lim_{h' \to 0} f(x+h') \;=\; f(x).$$

So, if we set

$$h \;=\; f(x+h') - f(x),$$

then $h \to 0$ when $h' \to 0$. Writing $y = f(x)$, we then get, after applying g to $y + h = f(x+h')$,

$$g(y+h) \;=\; x + h',$$

or in other words,

$$h' \;=\; g(y+h) - g(y).$$

We claim that, since f is one-to-one, we must also have $h' \to 0$ when $h \to 0$. Suppose not. Then for some $\varepsilon > 0$, $|h'| = |g(y+h) - g(y)|$ is $\geq \varepsilon$ for all h close to 0. But this will lead to f sending two distinct numbers, with $x + h'$ being one of them, to the same number $y + h$; the other number will be close to x, of distance less than ε. This contradicts the fact that f is one-to-one. Hence the Claim.

Hence

$$L \;=\; \lim_{h' \to 0} \frac{h'}{f(x+h') - f(x)},$$

which is the inverse of $f'(x)$. It makes sense because $f'(x)$ is by assumption non-zero.

$\square$

Note that this proof shows that g is not differentiable at any point $y = f(x)$ if f' is zero at x.

It is helpful to note that many a function is not one-to-one in its maximal domain, but becomes one when restricted to a smaller domain. To give a simple example, the **squaring function**

$$f(x) \;=\; x^2$$

is defined everywhere on $\mathbb{R}$. But it is not one-to-one, because $f(a) = f(-a)$. However, if we restrict to the subset $\mathbb{R}_+$ of non-negative real numbers, f is one-to-one and so we may define its inverse to be the **square-root function**

$$g(y) \;=\; \sqrt{y}, \;\; \forall\, y \in \mathbb{R}_+.$$

Another example is provided by the sine function, which is periodic of period 2π and hence not one-to-one on $\mathbb{R}$. But it becomes one when restricted to $[-\pi/2, \pi/2]$.

7.3 The natural logarithm

For any $x > 0$, its natural logarithm is defined by the definite integral

$$\log x \;=\; \int_1^x \frac{dt}{t}. \tag{7.3.1}$$

Some write $\ln(x)$ instead, and some others write $\log_e x$. When $0 < x < 1$, this signifies the negative of integral of $\frac{1}{t}$ from x to 1. Consequently, $\log x$ is positive if $x > 1$, negative if $x < 1$ and equals 0 at $x = 1$.

Proposition 3 *(a)* $\log 1 \;=\; 0$.

(b) $\log x$ *is differentiable everywhere in its domain* $\mathbb{R}_+ = (0, \infty)$ *with derivative* $\frac{1}{x}$.

*(c) (***addition theorem***) For all* $x, y > 0$,

$$\log(xy) \;=\; \log x + \log y.$$

(d) $\log x$ *is a strictly increasing function.*

(e) $\log x$ *becomes unbounded in the positive direction when* x *goes to* ∞ *and it is unbounded in the negative direction when* x *goes to* 0*.*

(f) $\log x$ *is integrable on any finite subinterval of* $\mathbb{R}_+$*, and its indefinite integral is given by*

$$\int \log x \, dx \; = \; x \log x - x + C.$$

(g) $\log x$ *goes to* $-\infty$ *(resp.* ∞*) slowly when* x *goes to* 0 *(resp.* ∞*); more precisely,*

(i)
$$\lim_{x \to 0} x \log x \; = \; 0$$

and

(ii)
$$\lim_{x \to \infty} \frac{\log x}{x} \; = \; 0.$$

(h) The **improper integral** *of* $\log x$ *over* $(0, b]$ *exists for any* $b > 0$*, with*

$$\int_0^b \log x \, dx \; = \; b \log b - b.$$

Property (c) is very important, because it can be used to **transform multiplicative problems into additive ones**. This was the motivation for their introduction by **Napier** in 1616. Property (b) is also important. Indeed, if we assume only the properties (a),(c),(d),(e) for a function f on $\mathbb{R}_+$, there are lots of functions (*logarithms*) which satisfy these properties. But the situation becomes rigidified with a **unique solution** once one requires (b) as well. This is why log is called the **natural logarithm**. The other (unnatural) choices will be introduced towards the end of the next section.

Proof. (**a**): This is immediate from the definition.
(**b**): For any x, the function $\frac{1}{t}$ is continuous on $[0, x]$, hence by the *First Fundamental Theorem of Calculus*, $\log x$ is differentiable with derivative $1/x$.
(**c**): Fix any $y > 0$ and consider the function of x defined by

$$\ell(x) \; = \; \log(xy) \quad \text{on} \quad \{x > 0\}.$$

Since it is the composite of the differentiable functions $x \to xy$ and $u \to \log u$, ℓ is also differentiable. Applying the *chain rule*, we get

$$\ell'(x) \;=\; y\left(\frac{1}{yx}\right) \;=\; \frac{1}{x}.$$

Thus ℓ and log both have the same derivative, and so their difference must be independent of x. Write

$$\ell(x) \;=\; \log x + c.$$

Evaluating at $x = 1$, and noting that $\log 1 = 0$ and $\lambda(1) = \log x$, we see that c must be $\log x$, proving the *addition formula*.
(**d**): As we saw in Chapter 4, a differentiable function f is strictly increasing iff its derivative is positive everywhere. When $f(x) = \log x$, the derivative, as we saw above, is $1/x$, which is positive for $x > 0$. This proves (d).
(**e**): As $\log x$ is strictly increasing and since it vanishes at 1, its value at any number $x_0 > 1$, for instance at $x_0 = 2$, is positive. By the addition theorem and induction, we see that for any positive integer n,

(7.3.2) $$\log(x_0^n) \;=\; n \log x_0.$$

Consequently, as n goes to ∞, $\log(x_0^n)$ goes to ∞ as well. This proves that $\log x$ is unbounded in the positive direction. For the negative direction, note that for any positive $x_1 < 1$, $\log x_1$ is negative (since $\log 1 = 0$ and $\log x$ is increasing). Applying (8,2,3) with x_0 replaced by x_1, we deduce that $\log(x_1^n)$ goes to $-\infty$ as n goes to ∞. Done.
(**f**) Since $\log x$ is continuous, it is integrable on any finite interval in $(0, \infty)$. Moreover, by *integration by parts*,

$$\int \log x \, dx \;=\; x \log x - \int x \frac{d}{dx}(\log x) dx.$$

The assertion (f) now follows since the expression on the right is $x \log x - x + C$.
(**g**): Put

$$L \;=\; \lim_{x \to 0} x \log x$$

and

$$u \;=\; \frac{1}{x}, \; f(u) \;=\; -\log\left(\frac{1}{u}\right), \quad \text{and} \quad g(u) \;=\; u.$$

Then we have
$$L = -\lim_{u\to\infty} \frac{f(u)}{g(u)}.$$
Then by (e), $f(u)$ and $g(u)$ approach ∞ as u goes to ∞, and both these functions are differentiable at any positive u. (All one needs is that they are differentiable for large u.) Since $u'(x) = -\frac{1}{x^2} = -u^2$ and $f(u) = -\log x$, we have by the *chain rule*,
$$f'(u) = \frac{df/dx}{du/dx} = \frac{-1/x}{-u^2} = \frac{1}{u}.$$
Since $g'(u) = 1$ for all u, we then get
$$\lim_{u\to\infty} \frac{f'(u)}{g'(u)} = 0.$$
So we may apply L'Hopital's rule (see the **Appendix** to this chapter), and conclude that
$$L = -\lim_{u\to\infty} \frac{f'(u)}{g'(u)} = 0,$$
giving (i). The proof of (ii) is similar and will be left for the reader to check.
(**h**): The improper integral of $\log x$ exists over $(0, b]$ iff the following limit exists:
$$L = \lim_{x\to 0} \int_x^b \log t\, dt.$$
Thanks to (f), we have
$$\int_x^b \log t\, dt = (t\log t - t)|_x^b = (b\log b - b) - x\log x + x.$$
To prove (h) we need to show that
$$\lim_{x\to 0} (x\log x - x) = 0,$$
which is a consequence of (g).

□

Remark on infinite products: Suppose $x_1, x_2, \ldots, x_n, \ldots$ is an infinite sequence of positive real numbers. Often we will want to consider infinite products of the form

$$P := \prod_{n=1}^{\infty} x_n,$$

which may not in general converge to a finite real number. Since the logarithm function converts multiplication into addition, it is convenient to consider the log of this product P, and consider the infinite series

$$\log P := \sum_{n=1}^{\infty} \log(x_n).$$

We will say that P converges when $\log P$ does.

7.4 The exponential function

In view of the discussion in section 7.2, and the fact (see parts (d), (e) of Proposition 3) that $\log x$ is a strictly increasing function with domain $\mathbb{R}_+$ and range $\mathbb{R}$, we can define the **exponential function**, $\exp(\mathbf{x})$ for short, to be the inverse function of $\log x$. Note that $\exp(x)$ has domain $\mathbb{R}$ and range $\mathbb{R}_+$.

Proposition 4 *(a)* $\exp(0) = 1$.

(b) $\exp(x)$ *is differentiable everywhere in* $\mathbb{R}$ *with derivative* $\exp(x)$.

*(c) (***addition theorem***) For all* $x, y \in \mathbb{R}$,

$$\exp(x+y) = \exp(x)\exp(y).$$

(d) $\exp(x)$ *is a strictly increasing function.*

(e) $\exp(x)$ *becomes unbounded in the positive direction when* x *goes to* ∞ *and it goes to* 0 *when* x *goes to* $-\infty$.

(f) $\exp(x)$ *is integrable on any finite subinterval of* $\mathbb{R}_+$*, and its indefinite integral is given by*

$$\int \exp(x)\,dx \;=\; \exp(x) + C.$$

(g) As x *goes to* ∞ *(resp.* $-\infty$*),* $\exp(x)$ *goes to* ∞ *(resp.* 0*) faster than any polynomial* $p(x)$ *goes to* ∞ *(resp.* $-\infty$*), i.e.,*

(i)
$$\lim_{x\to\infty} \frac{p(x)}{\exp(x)} \;=\; 0$$

and

(ii)
$$\lim_{x\to-\infty} p(x)\exp(x) \;=\; 0.$$

(h) The **improper integral** *of* $\exp(x)$ *over* $(-\infty, b]$ *exists for any* $b > 0$*, with*

$$\int_{-\infty}^{b} \exp(x)\,dx \;= \exp(b).$$

Proof. (**a**): Since $\log 1 = 0$, and as $\exp \circ \log$ is the identity function, we see that

$$\exp(0) \;=\; \exp(\log 1) \;=\; 1.$$

(**b**): Note that the derivative $\frac{1}{x}$ of $\log x$ is nowhere zero on its domain $\mathbb{R}_+$. So we may apply Proposition 2 with $f = \log$, $g = \exp$ and $y = \log x$ to get the everywhere differentiability of exp, with

$$\frac{d}{dy}\exp(y) \;=\; \frac{1}{(\log x)'} \;=\; \frac{1}{1/x} \;=\; x \;=\; \exp(y). \tag{7.4.1}$$

(**c**): Fix x, y in $\mathbb{R}$. Then we can find (unique) u, v in $\mathbb{R}_+$ such that

$$x \;=\; \log u \quad \text{and} \quad y \;=\; \log v.$$

Applying part (c) of Proposition 3, we then obtain

$$\exp(x+y) \;=\; \exp(\log u + \log v) \;=\; \exp(\log(uv)) \;=\; uv.$$

The assertion follows because $u = \exp(x)$ and $v = \exp(y)$.
(**d**): Let $x > y$ be arbitrary in $\mathbb{R}$. Write $x = \log u$, $y = \log v$ as above. Since log is strictly increasing, we need $u > v$. Since $u = \exp(x)$ and $v = \exp(v)$, we are done.
(**e**): Suppose $\exp(x)$ is bounded from above as x goes to infinity, i.e., suppose there is a positive number M such that $\exp(x) < M$ for all $x > 0$. Then, since log is a strictly increasing function, $x = \log(\exp(x))$ would be less than $\log(M)$ for all positive x, which is absurd. So $\exp(x)$ must be unbounded as x becomes large. To show that $\exp(x)$ approaches 0 as x goes to $-\infty$, we need to show that for any $\varepsilon > 0$, there exists $-T < 0$ such that

$$\exp(x) \, < \, \varepsilon \quad \text{whenever} \quad x < -T. \tag{7.4.2}$$

Applying the logarithm to both inequalities, writing $-T = \log u$ for a unique $u \in (0, 1)$ and $\varepsilon' = \exp(\varepsilon)$, and using the fact that log is a one-to-one function, we see that (7.4.2) is equivalent to the statement

$$x \, < \, \varepsilon' \quad \text{whenever} \quad \log x < \log u, \tag{7.4.3}$$

which evidently holds by the properties of the logarithm. Hence $\exp(x)$ goes to 0 as x goes to $-\infty$.
(**f**) Since $\exp(x)$ is continuous, it is integrable on any finite subinterval of $\mathbb{R}$. In fact, since $\exp(x)$ is its own derivative, it is its own **primitive** as well, proving the assertion.
(**g**): To prove (i), it suffices to show that for any $j \geq 0$, the limit

$$L \, = \, \lim_{x \to \infty} \frac{x^j}{\exp(x)}$$

exists and equals 0. When $j = 0$ this follows from (e), while the assertion for $j > 1$ follows from the case $j = 1$, which we will assume to be the case from here on. It then suffices to show that the function $\exp(x)/x$ goes to ∞ as x goes to ∞. By taking logarithms, this becomes equivalent to the statement that $x/\log x$ goes to ∞ as $\log x$, and hence x, goes to ∞, which is what we proved in our proof of the limit (i) of Proposition 3, part (g). (One can also apply, with care, L'Hopital's rule directly to the quotient $x/\exp(x)$ and thereby establish (i).) The proof of (ii) is similar and will be left for the reader to check.

(**h**): The improper integral of $\exp(x)$ exists over $(-\infty, b]$ iff the following limit exists:

$$L = \lim_{x \to -\infty} \int_x^b \exp(t)dt.$$

Thanks to (f), we have

$$\int_x^b \exp(t)\, dt = \exp(b) - \exp(x).$$

To prove (h) we need only show that

$$\lim_{x \to -\infty} \exp(x) = 0,$$

which is a consequence of (e).

□

How many differentiable functions are there which are derivatives (and hence primitives) of themselves? This is answered by the following

Lemma 7.2 *Let f be any differentiable function on an open interval (a, b), possibly of infinite length, satisfying $f'(x) = f(x)$ for all x in (a, b). Then there exists a scalar c such that*

$$f(x) = c \exp(x)\ \forall\, x \in (a, b).$$

Moreover, if $f(0) = 1$, then $f(x)$ equals $\exp(x)$ on this interval.

Proof. Define a function h on (a, b) by

$$h(x) = \frac{f(x)}{\exp(x)},$$

which makes sense because exp never vanishes anywhere.

Since f and exp are differentiable, so is h, and by the *quotient rule* we have

$$h'(x) = \frac{f'(x)\exp(x) - f(x)\exp'(x)}{\exp(x)^2} = \frac{f(x)\exp(x) - f(x)\exp(x)}{\exp(x)^2} = 0,$$

because f and exp are derivatives of themselves. Therefore $h(x)$ must be a constant c, say. The Lemma now follows.

$\square$

Definition *Put*

$$e = \exp(1).$$

Equivalently, e is the unique, positive real number whose natural logarithm is 1. It is not hard to see that, to a first approximation, $2 < e < 4$. One can do much better with some work, of course, and also show that e is **irrational**, even **transcendental**.

A natural question now arises. We have worked with the power function a^x before, for any positive real number a, which satisfies the same addition rule as $\exp x$, i.e.,

$$a^{x+y} = a^x \cdot a^y. \tag{7.4.4}$$

. What is the relationship between e^x and $\exp(x)$. The answer is very satisfying.

Proposition 5 *For all x in $\mathbb{R}$, we have*

$$\exp(x) = e^x.$$

We will need the following:

Lemma 7.3 $\log(e^x) = x$, *for all* $x \in \mathbb{R}$.

Proof. This is clear for x an integer m. Since we have for any $n > 0$,

$$1 = \log(e) = \log((e^{1/n})^n) = n\log(e^{1/n}),$$

we deduce that $\log(e^{1/n}) = 1/n$. Consequently,

$$\log(e^{m/n}) = \frac{m}{n}, \ \forall\ m/n \in \mathbb{Q}.$$

For any real number x, e^x is defined to be the limit $\lim_{\alpha_n \to x} e^{\alpha_n}$, where α_n is a sequence of rational numbers α_n coverging to x. Since $u \to \log(u)$ is a continuous function, we get

$$\log(\lim_{\alpha_n \to x} e^{\alpha_n}) = \lim_{\alpha_n \to x} \log(e^{\alpha_n}) = \lim_{\alpha_n \to x} \alpha_n = x.$$

$\square$

Proof of Proposition 5. Applying Lemma 7.3, and using the fact that $\exp(x)$ is the inverse function of logarithm, we get

$$\log(\exp(x)) \;=\; x \;=\; \log(e^x),\ \forall\, x \in \mathbb{R}.$$

Since log is one-to-one, we must have $\exp(x) = e^x$. $\square$

Just like $\exp(x)$, the power function a^x is, for any positive number a, a strictly increasing function. One calls the inverse of a^x by the name **logarithm to the base** a and denotes it by $\log_a y$. Then $\log_a$ satisfies many of the properties of log, but its derivative is not $1/x$ (if $a \neq e$).

Proposition 6 *Fix $a > 0$. Then*

$$\frac{d}{dx}(a^x) \;=\; a^x \log a,$$

and

$$\frac{d}{dy}(\log_a y) \;=\; \frac{1}{y \log a}.$$

Proof. Note that

$$a^x \;=\; \exp(\log(a^x)) \;=\; e^{x \log a},$$

which is the composite of $x \to x \log a$ and $u \to e^u$. Applying the chain rule, we obtain

$$\frac{d}{dx}(a^x) \;=\; e^{x \log a} \cdot \log a \;=\; a^x \log a.$$

The formula for the derivative of $\log_a y$ follows from this and Proposition 2. $\square$

The basic **hyperbolic functions** are defined as follows:

$$\sinh x \;=\; \frac{e^x - e^{-x}}{2},\ \cosh x \;=\; \frac{e^x + e^{-x}}{2},\ \tanh(x) \;=\; \frac{\sinh x}{\cosh x}. \tag{7.4.5}$$

The other hyperbolic functions are given by the inverses of these.

7.5 arcsin, arccos, arctan, et al

The sine function, which is defined and differentiable everywhere on $\mathbb{R}$, is periodic with period 2π and is therefore not a one-to-one function. It however becomes injective when the domain is restricted to $[-\pi/2, \pi/2]$. One calls the corresponding inverse function the **arcsine** function, denoted $\arcsin x$. Clearly, the domain of $\arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. Moreover, since the derivative $\cos x$ of $\sin x$ is non-zero, in fact positive, in $(-\pi/2, \pi/2)$, we may apply Proposition 2 and deduce that the arcsine function is differentiable on $(-\pi/2, \pi/2)$, with

$$\frac{d}{dy}(\arcsin y) \;=\; \frac{1}{\cos x} \;=\; \frac{1}{\sqrt{1-\sin^2 x}} \;=\; \frac{1}{\sqrt{1-y^2}}, \tag{7.5.1}$$

where $y = \sin x$.

One notes similarly that $\cos x$, resp. $\tan x$, is one-to-one on $[0, \pi]$, resp. $(-\pi/2, \pi/2)$, and defines its inverse function $\arccos x$, resp. $\arctan x$, with domain $[-1, 1]$ and range $[0, \pi]$. Arguing as above, we see that $\arccos x$ is differentiable on $(0, \pi)$ and $\arctan x$ is differentiable on $(-\pi/2, \pi/2)$, with

$$\frac{d}{dy}(\arccos x) \;=\; -\frac{1}{\sqrt{1-y^2}} \tag{7.5.2}$$

and

$$\frac{d}{dy}(\arctan x) \;=\; \frac{1}{1+y^2} \tag{7.5.3}$$

Consequently, we obtain

$$\int \frac{dy}{1+y^2} \;=\; \arctan y + C \tag{7.5.4}$$

and

$$\int \frac{dy}{\sqrt{1-y^2}} \;=\; \arcsin y + C. \tag{7.5.5}$$

This incidentally brings to a close our quest to integrate arbitrary rational functions, which we began in the previous chapter, where we reduced the problem to the evaluation of the integral on the left of (7.5.4).

It should be notes that it is a miracle that we can evaluate the reciprocal of the square root of $1-y^2$ (for $-1 \leq y \leq 1$) in terms of $\arcsin y$. If one tried to integrate $\frac{1}{\sqrt{f(y)}}$ for a polynomial f of degree $n > 2$, the problem becomes forbiddingly difficult. Even for $n = 3$, one needs to use **elliptic functions**.

7.6 A useful substitution

A very useful substitution to deal with trigonometric integrals is to set

$$u = \tan(x/2), \tag{7.6.1}$$

which implies that

$$x = 2\arctan u.$$

Note that

$$\frac{du}{dx} = \frac{1}{2}\sec^2(x/2) = \frac{1}{2}(1+\tan^2(x/2)) = \frac{1+u^2}{2}, \tag{7.6.2}$$

$$dx = \frac{2}{1+u^2}du, \tag{7.6.3}$$

$$\sin x = 2\sin(x/2)\cos(x/2) = 2\frac{\tan(x/2)}{\sec^2(x/2)} = \frac{2u}{1+u^2}, \tag{7.6.4}$$

and since $\cos^2(x/2) + \sin^2(x/2) = 1$,

$$\cos x = \frac{\cos^2(x/2) - \sin^2(x/2)}{\cos^2(x/2) + \sin^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-u^2}{1+u^2}. \tag{7.6.5}$$

For example, suppose we have to integrate

$$I = \int \frac{dx}{1-\sin x}.$$

Using the substitution above, which is justifiable here, we get

$$I = \int \frac{1}{1-2u/(1+u^2)}\frac{2}{1+u^2}du = \int 2\frac{du}{1-2u+u^2}.$$

Since

$$\frac{1}{1-2u+u^2} = \frac{1}{(1-u)^2} = \frac{d}{du}\left(\frac{1}{1-u}\right),$$

we get

$$I = \int d\left(\frac{1}{1-u}\right) = \frac{1}{1-u} + C = \frac{1}{1-\tan(x/2)} + C\,.$$

The idea behind this important substitution due to the Nineteenth century German mathematician *Weierstrass* is that there are **two natural paramatrizations of the unit circle** in the plane (with center at the origin). The first way is the well known one of representing any point on this circle as $(\cos\theta, \sin\theta)$, with $\theta \in [0, 2\pi)$. The other one is to represent it as $(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2})$, which has the advantage that the *coordinates are rational functions of the free parameter u*. This plays a fundamental role, not just in Calculus, but also in the mathematical area called *Algebraic Geometry.*

7.7 The integral test for infinite series

When we discussed the question of convergence of infinite series in chapter 2, we gave various tests one could use for this purpose, at least for series with non-negative coefficients. Here is another test, which can at times be helpful.

Proposition 7 *Consider an infinite series*

$$S = \sum_{n=1}^{\infty} a_n,$$

whose coefficients satisfy

$$a_n = f(n),$$

for some non-negative, monotone decreasing function f on the infinite interval $[1, \infty)$. Then S converges iff the improper integral

$$I = \int_1^\infty f(x)dx$$

converges.

Proof. For any integer $N > 1$, consider the partition

$$P_N : \ 1 < 2 < \ldots < N$$

of the closed interval $[0, N]$. Then, since f is monotone decreasing, the upper and lower sums are given by

$$U(f, P_N) \ = \ a_1 + a_2 + \ldots + a_{N-1}$$

and

$$L(f, P_N) \ = \ a_2 + \ldots + a_{N-1} + a_N.$$

Suppose f is integrable over $[1, \infty)$. Then it is integrable over $[1, N]$ and

$$a_2 + a_3 + \ldots + a_N \ \leq \ \int\limits_1^N f(x)dx \ \leq \ a_1 + \ldots + a_{N-2} + a_{N-1}.$$

As N goes to infinity, this gives

$$S - a_1 \ \leq \ \int\limits_1^\infty f(x)dx,$$

which implies that S is convergent.

To prove the converse we need to be a bit more wily. Suppose S converges. Note that f is integrable over $[1, \infty)$ iff the series

$$T \ = \ \sum_{n=1}^\infty b_n$$

converges, where

$$b_n \ = \ \int\limits_n^{n+1} f(x)dx$$

But since f is monotone decreasing over each interval $[n, n+1]$, the area under the graph of f is bounded above (resp. below) by the area under the constant function $x \mapsto f(n) = a_n$ (resp. $x \mapsto f(n+1) = a_{n+1}$). Thus we have, for every $n \geq 1$,

$$a_{n+1} \ \leq \ b_n \ \leq \ a_n.$$

Summing from $n = 1$ to ∞ and using the comparison test (see chapter 2), we get

$$S - a_1 \leq T \leq S.$$

Thus T converges as well.

$\square$

As a consequence we see that for any positive real number t, the series

$$S_t = \sum_{n=1}^{\infty} \frac{1}{n^t}$$

converges iff the improper integral

$$I_t = \int_1^{\infty} x^{-t} dx$$

converges. We have already seen that I_t is convergent iff $t > 1$. So the same holds for S_t. But recall that in the special case $t = 1$,we deduced the divergence of I_1 from that of S_1.

Appendix: L'Hôpital's Rule

It appears that the most popular mathematician for Calculus students is **Marquis de L'Hôpital**, who was prolific during the end of the seventeenth century. Everyone likes to use his **rule**, but two things must be taken due note of. The first is that, as with any other theorem, one has to make sure that all the hypotheses hold before applying it. The second is a bit more subtle. One should not use it when it leads to a circular reasoning, for example when the numerator or the denominator of the limit L in question, which goes to 0 or ∞ as the case might be, is differentiable as needed, but to prove it one needs the limit L to exist in the first place. Here is an example to illustrate this point. Consider the following two statements:

(I) The function e^x is differentiable with derivative e^x.

(II) The limit

$$L = \lim_{t \to 0} \frac{e^t - 1}{t}$$

equals 1.

When presented with the limit L one is tempted to prove that it is 1 by using L'Hôpital's rule. Indeed, if we accept that it is applicable here, then, since e^t is by (I) differentiable with derivative e^t, and since the limit

$$L_1 = \lim_{t \to 0} \frac{e^t}{1}$$

equals $e^0 = 1$, one thinks that the problem is solved. But not so fast! This method assumes (I) and how does one prove it? Well, one has to show the following:

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x.$$

But since e^{x+h} is $e^x e^h$, one has to show that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1,$$

which is the assertion (II). So (I) and (II) are equivalent and one cannot prove one using the other, unless one has found a different way to prove one of them. So if one uses the L'Hoˆpital's rule to evaluate L, one has to show why e^x is differentiable without using L as a tool, which can be done. Of course one defines the exponential function $\exp(x)$ as the inverse function of the logarithm and the fact that the derivative of $\log x$ is $1/x$ implies, as we saw earlier, that $\exp(x)$ is differentiable with derivative $\exp(x)$. But we then have to show, by another method, that e^x is the same as $\exp(x)$. If one is not careful one will be drawn into a delicate spider web.

This is not to scare you into not using L'Hôpital's rule. Just make sure before using it that you can satisfy the hypotheses and that there are no circular arguments. Make sure, in particular, that the numerator and the denominator of the limit L can be shown to be differentiable without using L. Indeed, the most important thing one has to learn in Ma 1a is to **think logically**.

Without further ado, let us now present the rule of L'Hôpital.

Proposition 8 *(***L'Hôpital's rule***)* *Consider a limit of the form*

$$L = \lim_{x \to a} \frac{f(x)}{g(x)},$$

where a is either 0 or ∞ or $-\infty$ or just any (finite) non-zero real number, and f, g are differentiable functions at all real numbers x with $|x-a|$ sufficiently small. Suppose both and f and g approach 0, or both approach ∞, or both tend to $-\infty$, as x goes to a. Then L exists **if** *the* **limit quotient of the derivatives**, *namely*

$$L' = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists. Moreover, when L' exists, L equals L'.

Proof of L'Hôpital's Rule. We will prove this in the case when $a = \infty$, with $f(x), g(x)$ both approaching ∞ as x approaches ∞. The other cases require only very slight modifications and will be left as exercises for the interested reader.

By hypothesis, $f(x)$ and $g(x)$ are defined and differentiable for large enough x. Suppose the limit L' exists. We have to show that L also exists, and prove that in fact $L = L'$.

The existence of L' as a (finite) real number implies that for every $\varepsilon > 0$, there is some $b > 0$ such that for all $x > b$,

$$(A1) \qquad \left|\frac{f'(x)}{g'(x)} - L'\right| < \varepsilon.$$

Since $g(x)$ goes to ∞ as $x \to \infty$, we may choose b large enough so that $g(x) \neq g(b)$ for all $x > b$.

Applying Cauchy's MVT to (f, g) on $[b, x]$, we get

$$(A2) \qquad \frac{f'(c)}{g'(c)} = \frac{f(x)-f(b)}{g(x)-g(b)},$$

for some c in (b, x). Combining with (A1), we then get

$$(A3) \qquad \left|\frac{f(x)-f(b)}{g(x)-g(b)} - L'\right| < \varepsilon.$$

In other words,

$$(A4). \qquad \lim_{x\to\infty} \frac{f(x)-f(b)}{g(x)-g(b)} = L.$$

Now we are almost there. To finish, note that since $f(x) \to \infty$ as $x \to \infty$,

$$\lim_{x\to\infty} \frac{f(x)}{f(x) - f(b)} = 1. \tag{A5}$$

Similarly,

$$\lim_{x\to\infty} \frac{g(x) - g(b)}{g(x)} = 1. \tag{A6}$$

The assertion now follows by combining (A4), (A5) and (A6).

$\square$

8 Approximations, Taylor Polynomials, and Taylor Series

Polynomials are the nicest possible functions. They are easy to differentiate and integrate, which is also true of the basic trigonometric functions, but more importantly, polynomials can be evaluated at any point, which is not true for general functions. So what one does in practice is to approximate any function f of interest by polynomials. When the approximation is done by *linear polynomials*, then it is called a *linear approximation*, which pictorially corresponds to *linearizing* the graph of f. It turns out that the more times one can differentiate f, the higher is the degree of the polynomial one can approximate it with, and more importantly, the better the approximation becomes, as one sees it intuitively. There is only one main theorem here, due to Taylor, but it is omnipresent in all the mathematical sciences, with a number of ramifications, and should be understood precisely.

8.1 Taylor polynomials

Suppose f is an N-times differentiable function on an open interval I. Fix any point a in I. Then for any non-negative integer $n \leq N$, the **nth Taylor polynomial of f at $x = a$** is given by

$$p_n(f(x); a) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j, \tag{8.1.1}$$

where $f^{(j)}(a)$ denotes the jth derivative of f at a. By convention, $f^{(0)}(a)$ just denotes $f(a)$. (f is the 0th derivative of itself!)

The coefficients $\frac{f^{(j)}(a)}{j!}$ are called the **Taylor coefficients** of f at a.

The definition has been rigged so that the following holds:

Lemma 8.1 *Suppose f is itself a polynomial, i.e.,*

$$f(x) = a_0 + a_1 x + \ldots + a_m x^m,$$

for some integer $m \geq 0$. Then f is infinitely differentiable (which means it can be differentiated any number of times), and

$$p_n(f(x);0) \; = \; \begin{cases} a_0 + a_1 x + \ldots + a_n x^n, & \text{if} \quad n < m \\ a_0 + a_1 x + \ldots + a_m x^m, & \text{if} \quad n \geq m \end{cases}$$

Proof. Clearly, f is differentiable any number of times and moreover, $f^{(n)}(x)$ vanishes if $n > m$. So we have only to show that for $n \leq m$,

$$f^{(n)}(0) \; = \; n! a_n. \tag{8.1.2}$$

When $m = 0$ this is clear. So let $m > 0$ and assume by induction that (8.1.2) holds for all polynomials of degree $m-1$ and $n \leq m-1$. Define a polynomial $g(x)$ by the formula

$$f(x) \; = \; a_0 + x g(x). \tag{8.1.3}$$

Then

$$g(x) \; = \; \sum_{j=0}^{m-1} a_{j+1} x^j$$

and by the inductive hypothesis,

$$g^{(n)}(0) \; = \; n! a_{n+1} \tag{8.1.4}$$

for all non-negative $n \leq m - 1$. But by the *product rule*,

$$f'(x) \; = \; g(x) + x g'(x), \; f''(x) \; = \; 2g'(x) + x g''(x), \ldots$$

By induction, we get

$$f^{(n)}(x) \; = \; n g^{(n-1)}(x) + x g^{(n)}(x),$$

so that

$$f^{(n)}(0) \; = \; n g^{(n-1)}(0) \; \forall \, n \leq m, n \geq 1. \tag{8.1.5}$$

The identity (8.1.2), and hence the Lemma, now follow by combining (8.1.4) and (8.1.5).

□

Lemma 8.2 (**Linearity**) *Let f, g be n-times differentiable at a, and let α, β be arbitrary scalars. Then*

$$p_n(\alpha f(x) + \beta g(x); a) \;=\; \alpha p_n(f(x); a) + \beta p_n(g(x); a).$$

This is easy to prove because the derivative is linear. In particular, we have

$$\frac{(\alpha f + \beta g)^{(j)}(a)}{j!} \;=\; \alpha \frac{f^{(j)}(a)}{j!} + \beta \frac{g^{(j)}(a)}{j!}.$$

It is helpful to look at some **examples**:

(**1**): Let

$$f(x) \;=\; \sin x,$$

which is infinitely differentiable, with

$$f'(x) \;=\; \cos x, \;\; f''(x) \;=\; -\sin x \;=\; -f(x).$$

Thus

$$f^{(n)}(x) \;=\; \begin{cases} (-1)^k \sin x, & \text{if} \quad n = 2k \\ (-1)^k \cos x, & \text{if} \quad n = 2k+1 \end{cases} \tag{8.1.6}$$

Since $\sin 0 = 0$ and $\cos 0 = 1$, the Taylor polynomials of $\sin x$ are given by

$$p_0(\sin x; 0) \;=\; 0, \;\; p_1(\sin x; 0) \;= p_2(\sin x; 0) \;=\; x, \;\; p_3(\sin x; 0) \;=\; p_4(\sin x; 0) \;=\; x - \frac{x^3}{6}, \ldots$$

More generally, for any positive integer k,

$$p_{2k-1}(\sin x; 0) \;=\; p_{2k}(\sin x; 0) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots - (-1)^k \frac{x^{2k-1}}{(2k-1)!}. \tag{8.1.7}$$

(**2**): Put

$$f(x) \;=\; \log x.$$

This function is not defined at 0, so we need to choose another point to evaluate the derivatives, and the easiest one is

$$a \;=\; 1.$$

We have

$$f'(x) \;=\; \frac{1}{x},\; f''(x) \;=\; -\frac{1}{x^2},\; f'''(x) \;=\; \frac{2!}{x^3}, \ldots$$

By induction, we have for any $n \geq 1$,

$$f^{(n)}(x) \;=\; (-1)^{n+1}\frac{(n-1)!}{x^n}.$$

So the nth Taylor coefficient is

$$\frac{f^{(n)}(1)}{n!} \;=\; (-1)^{n+1}\frac{1}{n},$$

where we have used the simple fact that $n!$ is n times $(n-1)!$. Consequently, since $\log 1 = 0$, the nth Taylor polynomial of $\log x$ is given by

$$p_n(\log x; 1) \;=\; x - \frac{x^2}{2} + \ldots + (-1)^{n+1}\frac{x^n}{n}. \tag{8.1.8}$$

(**3**): Consider

$$g(x) \;=\; \frac{1}{x}.$$

One has, for every $n \geq 0$,

$$g^{(n)}(x) \;=\; f^{(n+1)}(x),$$

where $f(x)$ is $\log x$. Thus for any $a > 0$,

$$\frac{g^{(n)}(1)}{n!} \;=\; (n+1)\frac{f^{(n+1)}(a)}{(n+1)!}. \tag{8.1.9}$$

As a consequence the Taylor polynomials of g at $a = 1$ are determinable from those of f. Let us make this idea precise.

Lemma 8.3 *Let f be a which is n times differentiable around a, with*

$$p_n(f(x); a)) \;=\; a_0 + a_1(x-a) + \ldots + a_n(x-a)^n.$$

Then

$$p_{n-1}(f'(x); a) \;=\; a_1 + 2a_2x + \ldots + na_n(x-a)^{n-1}.$$

Moreover, if ϕ is a primitive of f around a,

$$p_n(\phi(x); a) \;=\; \phi(a) + a_0(x-a) + a_1\frac{(x-a)^2}{2} + \ldots + a_{n-1}\frac{(x-a)^n}{n}.$$

The proof is immediate from the definition of Taylor polynomials.

For a general f, even for such a simple function like $\frac{1}{1+x^2}$, it is painful to work out the Taylor polynomials from scratch. One needs a better way to find them, and this will be accomplished in the next section.

8.2 Approximation to order n

Definition 8.4 *Let f, g be n times differentiable functions at a. We will say that f and g* **agree up to order** n *at a iff we have*

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

If g is a polynomial agreeing with f (or equalling f, as some would say) up to order n, then we would call g a **polynomial approximation of** $f(x)$ **to order** n **at** $x = a$. The immediate question which arises is whether the nth Taylor polynomial of f is a polynomial approximation to order n. The answer turns out to be **Yes**, but even more importantly, the Taylor polynomial is the only one which has this property. Here is the complete statement!

Proposition 1 *Let f be n times differentiable at a. Then*

(i) $p_n(f(x); a)$ is a polynomial approximation of f to order n;

(ii) If $q(x)$ is any polynomial in $(x-a)$ of degree $\leq n$ which agrees with f up to order n, then $q(x) = p_n(f(x); a)$.

Proof. (**i**): Put

(8.2.1) $$g(x) = p_{n-1}(f(x); a) \quad \text{and} \quad h(x) = (x-a)^n.$$

Then by definition,

(8.2.2) $$p_n(f(x); a) = g(x) + \frac{f^{(n)}(a)}{n!} h(x).$$

Hence

$$f(x) - p_n(f(x);a) \over (x-a)^n \;=\; \frac{f(x)-g(x)}{h(x)} - \frac{f^{(n)}(a)}{n!}. \tag{8.2.3}$$

So it suffices to prove the following:

$$\lim_{x\to a} \frac{f(x)-g(x)}{h(x)} \;=\; \frac{f^{(n)}(a)}{n!}. \tag{8.2.4}$$

Applying Lemma 8.1, we get

$$g^{(j)}(a) \;=\; f^{(j)}(a),\ \forall\, j < n. \tag{8.2.5}$$

Since g is a polynomial of degree $\leq n-1$, its $(n-1)$th derivative is a constant; so

$$g^{(n-1)}(x) \;=\; g^{(n-1)}(a). \tag{8.2.6}$$

Also,

$$h^{(j)}(x) \;=\; \frac{n!(x-a)^{n-j}}{(n-j)!}. \tag{8.2.7}$$

It follows from (8.2.5) and (8.2.7) that for every $j < n-1$,

$$\lim_{x\to a} f^{(j)}(x) - g^{(j)}(x) \;=\; \frac{f^{(j)}(a) - g^{(j)}(a)}{h^{(j)}(a)} \;=\; 0 \tag{8.2.8}$$

and

$$\lim_{x\to a} h^{(j)}(x) \;=\; h^{(j)}(a) \;=\; 0.$$

On the other hand, by (8.2.6) and (8.2.7),

$$\lim_{x\to a} \frac{f^{(n-1)}(x) - g^{(n-1)}(x)}{h^{(n-1)}(x)} \;=\; \lim_{x\to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{n!(x-a)} \;=\; \frac{f^{(n)}(a)}{n!}. \tag{8.2.9}$$

In view of (8.2.8) and (8.2.9), we may apply L'Hopital's rule as f and g are n-times differentiable (see the Appendix to Chapter 7) and deduce (8.2.4), which also proves part (i) of the Proposition.

(**ii**): By hypothesis, $q(x)$ approximates $f(x)$ to order n at a. By part (i), the Taylor polynomial $p_n(f(x);a)$ does the same thing. It follows, since the limit of a sum is the sum of the limits, that $q(x)$ and $p_n(x)$ agree up to order n. Put

$$u(x) \;=\; p_n(f(x);a) - q(x),$$

which is a polynomial of degree $\leq n$ and satisfies

$$\lim_{x\to a} \frac{u(x)}{(x-a)^n} \;=\; 0.$$

This implies in particular that

$$\lim_{x\to a} \frac{u(x)}{(x-a)^j} \;=\; 0 \;\forall\, j \leq n. \tag{8.2.10}$$

On the other hand, applying the *Euclidean algorithm* repeatedly, relative to the divisor $(x-a)$ (see the next chapter, section on *partial fractions*), we can find numbers $c_0, \ldots, c_n$ such that

$$u(x) \;=\; c_0 + c_1(x-a) + \ldots + c_n(x-a)^n.$$

It is then immediate that for any $j \leq n$,

$$\lim_{x\to a} \frac{u(x)}{(x-a)^j} \;=\; c_j.$$

In view of (8.2.10), this means that every coefficient c_j is zero. Thus the polynomial $u(x)$ is identically zero.

$\square$

Now let us apply this to compute the Taylor polynomials of

$$\phi(x) \;=\; \arctan x \tag{8.2.11}$$

at $a = 0$, where ϕ takes the value 0. (You may try as an educational exercise to compute directly with $\phi(x)$, and you will learn why this Proposition is helpful.)

Recall that ϕ is a primitive of

$$f(x) \;=\; \frac{1}{1+x^2} \tag{8.2.12}$$

for all x in the domain of $\arctan x$, namely the open interval $(-\pi/2, \pi/2)$. Also, f is infinitely differentiable everywhere.

We will compute the Taylor polynomials of $f(x)$ at 0 by the following trick. For each $n \geq 1$, look at the polynomial

$$g_n(x) \; = \; 1 - x^2 + \ldots + (-1)^n x^{2n}.$$

It is a *geometric sum* and so we can reexpress it as

$$g_n(x) \; = \; \frac{1 + x^{2(n+1)}}{1 + x^2} \; = \; f(x) + \frac{x^{2n+2}}{1 + x^2}.$$

Consequently, for $r = 2n, 2n+1$

$$\lim_{x \to 0} \frac{f(x) - g_n(x)}{x^r} \; = \; \lim_{x \to a} \frac{x^{2n+2-r}}{1 + x^2} \; = \; 0.$$

Thus $g_n(x)$ approximates $f(x)$ to order $2n$ and $2n+1$, so it must, by the Proposition above, equal the Taylor polynomials $p_{2n}(f(x); 0)$ and $p_{2n+1}(f(x); 0)$. Thus for any $n \geq 0$,

(8.2.13) $$p_{2n}(\frac{1}{x}; 0) \; = \; p_{2n+1}(\frac{1}{x}; 0) \; = \; 1 - x^2 + \ldots + (-1)^n x^{2n}.$$

Applying Lemma 8.1.10, we then deduce that for all $n \geq 0$,

(8.2.14)

$$p_{2n+1}(\arctan x; 0) \; = \; p_{2n+2}(\arctan x; 0) \; = \; x - \frac{x^3}{3} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Here we have used the fact that $p_0(\arctan x; 0) = \arctan 0 = 0$.

8.3 Taylor's Remainder Formula

Once one has looked at the Taylor polynomials $p_n(f; a)$ of a sufficiently differentiable function f at a point a, the natural question which arises immediately is how close an approximation to f does one get this way. To be precise, define the **nth remainder of f at a** to be

(8.3.1) $$R_n(f(x), a) \; = \; f(x) - p_n(f(x); a).$$

A very precise answer to this question wa supplied by Taylor. Here it is!

Theorem 8.5 *Let $n \geq 0$, $a < x \in \mathbb{R}$, and f an $(n+1)$-times differentiable function on an open interval containing $[a, x]$. Then we have the following:*

$$(8.3.2) \qquad R_n(f(x); a) \; = \; \frac{f^{(n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c in (a, x).

If $f^{(n+1)}$ is moreover integrable on $[a, x]$, then

$$(8.3.3) \qquad R_n(f(x); a) \; = \; \frac{1}{n!}\int\limits_a^x f^{(n+1)}(u)(x-u)^n du.$$

Corollary 8.6 *Let f be $(n+1)$-times differentiable on $[a, x]$. Suppose there are numbers m, M such that*

$$m \; \leq \; f^{(n+1)}(u) \; \leq \; M$$

for **all** *u in $[a, x]$. Then we have*

$$(i) \qquad m\frac{(x-a)^{n+1}}{(n+1)!} \; \leq \; R_n(f)x); a) \; \leq \; M\frac{(x-a)^{n+1}}{(n+1)!}.$$

In particular, if $C = \max\{|m|, |M|\}$,

$$(ii) \qquad |R_n(f)x); a)| \; \leq \; C\frac{(x-a)^{n+1}}{(n+1)!}.$$

Completely analogous assertions hold when $x < a$, in which case one should replace $[a, x]$ everywhere in the Theorem and Corollary with $[x, a]$.

Let us first look at the **example** of the exponential function. We know that

$$f(u) \; = \; \exp(u)$$

is infinitely differentiable on all of $\mathbb{R}$ with $f'(u) = f(u)$. Moreover, since e^u is an increasing function with $e^0 = 1$, we get, for $x > 0$, $u \in [1, x]$ and $n \geq 0$,

$$1 \; \leq \; f^{(n+1)}(u) \; \leq \; e^x.$$

Consequently, by Corollary 8.6, we have

$$(8.3.4) \qquad \frac{x^{n+1}}{(n+1)!} \;\le\; R_n(e^x;0) \;\le\; e^x \frac{x^{(n+1)}}{(n+1)!}.$$

Suppose we want to evaluate e to within an error of 10^{-4}. Then what we have to do is the following. Putting $x = 1$ in (8.3.4), and remembering the crude estimate that e is less than 3, we obtain

$$(8.3.5) \qquad \frac{1}{(n+1)!} \;\le R_n \;=\; R_n(e;0) \;\le\; \frac{3}{(n+1)!}.$$

Find the smallest n for which

$$\frac{3}{(n+1)!} \;<\; 10^{-4}.$$

Direct computation shows that

$$\frac{3}{7!} \;=\; \frac{3}{5040} \;=\; \frac{1}{1680} \;>\; 10^{-4}$$

and

$$\frac{3}{8!} \;=\; \frac{3}{40320} \;=\; \frac{1}{13440} \;<\; 10^{-4}.$$

So we take $n = 7$, and the error will be less than 10^{-4} if we approximate e by

$$p_7(e;0) \;=\; 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!},$$

which is

$$\frac{13700}{5040} \;=\; 2.7182539\ldots$$

The first four places after the decimal point are correct, as they should be. But at the fifth place the digit should be 8 instead of 5, and to get that one has to go to the n (namely 8) which makes R_n less than 10^{-5}.

The remainder formula applied to the functions $\sin x$ and $\cos x$ yields very similar estimates for the remainder. To be precise, we use the fact that the Taylor polynomials of $\sin x$, resp. $\cos x$, at $x = 0$, have only odd, resp. even, degree terms, and obtain the following:

$$(8.3.6 - i) \quad \sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + R_{2m+1}(\sin x;0),$$

with

$$|R_{2m+1}(\sin x; 0)| \le \frac{|x|^{2m+3}}{(2m+3)!};$$

and

$$(8.3.6 - ii) \qquad \cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m}(\cos x; 0),$$

with

$$|R_{2m+1}(\cos x; 0)| \le \frac{|x|^{2m+2}}{(2m+2)!}.$$

It is a simple exercise to approximate numbers like $\sin 1$ or $\cos(1/2)$ to any number of decimal places.

Taylor's formula is not very useful, however, for estimating the remainders of functions f for which it is hard to get a nice expression for $f^{(n+1)}(u)$. A very **important example** illustrating this phenomenon is the function

$$f(x) \;=\; \arctan x, \;\; x \in (-\pi/2, \pi/2).$$

So what does one do? After some reflection, one remembers the method by which one found the Taylor polynomials of this functions. Luckily, this method also leads to a good estimate for the remainder. Let us see how.

Recall that

$$\frac{d}{dx}(\arctan x) \;=\; \frac{1}{1+x^2},$$

and that

$$\frac{1 - (-1)^m x^{2m+2}}{1+x^2} \;=\; 1 - x^2 + x^4 - \ldots + (-1)^m x^{2m}.$$

The second formula can be rewritten as

$$\frac{1}{1+x^2} \;=\; 1 - x^2 + x^4 - \ldots + (-1)^m \frac{x^{2m+2}}{1+x^2}.$$

Integrating this expression and using the fact that $\arctan 0 = 0$, we get by the fundamental theorem of Calculus,

$$(8.3.8 - i) \;\; \arctan x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + R_{2m+1}(\arctan x; 0),$$

where

$$R_{2m+1}(\arctan x; 0) \;=\; (-1)^m \int_0^x \frac{u^{2m+3}}{1+u^2}.$$

I am using here the fact that we have already seen that the polynomial $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1}$ is the Taylor polynomial by the criterion given by part (ii) of Proposition 1. Suppose $x > 0$. Then

$$\frac{u^{2m+3}}{1+u^2} \;\le\; u^{2m+3}, \;\; \forall\, u \in [0, x],$$

in fact with equality only for $u = 0$, and hence

$$|R_{2m+1}(\arctan x; 0)| \;\le\; \int_0^x u^{2m+3} du.$$

Since the integral of u^{2m+3} is $u^{2m+4}/(2m+4)$, we get the desired bound

$$(8.3.8 - ii) \qquad |R_{2m+1}(\arctan x; 0)| \;<\; \frac{x^{2m+3}}{2m+3}.$$

Note that for any fixed $x > 0$, this expression goes to 0 as m goes to ∞. By taking $x = 1$, and letting $m \to \infty$, one gets the **Leibniz formula**:

$$\frac{\pi}{4} \;=\; \arctan 1 \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^m \frac{1}{2m+1} + \ldots$$

This is no doubt a beautiful formula, but it is not quite useful for computations, because $1/m$ goes to 0 rather slowly, at least compared to $1/n!$, which is what one had for the exponential or the sine function. The silver lining is that while (8.3.8-ii) is not decreasing fast for $x = 1$, it converges faster when x is small. To exploit this, one appeals to the **addition theorem for the arctangent**, namely

$$(8.3.9) \qquad \arctan x + \arctan y \;=\; \arctan\left(\frac{x+y}{1-xy}\right),$$

which follows by applying the inverse function arctan to the addition theorem for the tangent function (with $x = \tan u, y = \tan v$):

$$\tan(u+v) \;=\; \frac{\tan u + \tan v}{1 - \tan u \tan v}.$$

From this one can derive, for example, the following identities:

$$\frac{\pi}{4} = \arctan 1 = \arctan(1/2) + \arctan(1/3)$$

and

$$\frac{\pi}{4} = \arctan 1 = 4\arctan(1/5) - \arctan(1/239).$$

The second formula, proved by Machin in 1706, can be used to find the first five or six decimal places of π very fast. (Of course Mathematica or Maple can spew out the first 10,000 digits in a few seconds, but the methods used there are very sophisticated and appeal to formulas involving the mysterious and beautiful class of functions called the *elliptic functions*.)

Proof of Theorem 8.5, For every u in $[a,x]$, we have

(8.3.10)
$$f(x) = p_n(f(x);u) + R_n(f(x);u),$$

where
(8.3.11)
$$p_n(f(x);u) = f(u) + f'(u)(x-u) + \frac{f''(u)}{2!}(x-u)^2 + \ldots + \frac{f^{(n)}(u)}{n!}(x-u)^n.$$

Note that $\frac{d}{du}p_n(f(x);u)$ equals
(8.3.12)
$$f'(u)+(-f'(u) + f''(u)(x-u))+\left(-f''(u)(x-u) + \frac{f^{(3)}(u)}{2!}(x-u)^2\right)+\ldots+\left(-\frac{-f^{(n)}(u)}{(n-1)!}(x-u\right.$$

Differentiating both sides of (8.3.10) and making use of (8.3.12), we obtain, for every u in $[a,x]$,

(8.3.13)
$$0 = \frac{f^{(n+1)}(u)}{n!}(x-u)^n + \frac{d}{du}R_n(f(x);u).$$

The function $R_n(f(x);u)$ is continuous on $[a,x]$ and differentiable on (a,x), because $f(x)$ and $p_n(f(x);u)$ are. Of course the polynomial function $\phi(u) = (x-u)^{n+1}$ has the same properties. So we may apply the Cauchy Mean Value Theorem (see the Appendix to chapter 9) to $R_n(f(x);u)$ and $\phi(u)$ and get a number c in (a,x) such that
(8.3.15)
$$\frac{d\phi}{du}(c)(R_n(f(x);x) - R_n(f(x);a)) = \frac{d}{du}R_n(f(x);u)(c)(\phi(x) - \phi(a)).$$

By (8.3.13),

$$\frac{d}{du}R_n(f(x);u)(c) \;=\; -\frac{f^{(n+1)}(c)}{n!}(x-c)^n. \tag{8.3.16}$$

And

$$\frac{d\phi}{du}(c) \;=\; -(n+1)(x-c)^n. \tag{8.3.17}$$

Combining (8.3.15), (8.3.16) and (8.3.17), cancelling $-(x-u)^n$, and dividing by $(n+1)$, we obtain the formula (i). (This particular form of the remainder was in fact derived by Lagrange.)

Now suppose $f^{(n+1)}$ is integrable on $[a,x]$. Then applying the fundamental theorem of Calculus, and remembering that $R_n(f(x);x)=0$, we get

$$-R_n(f(x);a) \;=\; \int_a^x \left(\frac{d}{du}R_n(f(x);u)\right)du,$$

whose right hand side expression is, by (8.3.13),

$$-\int_a^x \frac{f^{(n+1)}(u)}{n!}(x-u)^n du.$$

Hence we get (ii).

□

8.4 The irrationality of e

Now let us prove (by contradiction) that e is irrational. It is even transcendental, but that is much harder to prove.

Suppose $e = p/q$ for some positive integers p, q. Choose an integer $n > 3$ which is greater than q. Using (8.3.4) and (8.3.5), we get

$$e \;=\; \frac{p}{q} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n,$$

with

$$R_n \;\le\; \frac{3}{(n+1)!}.$$

Multiplying throughout by $n!$, we get

$$n!\frac{p}{q} \;=\; n! + n! + \frac{n!}{2!} + \ldots + \frac{n!}{n!} + n!R_n.$$

But since $n > q$, $\frac{n!}{q}$ is an integer; so is $\frac{n!}{j!}$ for any positive integer $j \leq n$. This implies that $n!R_n$ is an integer. But

$$0 \;<\; n!R_n \;<\; \frac{(n!)(3)}{(n+1)!} \;=\; \frac{3}{n+1},$$

and this gives a contradiction because $n > 3$, implying that $3/(n+1)$ is < 1.

Hence e must be irrational!

□

8.5 Taylor Series

If f is an infinitely differentiable function around a point $x = a$, then we may consider the associated infinite series (for x near a):

$$\text{(8.5.1)} \qquad \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \;=\; \lim_{n\to\infty} p_n(f(x); a),$$

called the Taylor series of $f(x)$ at a. It may not converge in general, and even when it does, it may not equal $f(x)$.

A $\mathcal{C}^\infty$-function which is represented by its Taylor series near a is said to be **analytic** there. A standard example of a $\mathcal{C}^\infty$-function $g(x)$ which is not analytic at 0 is given by the following: (It is easy to modify it to produce an example at any $a \in \mathbb{R}$.) Define $g(x)$ to be the zero function on $\{x \leq 0\}$, and set

$$g(x) \;=\; e^{-1/x^2}, \quad \text{if } x > 0.$$

Note that e^{-1/x^2} goes fast to zero (from the right) as x goes to zero, and becomes *asymptotic* to the x-axis in the limit. The graph of g really looks flat at $x = 0$, and for good reason. It will be left for you to check that g is infinitely differentiable at 0, with $g^{(n)}(0) = 0$, for every $n \geq 0$. Hence the Taylor series of g at $x = 0$ is identically zero and hence does not represent $g(x)$ near 0.

It is clear, from the discussion in the previous sections, that the Taylor series of an infinitely differentiable function around a point a represents f near a iff we have

$$R_n(f(x);a) \;=\; f(x) - p_n(f(x);a) \;\to\; 0, \tag{8.5.2}$$

for all x near a.

Limits of functions is a bit subtle, and there are in fact two notions of convergence. To be precise, Let $\{f_n\}$ be a sequence or $\mathbb{C}$. we will say that a sequence $\{f_n\}$ of functions on a subset X of $\mathbb{R}$ is **pointwise convergent** with limit f on X iff for every $x \in X$, the sequence $\{f_n(x)\}$ of numbers converges to $f(x)$. In other words, for every $\varepsilon > 0$, there is a positive number $N(x)$ such that

$$n \;>\; N(x) \;\Longrightarrow\; |f(x) - f_n(x)| \;<\; \varepsilon. \tag{8.5.3}$$

It is natural to wonder if $N(x)$ can be taken to be a number N, say, which is **independent of** x. In such a case, we will say that f_n is **uniformly convergent** with limit f on X.

Suppose we are dealing with the situation where the Taylor series of a function f converges and represents f near a. We will now state without proof a key result, which tells us when we can differentiate (or integrate) the Taylor series term by term to get the derivative (or the integral) of f.

Theorem 8.7 *Let $\{a_n\}$ be a sequence of real numbers. Suppose there is a positive real number c such that*

$$\sum_{n=0}^{\infty} a_n c^n$$

converges. Pick any positive number $b < c$. Then each of the power series

$$S(x) \;=\; \sum_{n=0}^{\infty} a_n x^n$$

and

$$T(x) \;=\; \sum_{n=1}^{\infty} n a_n x^{n-1}$$

converges absolutely and uniformly in $[-b, b]$. Moreover,

$$S'(x) \;=\; T(x) \quad \forall\, x \in (-c, c).$$

Let us try to understand the **exponential function** using this Theorem. We have seen that

$$e^x \,=\, 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n(e^x; 0), \tag{8.5.4}$$

with

$$|R_n(e^x; 0)| \,\leq\, e^x \frac{|x|^{n+1}}{(n+1)!}.$$

Lemma 8.8 *Let t be any positive real number. Then we have*

$$\lim_{n\to\infty} \frac{t^n}{n!} \,=\, 0.$$

Granting this Lemma (for the moment) we see for any positive c, the remainder term $R_n(e^c; 0)$ goes to zero as n goes to infinity. This gives us the convergent infinite series expression

$$e^c \,=\, 1 + c + \frac{c^2}{2!} + \frac{c^3}{3!} + \ldots$$

Applying Theorem 8.7 we see then that the **Taylor series**

$$e^x \,=\, 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{8.5.5}$$

converges absolutely and uniformly on $[-c, c]$ for any $c > 0$. Moreover, its derivative is given by differentiating the series term by term, which gives back e^x, as expected.

One proves Lemma 8.8 by appealing to a famous limit formula in Mathematics involving the factorial function, namely the following:

Stirling's formula *One has*

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} \,=\, \sqrt{2\pi}. \tag{8.5.6}$$

There will be a separate posting on the class webpage (under *Notes*), giving a proof of this important formula.

We can write

$$\frac{t^n}{n!} = \left(\frac{n^{n+1/2}e^{-n}}{n!}\right)\left(\frac{u^n}{n^{n+1/2}}\right)$$

where $u = et$. So, thanks to Stirling and the easy fact that $\frac{1}{n^{1/2}}$ goes to 0 as $n \to \infty$, Lemma 8.8 will follow from knowing that

$$\frac{u^n}{n^n} < 1$$

for n sufficiently large. (If $u < 1$, u^n goes to zero as $n \to \infty$, so we may assume, if necessary, that u is greater than 1.) Applying the (one-to-one function) logarithm to both sides, we see that we need to prove that

$$n \log u - n \log n < 0,$$

i.e., that $\log u < \log n$, which holds as soon as $n > u$. Done.

□

By a similar argument, we also get the Taylor series expressions for the sine and cosine functions, valid for all x:

$$(8.5.7-i) \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$(8.5.7-ii) \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

We can also apply Theorem 8.7 and differentiate the expression for $\sin x$ term by term, and as expected, one gets the series expression for $\cos x$.

The series

$$(8.5.8) \qquad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

is, as we saw before, convergent at $x = 1$. Thus by Theorem 8.7, it converges absolutely in $-1 < x < 1$ and uniformly in $[-b, b]$ for any positive $b < 1$.

We claim that this Taylor series for $\arctan x$ does not converge at any $x > 1$. Indeed, if it did, then by Theorem 8.7, it would converge absolutely at $x = 1$ which it does not. So the number 1, which some call the *radius of convergence* of the Taylor series (8.5.8), is the boundary beyond which there is no convergence.

Taking the derivative of the series (8.5.8) term by term, and applying Theorem 11.2..4, we get (as expected)

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots, \tag{8.5.9}$$

which is valid in $\{|x| < 1\}$. At $x = \pm 1$ the series makes no sense, but the left hand side makes sense there and equals $\frac{1}{2}$.

Incidentally, the Taylor series for $\arctan x$ converges at $x = -1$, with

$$\arctan(-1) = -\arctan 1 = -\frac{\pi}{4}.$$

It should be noted, despite what the arctan function may suggest, that in Theorem 8.7, the absolute convergence of $S(x)$ is asserted only for $|x| < c$, and one can say nothing in general about the convergence, or the lack of it, at $x = -c$. Sometimes it does happen that $S(x)$ is divergent at $-c$. The simplest example illustrating this is the Taylor series for $\log(1+x)$ at $x = 0$, which we claim to be

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \tag{8.5.10}$$

absolutely convergent in $-1 < x < 1$ and uniformly convergent in $[-b, b]$ for any positive $b < 1$. One way to see this is the following:

Since $\log(1+x)$ is (defined and) differentiable in $x > -1$ with derivative $\frac{1}{1+x}$, we could find the Taylor series expansion of $\frac{1}{1+x}$ first and then differentiate.

The identity (8.5.9) implies that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \tag{8.5.11}$$

which is absolutely convergent in $0 \leq x < 1$. Then it is also absolutely convergent in $(-1, 1)$. Pick any positive $b < 1$. Then since the series (8.5.11) converges at c for any c with $b < c < 1$. So we may apply Theorem 8.7 and conclude that (8.5.11) converges uniformly in $[-b, b]$. We can also differentiate term by term, by this Theorem, and obtain the desired expansion (8.5.10), which is valid in $-b \leq x \leq b$.

At $x = 1$, the series (8.5.10) is alternating and converges to $\log 2$. But it is important to note, however, that it is divergent at $x = -1$.

A consequence of (8.5.10) is the series for $\log(1-x)$, given by

$$-\log(1-x) = \log\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \tag{8.5.12}$$

9 Complex numbers and functions, factoring, and integration via partial fractions

9.1 Complex Numbers

Recall that for every non-zero real number x, its square $x^2 = x \cdot x$ is always positive. Consequently, $\mathbb{R}$ does not contain the square roots of any negative number. This is a serious problem which rears its head all over the place.

It is a non-trivial fact, however, that any positive number has two square roots in $\mathbb{R}$, one positive and the other negative; the positive one is denoted $\sqrt{x}$. One can show that for any x in $\mathbb{R}$,

$$|x| = \sqrt{x \cdot x}.$$

So if we can somehow have at hand a square root of -1, we can find square roots of any real number.

This motivates us to declare a new entity, denoted i, to satisfy

$$i^2 = -1.$$

One defines the set of *complex numbers* to be

$$\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\}$$

and defines the basic arithmetical operations in $\mathbb{C}$ as follows:

$$(x + iy) \pm (x' + iy') = (x \pm x') + i(y \pm y'),$$

and

$$(x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y).$$

There is a natural one-to-one function

$$\mathbb{R} \to \mathbb{C},\ x \to x + i.0,$$

compatible with the arithmetical operations on both sides.

It is an easy exercise to check all the field axioms, except perhaps for the existence of multiplicative inverses for non-zero complex numbers. To this end one defines the *complex conjugate* of any $z = x + iy$ in $\mathbb{C}$ to be

$$\overline{z} = x - iy.$$

Clearly,

$$\mathbb{R} = \{z \in \mathbb{C} \,|\, \overline{z} = z\}.$$

If $z = x + iy$, we have by definition,

$$z\overline{z} = x^2 + y^2.$$

In particular, $z\overline{z}$ is either 0 or a positive real number. Hence we can find a non-negative square root of $z\overline{z}$ in $\mathbb{R}$. Define the *absolute value*, sometimes called *modulus* or *norm*, by

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

If $z = x + iy$ is not 0, we will put

$$z^{-1} = \frac{\overline{z}}{z\overline{z}} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

It is a complex number satisfying

$$z(z^{-1}) = z\frac{\overline{z}}{z\overline{z}} = 1.$$

Done.

It is natural to think of complex numbers $z = x + iy$ as being ordered pairs (x, y) of real numbers. So one can try to visualize $\mathbb{C}$ as a plane with two perpendicular coordinate directions, namely giving the x and y parts. Note in particular that 0 corresponds to the origin $O = (0, 0)$, 1 to $(1, 0)$ and i with $(0, 1)$. Geometrically, one can think of getting from -1 to 1 (and back) by rotation about an angle π, and similarly, one gets from i to its square -1 by rotating by half that angle, namely $\pi/2$, in the counterclockwise direction. To get from the other square root of -1, namely $-i$, one rotates by $\pi/2$ in the clockwise direction. (Going counterclockwise is considered to be in the positive direction in Math.)

Addition of complex numbers has then a simple geometric interpretation: If $z = x+iy$, $z' = x'+iy'$ are two complex numbers, represented by the points $P = (x, y)$ and $Q = (x', y')$ on the plane, then one can join the origin O to P and Q, and then draw a parallelogram with the *line segments* OP and OQ as a pair of adjacent sides. If R is the fourth vertex of this parallelogram, it corresponds to $z + z'$. This is called the *parallelogram law.*

Complex conjugation corresponds to *reflection* about the x-axis.

The *absolute value* or *modulus* $|z|$ of a complex number $z = x + iy$ is, by the Pythagorean theorem applied to the triangle with vertices $O, P = (x, y)$ and $R = (x, 0)$, simply the *length*, often denoted by r, $\sqrt{x^2 + y^2}$ of the line OP.

The *angle* θ between the line segments OR and OP is called the *argument* of z. The pair (r, θ) determines the complex number z. Indeed High school trigonometry allows us to show that the coordinates of z are given by

$$x \;=\; r\text{cos}\theta \quad \text{and} \quad y \;=\; r\text{sin}\theta,$$

where cos (or cosine) and sin (or sine) are the familiar trigonometric functions. Consequently,

$$z \;=\; r(\text{cos}\theta + i\text{sin}\theta).$$

Those who know about *exponentials* (to be treated below in section 9.4) will recognize the identity

$$e^{i\theta} \;=\; \text{cos}\theta + i\text{sin}\theta.$$

(This can also be taken as a definition of $e^{i\theta}$, for any $\theta \in \mathbb{R}$.)

Note that $e^{i\theta}$ has absolute value 1 and hence lies on the *unit circle* in the plane given by the equation $|z| = 1$.

It is customary for the angle θ to be called the **argument** of z, denoted $\arg(z)$, taken to lie in $[0, 2\pi)$.

9.2 Cardano's formula

This section is mainly for motivational purposes. Recall the well known **quadratic formula** from the days of old, which asserts that the roots of the quadratic equation

$$ax^2 + bx + c \;=\; 0, \quad \text{with} \quad a, b, c \in \mathbb{R},$$

are given by

$$\alpha_{\pm} \;=\; \frac{-b \pm \sqrt{D}}{2a},$$

where the **discriminant** D is $b^2 - 4ac$. Note that

$$D > 0 \implies \exists\, 2 \text{ real roots;}$$

$$D = 0 \implies \exists \text{ a unique real root (with multiplicity 2);}$$

$D < 0 \implies \nexists$ real root.

There were several people in the old days (up to the middle of the nineteenth century), some of them even quite well educated, who did not believe in imaginary numbers, such as square-roots of negative numbers. Their reaction to the quadratic formula was to just ignore the case when $D < 0$ and thus not deal with the possibility of non-real roots. They said they were only interested in real roots. Their argument was shattered when one started looking at the cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad \text{with} \quad a, b, c, d \in \mathbb{R}.$$

Thanks to a beautiful formula of the Italian mathematician Cardano, the roots are given by

$$\begin{aligned}
\alpha_1 &= S + T - \frac{b}{3a}, \\
\alpha_2 &= -(S+T)/2 - \frac{b}{3a} + \frac{\sqrt{-3}}{2}(S-T), \\
\alpha_3 &= -(S+T)/2 - \frac{b}{3a} - \frac{\sqrt{-3}}{2}(S-T),
\end{aligned}$$

with

$$S = (R + \sqrt{D})^{1/3}, \ T = (R - \sqrt{D})^{1/3},$$

where the **discriminant** D is $Q^3 + R^2$, and

$$R = \frac{9abc - 27a^2d - 2b^3}{54a^3}, \ Q = \frac{3ac - b^2}{9a^2}.$$

One has

$D > 0 \implies \exists$ a unique real root;

$D = 0 \implies \exists\, 3$ real roots with 2 of them equal;

$D < 0 \implies \exists\, 3$ distinct real roots.

This presented a problem for the Naysayers. One is for sure interested in the case when there are three real roots, but Cardano's formula for the roots goes through an auxiliary computation, namely that of the square-root of D, which involves imaginary numbers!

9.3 Complex sequences and series

As with real sequences, given a sequence $\{z_n\}$ of complex numbers z_n, we say that it converges to a limit L, say, in $\mathbb{C}$ iff we have, for every $\varepsilon > 0$, we can find an integer $N > 0$ such that

$$n \geq N \implies |L - z_n| < \varepsilon.$$

Proposition 1 *(i) If $\{a_n\}$ is a convergent sequence with limit L, then for any scalar c, the sequence $\{ca_n\}$ is convergent with limit cL;*

(ii) If $\{a_n\}$, $\{b_n\}$ are convergent sequences with respective limits L_1, L_2, then their sum $\{a_n + b_n\}$ and their product $\{a_n b_n\}$ are convergent with respective limits $L_1 + L_2$ and $L_1 L_2$.

The proof is again a simple application of the properties of absolute values. The following Corollary allows the convergence questions for complex sequences to b reduced to real ones.

Corollary 9.1 *Let $\{z_n = x_n + iy_n\}$ be a sequence of complex numbers, with x_n, y_n real for each n. Then $\{z_n\}$ converges iff the real sequences $\{x_n\}$ and $\{y_n\}$ are both convergent.*

Proof. Suppose $\{x_n\}, \{y_n\}$ are both convergent, with respective limits u, v. We claim that $\{z_n\}$ then converges to $w = u + iv$. Indeed, by the Proposition above, $\{iy_n\}$ is convergent with limit iv, and so is $\{x_n + iy_n\}$, with limit w. Conversely, suppose that $\{z_n\}$ converges, say to w. We may write w as $u + iv$, with u, v real. For any complex number $z = x + iy$, $|x|$ and $|y|$ are both bounded by $\leq \sqrt{x^2 + y^2}$, i.e., by $|z|$. Since $w - z_n = (u - x_n) + i(v - y_n)$, we get

$$|u - x_n| \leq |w - z_n| \quad \text{and} \quad |v - y_n| \leq |w - z_n|.$$

For any $\epsilon > 0$, pick $N > 0$ such that for all $n > N$, $|w - z_n|$ is $< \epsilon$. Then we also have $|u - x_n| < \epsilon$ and $|v - y_n|$ is $< \epsilon$ for all $n > N$, establishing the convergence of $\{x_n\}$ and $\{y_n\}$ with respective limits u and v. $\square$

One can define Cauchy sequences as in the real case, and it is immediate that $\{z_n = x_n + iy_n\}$ is Cauchy iff $\{x_n\}$ and $\{y_n\}$ are Cauchy. We have

Theorem 9.2 *A complex sequence $\{z_n\}$ converges iff it is Cauchy.*

Hence $\mathbb{C}$ is also a complete field like $\mathbb{R}$.

An infinite **series** $\sum_{n=n_0}^{\infty} z_n$ of complex numbers is said to be **convergent** iff the sequence of partial sums $\{\sum_{m=n_0}^{n} z_m\}$ is convergent. (Here n_0 is any integer, usually 0 or 1.)

We will say that $\sum_n z_n$ is **absolutely convergent** iff the series of its absolute values, namely $\sum_n |z_n|$ converges.

Note that the question of absolute convergence of a complex series, one is reduced to a real series, since $|z_n|$ is real, even non-negative.

Check that if a complex series $\sum_n z_n$ is absolutely convergent, then it is convergent.

9.4 The complex exponential function, and logarithm

For any complex number z, we will define its exponential to be given by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This series is absolutely convergent at any z, because the real sequence $\sum_n \frac{r^n}{n!}$ is convergent, with $r = |z|$.

The exponential function has some nice properties, which we state without proof:

$$e^0 = 1,\ e^z = e^x e^{iy},\ e^{z+z'} = e^z e^{z'},$$

for all $z = x + iy, z' \in \mathbb{C}$.

Lemma 9.3 $e^{i\theta} = \cos\theta + i\sin\theta$, *for any real number θ. In particular, $e^{i\theta}$ is periodic of period 2π like the trig functions, and moreover,*

$$|e^{i\theta}| = 1,\ e^{i\pi} = -1.$$

Proof By definition,

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots,$$

where the even power terms are real and the odd ones are purely imaginary, since $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$. Since the series is absolutely convergent, we may rearrange and express it as a sum of two series as follows:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \dots\right).$$

From the Taylor series expansions for the sine and cosine functions, we then see that the right hand side is the sum of $\cos\theta$ and i times $\sin\theta$, as asserted.

The periodicity relative to 2π is now clear. Moreover,

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Finally,

$$\cos(\pi) = -1, \sin(\pi) = 0 \implies e^{i\pi} = -1.$$

$\square$

Now let $z = x + iy \in \mathbb{C}$. Note that, since $|e^{iy}| = 1$,

$$|e^z| = e^x.$$

Here we have used the fact that the real exponential is always positive, so $|e^x| = e^x$.

Furthermore, by the periodicity of e^{iy},

$$e^{z+2i\pi} = e^x e^{i(y+2\pi)} = e^x e^{iy} = e^z.$$

So, the complex exponential function is **not one-to-one**, and is in fact periodic of period $2i\pi$. This presents a problem for us, since we would like to define the logarithm as its *inverse*. However, note that e^z is one-to-one if we restrict $z = x + iy$ to lie in the rectangular strip Φ in the complex plane defined by $0 \leq y < 2\pi$.

The **complex logarithm** is defined, for $z \neq 0$, to be

$$\log(z) = \log|z| + i\arg(z),$$

where $\arg(z)$ is taken to lie (as usual) in $[0, 2\pi)$. Note that since $|z| > 0$ if $z \neq 0$, $\log|z|$ makes sense.

Since $|e^z| = e^x$ (as seen above) and $\arg(e^z) = y$ if $y \in [0, 2\pi)$, we see that

$$\log(e^z) = \log(e^x) + iy = x + iy = z, \forall z \in \Phi,$$

as desired.

9.5 Differentiability, Cauchy-Riemann Equations

Let $f(z)$ be a complex valued function of a complex variable $z = x + iy$, with $x, y \in \mathbb{R}$. We will say that f is **differentiable** at a point z_0 in $\mathbb{C}$ iff the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

when this limit exists, we call it $f'(z_0)$.

It is important to note that **the existence of this limit is a stringent condition**, because, in the complex plane, one can approach a point $z_0 = x_0 + iy_0$ from infinitely many directions. In particular, there are the two independent directions given, for $h \in \mathbb{R}$, by the horizontal one $z_0 + h \to z_0$, and the vertical one $z_0 + ih \to z_0$. The former corresponds to having $x_0 + h \to x_0$ with the y-coordinate fixed, and the latter $y_0 + h \to y_0$ with the x-coordinate fixed. So we must have

$$f'(z_0) \;=\; \lim_{h \to 0} \frac{f(x_0 + h + iy_0) - f(x_0 + iy_0)}{h} \;=\; \lim_{h \to 0} \frac{f(x_0 + i(y_0 + h)) - f(x_0 + iy_0)}{ih}.$$

The two limits on the right define the *partial derivatives* of f, denoted respectively by $\frac{\partial f}{\partial x}(z_0)$ and $-i\frac{\partial f}{\partial y}(z_0)$.

Clearly, given any function φ, real or complex, depending on x, y, we can define the partial derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$. In any case, we get the equation (when f is differentiable at z_0)

$$\frac{\partial f}{\partial x}(z_0) \;=\; -i\frac{\partial f}{\partial y}(z_0),$$

which is sometimes written as

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z_0) \;=\; 0.$$

It is also customary to write

$$f(z) \;=\; u(x, y) + iv(x, y),$$

where u, v are real-valued functions of x, y, and taking the real and imaginary parts of the equation above becomes a pair of differential equations, called the *Cauchy-Riemann equations*, at $z_0 = x_0 + iy_0$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

We will now state the following two (amazing) theorems about complex functions without proof:

Theorem 9.4 *Let f be a differentiable function on an open circular disk $D_c(z_0)$ defined by $|z - z_0| < c$, for some $c > 0$. Then f is infinitely differentiable on $D_c(z_0)$. In fact, it is analytic there, meaning that it is represented by its Taylor series in $z - z_0$.*

This is a tremendous contrast from the real situation.

Theorem 9.5 *Let f be a differentiable complex function on all of $\mathbb{C}$. Suppose f is also bounded. Then it must be a constant function.*

Note that this is false in the real case. Indeed, the real function $f(x) = \frac{1}{1+x^2}$ is analytic and bounded, but is not a constant.

9.6 Factorization over $\mathbb{C}$

The most important result over $\mathbb{C}$, which is the reason people are so interested in working with complex numbers, is the following:

Theorem 9.6 *(**The Fundamental Theorem of Algebra**) Every non-constant polynomial with coefficients in $\mathbb{C}$ admits a root in $\mathbb{C}$.*

We will not prove this result here. But one should become aware of its existence if it is not already the case! We will now give an important consequence.

Corollary 9.7 *Let f be a polynomial of degree $n \geq 1$ with $\mathbb{C}$-coefficients. Then there exist complex numbers $\alpha_1, \ldots, \alpha_r$, with $\alpha_i \neq \alpha_j$ if $i \neq j$, positive integers $m_1, \ldots, m_r$, and a scalar c, such that*

$$f(x) \, = \, c \prod_{j=1}^{r} (x - \alpha_j)^{m_j},$$

and

$$\sum_{j=1}^{r} m_j \, = \, n.$$

In other words, any non-constant polynomial f with $\mathbb{C}$-coefficients **factorizes completely into a product of linear factors**. For each $j \le r$, the associated positive integer m_j is called the **multiplicity** of α_j as a root of f, which means concretely that m_j is the highest power of $(x - \alpha_j)$ dividing $f(x)$.

Proof of Corollary. Let $n \ge 1$ be the degree of f and let a_n be the non-zero **leading coefficient**, i.e, the coefficient of x^n. Let us set

$$c = a_n. \tag{9.6.1}$$

If $n = 1$,

$$f(x) = a_1 x + a_0 = c(x - \alpha_1) \quad \text{with} \quad \alpha_1 = -\frac{a_0}{a_1}.$$

So we are done in this case by taking $r = 1$ and $m_1 = 1$.

Now let $n > 1$ and assume by induction that we have proved the assertion for all $m < n$, in particular for $m = n - 1$. By Theorem 9.6, we can find a root, call it β, of f. We may then write

$$f(x) = (x - \beta)h(x), \tag{9.6.2}$$

for some polynomial $h(x)$ necessarily of degree $n-1$. The leading coefficients of f are evidently the same. By induction we may write

$$h(x) = c\prod_{i=1}^{s} (x - \alpha_i)^{k_i},$$

for some roots $\alpha_1, \dots, \alpha_s$ of h with respective multiplicities $n_1, \dots, n_s$, so that

$$\sum_{i=1}^{s} k_i = n - 1.$$

But by (9.6.2), every root of h is also a root of f, and the assertion of the Corollary follows.

$\square$

9.7 Factorization over $\mathbb{R}$

The best way to understand polynomials f with real coefficients is to first look at their complex roots and then determine which ones of them could be real. To this end recall first the baby fact that a complex number $z = u + iv$ is real iff z equals its **complex conjugate** $\overline{z} = u - iv$, where $i = \sqrt{-1}$.

Proposition 2 *Let*

$$f(x) \;=\; a_0 + a_1 x + \ldots + a_n x^n \quad \text{with} \quad a_j \in \mathbb{R},\ \forall j \leq n, \quad \text{and} \quad a_n \neq 0,$$

for some $n \geq 1$. Suppose α is a **complex root** *of f. Then $\overline{\alpha}$ is also a root of f. In particular, if r denotes the number of real roots of f and s the non-real (complex) roots of f, then we must have*

$$n \;=\; r + 2s.$$

We get the following consequence, which we proved earlier using the *Intermediate value theorem.*

Corollary 9.8 *Let f be a real polynomial of odd degree. Then f must have a real root.*

Proof of Proposition. Let α be a complex root of f. Recall that for all complex numbers z, w,

$$\overline{zw} \;=\; \overline{z}\,\overline{w} \quad \text{and} \quad \overline{z+w} \;=\; \overline{z} + \overline{w}. \tag{9.7.1}$$

Hence for any $j \leq n$,

$$(\overline{\alpha})^j \;=\; \overline{\alpha^j}.$$

Moreover, since $a_j \in \mathbb{R}$ $(\forall j)$, $\overline{a}_j \;=\; a_j$, and therefore

$$a_j(\overline{\alpha})^j \;=\; \overline{a_j \alpha^j}.$$

Consequently, using (9.7.1) again, we get

$$f(\overline{\alpha}) \;=\; \sum_{j=0}^{n} a_j(\overline{\alpha})^j \;=\; \overline{f(\alpha)}. \tag{9.7.2}$$

But α is a root of f (which we have not used so far), $f(\alpha)$ vanishes, as does its complex conjugate $\overline{f(\alpha)}$. So by (9.7.2), $f(\overline{\alpha})$ is zero, showing that $\overline{\alpha}$ is a root of f.

So the **non-real roots** come in **conjugate pairs**, and this shows that n minus the number r, say, of the **real roots** is even. Done.

$\square$

Given any complex number z, we have

$$z + \overline{z},\ z\overline{z}\ \in \mathbb{R}. \tag{9.7.3}$$

This is clear because both the **norm** $z\overline{z}$ and the **trace** $z + \overline{z}$ are unchanged under complex conjugation.

Proposition 3 *Let f be a real polynomial of degree $n \geq 1$ with real roots $\alpha_1, \ldots, \alpha_k$ with multiplicities $n_1, \ldots, n_k$, and non-real roots $\beta_1, \overline{\beta}_1, \ldots, \beta_\ell, \overline{\beta}_\ell$ with multiplicities $m_1, \ldots, m_\ell$ in $\mathbb{C}$. Then we have the factorization*

$$f(x) \;=\; c\prod_{i=1}^{k}(x-\alpha_i)^{n_i} \cdot \prod_{j=1}^{\ell}(x^2 + b_j x + c_j)^{m_j}, \tag{$*$}$$

where for each $j \leq \ell$,

$$b_j \;=\; -(\beta_j + \overline{\beta}_j) \quad \text{and} \quad c_j \;=\; \beta_j\overline{\beta}_j,$$

Each of the factors occurring in $(*)$ is a real polynomial, and the polynomials $x - \alpha_i$ and $x^2 + b_j x + c_j$ are all irreducible over $\mathbb{R}$.

Proof. In view of Corollary 9.7 and Proposition 2, the only thing we need to prove is that for each $j \leq \ell$, the polynomial

$$h_j(x) \;=\; x^2 + b_j x + c_j$$

is real and irreducible over $\mathbb{R}$. The reality of the coefficients $b_j = -(\beta_j + \overline{\beta}_j)$ and $c_j = \beta_j\overline{\beta}_j$ follows from (9.7.3). Suppose it is reducible over $\mathbb{R}$. Then we can write

$$h_j(x) \;=\; (x - t_j)(x - t_j')$$

for some real numbers t_j, t_j'. On the other hand $\beta_j, \overline{\beta}_j$ are roots of h_j. This forces the equality of the sets $\{t_j, t_j'\}$ and $\{\beta_j, \overline{\beta}_j\}$, contradicting the fact that β_j is non-real. So h_j must be irreducible over $\mathbb{R}$.

$\square$

9.8 The partial fraction decomposition

Here is the main result.

Theorem 9.9 *Let*

$$g(x) = \prod_{i=1}^{k}(x-\alpha_i)^{n_i} \cdot \prod_{j=1}^{\ell}(x^2+b_jx+c_j)^{m_j},$$

where the α_i, b_j, c_j are real, and the n_i, m_j are positive integers. Then there exist real numbers $A_i^{(p)}, B_j^{(q)}, C_j^{(q)}$, with $1 \leq i \leq k$, $1 \leq p \leq n_i$, $1 \leq j \leq \ell$ and $1 \leq q \leq m_j$, such that

$$\frac{1}{g(x)} = \sum_{i=1}^{k}\sum_{p=1}^{n_i}\frac{A_i^{(p)}}{(x-\alpha_i)^p} + \sum_{j=1}^{\ell}\sum_{q=1}^{m_j}\frac{B_j^{(q)}x+C_j^{(q)}}{(x^2+b_jx+c_j)^q}. \tag{9.4.2}$$

We will not prove this here. But here is the basic idea of the proof. Cross multiply (9.4.2) and get a polynomial equation of degree $n = \sum_{i=1}^{k} n_i + \sum_{j=1}^{\ell} m_j$ (which is the degree of g) where the coefficients involve the n indeterminates $A_i^{(p)}, B_j^{(q)}, C_j^{(q)}$. One solves for them by comparing the coefficients of x^i, for $1 \leq i \leq n$. This results in an $n \times n$ **linear system**, i.e., a system of n linear equations in the n unknowns. In Ma1b you will learn to determine when such a linear system has a solution.

Let us try to understand this procedure in the simple case when

$$g(x) = (x-\alpha)^2(x^2+bx+c).$$

We want to show that there exist numbers A^1, A^2, B, C such that

$$\frac{1}{g(x)} = \frac{A^{(1)}}{x-\alpha} + \frac{A^{(2)}}{(x-\alpha)^2} + \frac{Bx+C}{x^2+bx+c}.$$

Cross multiplying, this gives the equation

$$1 = A^{(1)}(x-\alpha)(x^2+bx+c) + A^{(2)}(x^2+bx+c) + (Bx+C)(x-\alpha)^2.$$

Multiplying the right hand side out, we obtain

$$1 = A^{(1)}(x^3+(b-\alpha)x^2+(c-\alpha)x-c\alpha)+A^{(2)}(x^2+bx+c)+B(x^3-2\alpha x^2+\alpha^2x)+C(x^2-2\alpha x+\alpha^2).$$

Comparing coefficients, we get

$$(i) \qquad A^{(1)} + B \;=\; 0,\; A^{(1)}(b-\alpha) + A^{(2)} - 2B\alpha + C \;=\; 0,$$

$$(ii)\;\; A^{(1)}(c-\alpha)+A^{(2)}b+B\alpha^2-2C\alpha \;=\; 0, \quad \text{and} -A^{(1)}c\alpha+A^{(2)}c+C\alpha^2 \;=\; 1.$$

This gives four linear equations in four unknowns, namely in $A^{(1)}, A^{(2)}, B$ and C. The equations (i) imply

$$(iii)\;\; A^{(1)} \;=\; -B \quad \text{and} \quad A^{(2)} \;=\; -A^{(1)}(b-\alpha)+2B\alpha - C \;=\; B(b+\alpha) - C.$$

Eliminating $A^{(1)}, A^{(2)}$ from (ii) using (iii), we get

$$(iv) \qquad B(\alpha^2 + (b+1)\alpha + b^2 - c) - C(b+2\alpha) \;=\; 0$$

and

$$(v) \qquad B(b+2\alpha)c + C(\alpha^2 - c) \;=\; 1.$$

It can be checked that the linear equations (iv), (v) are independent, so that we can solve for B, C in terms of α, b, c. Then we can find $A^{(i)}, i = 1, 2$ by using (iii).

To have a **numerical example**, take

$$\alpha = 1, b = 0, c = 1.$$

Then (iv) becomes $B - 2C = 0$ and (v) becomes $2B = 1$, so the solution we seek is given by

$$B = \frac{1}{2},\; C = \frac{1}{4},\; A^{(1)} = -\frac{1}{2},\; A^{(2)} = \frac{1}{2}.$$

Therefore

$$\frac{1}{(x-1)^2(x^2+1)} \;=\; -\frac{1}{2(x-1)} + \frac{1}{2(x-1)^2} + \frac{2x+1}{4(x^2+1)}.$$

9.9 Integration of rational functions

Let us begin discussing a simple situation. The numerical example at the end of section 9.8 implies, by the additivity of the integral, that

$$\int \frac{dx}{(x-1)^2(x^2+1)} = I_1 + I_2 + I_3, \tag{9.9.1}$$

where

$$I_1 = -\frac{1}{2}\int \frac{dx}{x-1},$$
$$I_2 = \frac{1}{2}\int \frac{dx}{(x-1)^2}$$

and

$$I_3 = \frac{1}{4}\int \frac{2x+1}{x^2+1}dx.$$

We know that

$$I_1 = -\frac{1}{2}\log|x-1| + C.$$

Using substitution and the knowledge of the integral of x^t, we get

$$I_2 = -\frac{1}{2(x-1)} + C.$$

And

$$I_3 = I_{3,1} + I_{3,2}, \tag{9.9.2}$$

where

$$I_{3,1} = \frac{1}{4}\int \frac{2x}{x^2+1}dx = \frac{1}{4}\log(x^2+1) + C,$$

which was evaluated by using the substitution $u = x^2 + 1$, and

$$I_{3,2} = \frac{1}{4}\int \frac{dx}{x^2+1} = \frac{1}{4}\arctan x + C.$$

Suppose we want to integrate a **general rational function**. We have the following result.

Proposition 4 *Let $\frac{f(x)}{g(x)}$ be a rational function, i.e., a quotient of polynomials $f(x), g(x)$, with real coefficients. Then the (indefinite)*

$$I = \int \frac{f(x)}{g(x)} dx$$

can be written as a real linear combination of integrals of the following types: (with $a, b, c \in \mathbb{R}, m \in \mathbb{N}$)

$$\int \frac{dx}{(x-a)^m},$$

$$\int \frac{dx}{(x^2+bx+c)^m},$$

and

$$\int \frac{xdx}{(x^2+bx+c)^m},$$

and the integral of a polynomial (which shows up only when deg(f) $\geq$deg(g)).

Proof. Thanks to Proposition 3 and Theorem 9.9, we can write I as a linear combination of integrals of the form

$$I_1 = \int \frac{h(x)}{(x-a)^m} dx \tag{9.9.3}$$

and

$$I_2 = \int \frac{h(x)}{(x^2+bx+c)^m} dx,$$

where $h(x)$ denotes a polynomial with real coefficients. In fact, $h(x)$ is in I_1, resp. I_2, a multiple of $f(x)$, resp. $(Ax+B)f(x)$ for some A, B.

There is nothing to prove if $f(x)$ is a constant. So let us take the degree of f to be ≥ 1. The Proposition is clearly a consequence of the following

Lemma 9.10 *Let $\phi(x)$ be a real polynomial of degree ≥ 1, and let a, b, c be real numbers with $a, b \neq 0$. Then*

(i) We can write $\phi(x)$ as a polynomial in $(x-a)$ with real coefficients.

(ii) If $\phi(x)$ has degree ≥ 2, then we can write

$$\phi(x) \;=\; \sum_{j=0}^{r} \lambda_j(x)(x^2+bx+c)^j,$$

where each $\lambda_j(x)$ is a real polynomial of degree ≤ 1.

Proof of Lemma 9.10. Let the degree of $\phi(x)$ be n.

(i) The assertion is obvious if $n = 1$, So take n to be > 1 and assume by induction that the assertion holds for $n-1$. We can write

$$\phi(x) \;=\; Q(x)(x-a) + c_1, \tag{9.9.4}$$

where c_1 is a constant. Since $Q(x)$ is of degree $n-1$, we may apply the inductive hypothesis and conclude that $Q(x)$ is a real polynomial in $x-a$. Then (9.9.4) shows that $\phi(x)$ is also a polynomial in $x-a$, as claimed. Done.
(ii) We will apply the *Principle of Induction* to the set of all integers ≥ 2. Suppose $n = 2$, with $\phi(x) = Ax^2+Bx+C$, $A \neq 0$. Then $\phi(x)$ can be written as $A(x^2+bx+c) + ((B-Ab)x + (C-Ac))$, so the assertion holds with $\lambda_0(x) \;=\; (B-Ab)x+(C-Ac)$ and $\lambda_1(x) \;=\; A$. So take n to be greater than 2 and assume by induction that the assertion holds for all $m < n$. Now we may write

$$\phi(x) \;=\; Q(x)(x^2+bx+c) + \lambda_0(x), \tag{9.9.5}$$

where $\lambda_0(x)$ is a real polynomial of degree < 2. Since the degree of $Q(x)$ is of degree $n-2$, we may apply the inductive hypothesis and conclude that

$$Q(x) \;=\; \sum_{i=0}^{k} \mu_i(x)(x^2+bx+c),$$

with each $\mu_i(x)$ if a real polynomial of degree ≤ 1. Combining this with (9.9.5), we get what we want with $r = k+1$ and $\lambda_j(x) = \mu_{j-1}(x)$ for each $j \geq 1$.

$\square$

The next key step is to **complete the square**. Explicitly, we can, given any pair of real numbers b, c, write

$$x^2+bx+c \;=\; (x+\frac{b}{2})^2 + (c-\frac{b^2}{4}). \tag{9.9.6}$$

When x^2+bx+c is irreducible over $\mathbb{R}$, which we may assume to be the case, thanks to Proposition 3, its discriminant b^2-4c is necessarily negative, and so $c-\frac{b^2}{4}$ is positive and thus can be expressed as e^2, for a positive real number e. Consequently, by using the substitution $u = x+\frac{b}{2}$, which gives $u'(x) = 1$, we can transform the integrals (9.0.2) and (9.0.3) into linear combinations of integrals of the following form:

$$I_1 = \int \frac{du}{(u^2+e^2)^m} \tag{9.9.7}$$

and

$$I_2 = \int \frac{udu}{(u^2+e^2)^m}.$$

There is no problem at all in evaluating I_2. If we put $v = u^2+e^2$, then $dv = 2udu$, and

$$I_2 = \frac{1}{2}\int \frac{dv}{v^m},$$

which equals

$$\frac{1}{2}\log|v|+C = \frac{1}{2}\log|u^2+e^2|+C, \quad \text{if} \quad m=1,$$

and

$$-\frac{1}{2(m-1)v^{m-1}}+C = -\frac{1}{2(m-1)(u^2+e^2)^{m-1}}+C, \quad \text{if} \quad m>1.$$

We may use substitution again to simplify I_1. Indeed, if we set $y = u/e$, we have

$$dy = \frac{1}{e}du, \quad \text{and} \quad u^2+e^2 = e^2(y^2+1).$$

Consequently,

$$I_1 = \frac{1}{e}\int \frac{dy}{(y^2+1)^m}. \tag{9.9.8}$$

Of course, this is just $\frac{1}{e}\arctan y + C$ when $m=1$.

It is left to discuss a **reduction process** which allows us to compute

$$J_m = \int \frac{dy}{(y^2+1)^m}$$

for $m > 1$.

Let us try the substitution

$$y = \tan t.$$

Since $\tan^2 t + 1 = \sec^2 t$, we have

$$(y^2+1)^m = \sec^{2m} t \quad \text{and} \quad dy = \sec^2 t dt.$$

Consequently, since $\frac{1}{\sec t} = \cos t$,

$$J_m = \int \cos^{2(m-1)} t \, dt.$$

We have already evaluated it in a previous chapter. Finally, to write the answer in the variable y, we will need to write t as $\arctan y$.