Chapter 8

Change of Variables, Parametrizations, Surface Integrals

§0. The transformation formula

In evaluating any integral, if the integral depends on an auxiliary function of the variables involved, it is often a good idea to change variables and try to simplify the integral. The formula which allows one to pass from the original integral to the new one is called the **transformation formula** (or **change of variables formula**). It should be noted that certain conditions need to be met before one can achieve this, and we begin by reviewing the one variable situation.

Let $\mathcal{D}$ be an open interval, say (a, b), in $\mathbb{R}$, and let $\varphi : \mathcal{D} \to \mathbb{R}$ be a 1-1 , $\mathcal{C}^1$ mapping (function) such that $\varphi' \neq 0$ on $\mathcal{D}$. Put $\mathcal{D}^* = \varphi(\mathcal{D})$. By the hypothesis on φ, it's either increasing or decreasing everywhere on $\mathcal{D}$. In the former case $\mathcal{D}^* = (\varphi(a), \varphi(b))$, and in the latter case, $\mathcal{D}^* = (\varphi(b), \varphi(a))$. Now suppose we have to evaluate the integral

$$I = \int_a^b f(\varphi(u))\varphi'(u)\ \mathrm{d}u,$$

for a nice function f. Now put $x = \varphi(u)$, so that $\mathrm{d}x = \varphi'(u)\ \mathrm{d}u$. This change of variable allows us to express the integral as

$$I = \int_{\varphi(a)}^{\varphi(b)} f(x)\ \mathrm{d}x = \operatorname{sgn}(\varphi') \int_{\mathcal{D}^*} f(x)\ \mathrm{d}x,$$

where $\operatorname{sgn}(\varphi')$ denotes the sign of φ' on $\mathcal{D}$. We then get the transformation formula

$$\int_{\mathcal{D}} f(\varphi(u))|\varphi'(u)|\ \mathrm{d}u = \int_{\mathcal{D}^*} f(x)\ \mathrm{d}x$$

This generalizes to higher dimensions as follows:

Theorem *Let $\mathcal{D}$ be a bounded open set in $\mathbb{R}^n$, $\varphi : \mathcal{D} \to \mathbb{R}^n$ a $\mathcal{C}^1$, 1-1 mapping whose Jacobian determinant $\det(D\varphi)$ is everywhere non-vanishing on $\mathcal{D}$, $\mathcal{D}^* = \varphi(\mathcal{D})$, and f an integrable function on $\mathcal{D}^*$. Then we have the* **transformation formula**

$$\int \cdots \int_{\mathcal{D}} f(\varphi(u))|\det D\varphi(u)|\ du_1 \ldots\ du_n = \int \cdots \int_{\mathcal{D}^*} f(x)\ dx_1 \ldots\ dx_n.$$

Of course, when $n = 1$, $\det D\varphi(u)$ is simply $\varphi'(u)$, and we recover the old formula. This theorem is quite hard to prove, and we will discuss the 2-dimensional case in detail in §1. In any case, this is one of the most useful things one can learn in Calculus, and one should feel free to (properly) use it as much as possible.

It is helpful to note that if φ is **linear**, i.e., given by a linear transformation with associated matrix M, then $D\varphi(u)$ is just M. In other words, the Jacobian determinant is constant in this case. Note also that φ is $\mathcal{C}^1$, even $\mathcal{C}^\infty$, in this case, and moreover φ is $1-1$ iff M is invertible, i.e., has non-zero determinant. Hence φ is $1-1$ iff $\det D\varphi(u) \neq 0$. But this is special to the linear situation. Many of the cases where change of variables is used to perform integration are when φ is not linear.

§1. The formula in the plane

Let $\mathcal{D}$ be an open set in $\mathbb{R}^2$. We will call a mapping $\varphi : \mathcal{D} \to \mathbb{R}^2$ as above **primitive** if it is either of the form

$$(P1) \qquad (u, v) \to (u, g(u, v))$$

or of the form

$$(P2) \qquad (u, v) \to (h(u, v), v),$$

with $\partial g/\partial v, \partial h/\partial u$ nowhere vanishing on $\mathcal{D}$. (If $\partial g/\partial v$ or $\partial h/\partial u$ vanishes at a finite set of points, the argument below can be easily extended.)

We will now prove the transformation formula when φ is a composition of two primitive transformations, one of type (P1) and the other of type (P2).

For simplicity, let us assume that the functions $\partial g/\partial v(u, v)$ and $\partial h/\partial u(u, v)$ are always positive. (If either of them is negative, it is elementary to modify the argument.) Put $\mathcal{D}_1 = \{(h(u, v), v) | (u, v) \in \mathcal{D}\}$ and $\mathcal{D}^* = \{(x, g(x, v)) | (x, v) \in \mathcal{D}_1\}$. By hypothesis, $\mathcal{D}^* = \varphi(\mathcal{D})$.

Enclose $\mathcal{D}_1$ in a closed rectangle R, and look at the intersection P of $\mathcal{D}_1$ with a partition of R, which is bounded by the lines $x = x_m, m = 1, 2, ...,$ and $v = v_r, r = 1, 2, ...,$ with the subrectangles R_{mr} being of sides $\triangle x = l$ and $\triangle v = k$. Let R^*, respectively R^*_{mr}, denote the image of R, respectively R_{mr}, under $(u, v) \to (u, g(u, v))$. Then each R^*_{mr} is bounded by the parallel lines $x = x_m$ and $x = x_m + l$ and by the arcs of the two curves $y = g(x, v_r)$ and $y = g(x, v_r + k)$. Then we have

$$\text{area}(R^*_{mr}) = \int_{x_m}^{x_m+l} (g(x, v_r + k) - g(x, v_r)) \, \mathrm{d}x.$$

By the mean value theorem of 1-variable integral calculus, we can write

$$\text{area}(R^*_{mr}) = l[g(x'_m, v_r + k) - g(x'_m, v_r)],$$

for some point x'_m in (x_m, x_m+l). By the mean value theorem of 1-variable differential calculus, we get

$$\text{area}(R^*_{mr}) = lk\frac{\partial g}{\partial v}(x'_m, v'_r),$$

for some $x'_m \in (x_m, x_m + l)$ and $v'_r \in (v_r, v_r + k)$. So, for any function f which is integrable on $\mathcal{D}^*$, we obtain

$$\iint\limits_{\mathcal{D}^*} f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \lim_P \sum_{m,r} klf(x'_m, g(x'_m, v'_r))\frac{\partial g}{\partial v}(x'_m, v'_r).$$

The expression on the right tends to the integral $\iint\limits_{\mathcal{D}_1} f(x, g(x, v))\frac{\partial g}{\partial v}(x, v)\ \mathrm{d}x\ \mathrm{d}v$. Thus we get the identity

$$\boxed{\iint\limits_{\mathcal{D}^*} f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}_1} f(x, g(x, v))\tfrac{\partial g}{\partial v}(x, v)\ \mathrm{d}x\ \mathrm{d}v}$$

By applying the same argument, with the roles of x, y (respectively g, h) switched, we obtain

$$\boxed{\iint\limits_{\mathcal{D}_1} f(x, g(x, v))\tfrac{\partial g}{\partial v}(x, v)\ \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}} f(h(u, v), g(h(u, v), v))\tfrac{\partial g}{\partial v}(h(u, v), v)\tfrac{\partial h}{\partial u}(u, v)\ \mathrm{d}u\ \mathrm{d}v}$$

Since $\varphi = g \circ h$ we get by chain rule that

$$\det D\varphi(u, v) = \frac{\partial h}{\partial u}(u, v)\frac{\partial g}{\partial v}(h(u, v), v),$$

which is by hypothesis > 0. Thus we get

$$\boxed{\iint\limits_{\mathcal{D}^*} f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}} f(\varphi(u, v))|\det D\varphi(u, v)|\ \mathrm{d}u\ \mathrm{d}v}$$

as asserted in the Theorem.

How to do the general case of φ? The fact is, we can subdivide $\mathcal{D}$ into a finite union of subregions, on each of which φ can be realized as a composition of primitive transformations. We refer the interested reader to chapter 3, volume 2, of "Introduction to Calculus and Analysis" by R. Courant and F. John.

§2. Examples

(1) Let $\Phi = \{(x, y) \in \mathbb{R}^2 | x^2+y^2 \leq a^2\}$, with $a > 0$. We know that $A = \text{area}(\Phi) = \pi a^2$. But let us now do this by using **polar coordinates.** Put

$$\mathcal{D} = \{(r, \theta) \in \mathbb{R}^2 | 0 < r < a, 0 < \theta < 2\pi\},$$

and define $\varphi : \mathcal{D} \to \Phi$ by

$$\varphi(r, \theta) = (r \cos\theta, r \sin\theta).$$

Then $\mathcal{D}$ is a connected, bounded open set, and φ is $\mathcal{C}^1$, with image $\mathcal{D}^*$ which is the complement of a negligible set in Φ; hence any integration over $\mathcal{D}^*$ will be the same as doing it over Φ. Moreover,

$$\frac{\partial\varphi}{\partial r} = (\cos\theta, \sin\theta) \text{ and } \frac{\partial\varphi}{\partial\theta} = (-r\sin\theta, r\cos\theta).$$

Hence

$$\det(D\varphi) = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r,$$

which is positive on $\mathcal{D}$. So φ is $1 - 1$.
By the transformation formula, we get

$$A = \iint\limits_{\Phi} \mathrm{d}x \ \mathrm{d}y = \iint\limits_{\mathcal{D}} |\det D\varphi(r, \theta)| \ \mathrm{d}r \ \mathrm{d}\theta = \int_0^a \int_0^{2\pi} r \ \mathrm{d}r \ \mathrm{d}\theta =$$

$$= \int_0^a r \ \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta = 2\pi \int_0^a r \ \mathrm{d}r = 2\pi \left. \frac{r^2}{2} \right]_0^a = \pi a^2.$$

We can justify the iterated integration above by noting that the *coordinate function* $(r, \theta) \to r$ on the open rectangular region $\mathcal{D}$ is continuous, thus allowing us to use Fubini.

(2) Compute the integral $I = \int\int_R x dx dy$ where R is the region $\{(r, \phi) \mid 1 \leq r \leq 2, \ 0 \leq \phi \leq \pi/4\}$.

We have $I = \int_1^2 \int_0^{\pi/4} r\cos(\phi) r dr d\phi = \sin(\phi)]_0^{\pi/4} \cdot \left. \frac{r^3}{3} \right]_1^2 = \frac{\sqrt{2}}{2}(8/3 - 1/3) = \frac{7\sqrt{2}}{6}$.

(3) Let Φ be the region inside the parallelogram bounded by $y = 2x, y = 2x-2, y = x$ and $y = x + 1$. Evaluate $I = \iint\limits_{\Phi} xy \ \mathrm{d}x \ \mathrm{d}y$.

The parallelogram is spanned by the vectors $(1, 2)$ and $(2, 2)$, so it seems reasonable to make the *linear* change of variable

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

since then we'll have to integrate over the box $[0,1]\times[0,1]$ in the $u-v$-region. The Jacobian matrix of this transformation is of course constant and has determinant -2. So we get

$$\begin{aligned} I &= \int_0^1\int_0^1 (u+2v)(2u+2v)|-2|dudv = 2\int_0^1\int_0^1 (2u^2+6uv+4v^2)dudv \\ &= 2\int_0^1 2u^3/3+3u^2v+4v^2u\Big]_0^1 dv = 2\int_0^1 (2/3+3v+4v^2)dv \\ &= 2(2v/3+3v^2/2+4v^3/3\Big]_0^1) = 2(2/3+3/2+4/3) = 7. \end{aligned}$$

(4) Find the volume of the cylindrical region in $\mathbb{R}^3$ defined by

$$W = \{(x,y,z)|x^2+y^2\le a^2, 0\le z\le h\},$$

where a,h are positive real numbers. We need to compute

$$I = \ \mathrm{vol}(W) = \iiint\limits_W \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z.$$

It is convenient here to introduce the **cylindrical coordinates** given by the transformation

$$\varphi : \mathcal{D}\to\mathbb{R}^3$$

given by

$$\varphi(r,\theta,z) = (r\cos\theta, r\sin\theta, z),\ \text{ where } \mathcal{D} = \{0<r<a, 0<\theta<2\pi, 0<z<h\}.$$

It is easy to see φ is 1-1, $\mathcal{C}^1$ and onto the interior $\mathcal{D}^*$ of W. Again, since the boundary of W is negligible, we may just integrate over $\mathcal{D}^*$. Moreover,

$$|\det D\varphi| = |\det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}| = r$$

By the transformation formula,

Since the function $(r,\theta,z)\to r$ is continuous on $\mathcal{D}$, we may apply Fubini and obtain

$$I = \int_0^h\int_0^a\int_0^{2\pi} r\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}z = \pi a^2 h,$$

which is what we expected.

(5) Let W be the **unit ball** $\overline{B}_0(1)$ in $\mathbb{R}^3$ with center at the origin and radius 1. Evaluate

$$I = \iiint\limits_{W} e^{(x^2+y^2+z^2)^{3/2}} \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z.$$

Here it is best to use the **spherical coordinates** given by the transformation

$$\psi : \mathcal{D} \to \mathbb{R}^3, \ (\rho, \theta, \phi) \to (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi),$$

where $\mathcal{D} = \{0 < \rho < 1, 0 < \theta < 2\pi, 0 < \phi < \pi\}$. Then ψ is $\mathcal{C}^1$, 1-1 and onto W minus a negligible set (which we can ignore for integration). Moreover

$$\det(D\psi) = \det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} =$$

$$= \cos\phi \det \begin{pmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{pmatrix} - \rho\sin\phi \det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{pmatrix} =$$

$$= -\rho^2(\cos\phi)^2\sin\phi - \rho^2(\sin\phi)^3 = -\rho^2\sin\phi.$$

Note that $\sin\phi$ is > 0 on $(0, \pi)$. Hence $|\det(D\psi)| = \rho^2\sin\phi$, and we get (by the transformation formula):

$$I = \int_0^1 \int_0^\pi \int_0^{2\pi} e^{\rho^3}\rho^2 \sin\phi \ \mathrm{d}\rho \ \mathrm{d}\phi \ \mathrm{d}\theta = \int_0^1 e^{\rho^3}\rho^2 \ \mathrm{d}\rho \int_0^\pi \sin\phi \ \mathrm{d}\phi \int_0^{2\pi} \ \mathrm{d}\theta =$$

The function $(\rho, \theta, \phi) \to e^{\rho^3}\rho^2\sin\phi$ is continuous on $\mathcal{D}$, and so we may apply Fubini and perform ietrated integration (on $\mathcal{D}$). We obtain

$$= 2\pi \int_0^1 e^{\rho^3}\rho^2 \ \mathrm{d}\rho \ (-\cos\phi)|_0^\pi = \frac{4\pi}{3} \int_0^1 e^u \ \mathrm{d}u,$$

where $u = \rho^3$

$$= \frac{4\pi}{3} e^u \Big]_0^1 = \frac{4\pi}{3}(e-1).$$

§3. Parametrizations

Let n, k be positive integers with $k \leq n$. A subset Φ of $\mathbb{R}^n$ is called a **parametrized k-fold** iff there exist a connected region T in $\mathbb{R}^k$ together with a $\mathcal{C}^1$, 1-1 mapping

$$\varphi : T \to \mathbb{R}^n, \ u \to (x_1(u), x_2(u), ..., x_n(u)),$$

such that $\varphi(T) = \Phi$.

It is called a **parametrized surface** when $k = 2$, and a parametrized curve when $k = 1$.

Example: Let $T = \{(u, v) \in \mathbb{R}^2 | 0 \leq u < 2\pi, -\frac{\pi}{2} \leq v < \frac{\pi}{2}\}$. Fix a positive number a, and define

$$\varphi : T \to \mathbb{R}^3 \text{ by } \varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where

$$x(u, v) = a \cos u \cos v, \ \ y(u, v) = a \sin u \cos v, \ \text{ and } z(u, v) = a \sin v.$$

Then

$$x(u, v)^2 + y(u, v)^2 + z(u, v)^2 = a^2(\cos^2 u \cos^2 v + \sin^2 u \cos^v + \sin^2 v) =$$
$$= a^2(\cos^2 v + \sin^2 v) = a^2.$$

Also, given $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = a^2$ and $(x, y, z) \neq (0, 0, \pm 1)$, we can find $u, v \in T$ such that $x = x(u, v), y = y(u, v)$ and $z = z(u, v)$. So we see that φ is $\mathcal{C}^1$, 1-1 mapping onto the **standard sphere** $S_0(a)$ **of radius** a in $\mathbb{R}^3$, minus 2 points. If we want to integrate over the sphere, removing those two points doesn't make a difference because they form a set of content zero.

§4. Surface integrals in $\mathbb{R}^3$

Let Φ be a parametrized surface in $\mathbb{R}^3$, given by a $\mathcal{C}^1, 1 - 1$ mapping

$$\varphi : T \to \mathbb{R}^3, \quad T \subset \mathbb{R}^2, \quad \varphi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

(When we say $\varphi \in \mathcal{C}^1$, we mean that it is so on an open set containing T.) Let $\xi = (x, y, z)$ be a point on Φ. Consider the curve C_1 on Φ passing through ξ on which v is constant. Then the tangent vector to C_1 at ξ is simply given by $\frac{\partial \varphi}{\partial u}(\xi)$. Similarly, we may consider the curve C_2 on Φ passing through ξ on which u is constant. Then the tangent vector to C_2 at ξ is given by $\frac{\partial \varphi}{\partial v}(\xi)$. So the surface Φ has a **tangent plane** $\mathcal{J}_\Phi(\xi)$ at ξ iff $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are linearly independent there. From now on we will assume this to be the case (see also the discussion of tangent spaces in Ch. 4 of the class notes). When this happens at every point, we call Φ **smooth.** (In fact, for integration purposes, it suffices to know that $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are independent except at a set $\{\varphi(u, v)\}$ of content zero.)

By the definition of the cross product, there is a natural choice for a **normal vector to Φ at ξ** given by:

$$\frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi).$$

Definition : Let f be a bounded scalar field on the parametrized surface Φ. The **surface integral of f over Φ,** denoted $\iint\limits_{\Phi} f \, \mathrm{d}S$, is given by the formula

$$\iint\limits_{\Phi} f \, \mathrm{d}S = \iint\limits_{T} f(\varphi(u, v)) || \frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi) || \, \mathrm{d}u \, \mathrm{d}v.$$

We say that f is **integrable** on Φ if this integral converges. An important special case is when $f = 1$. In this case, we get

$$\boxed{\operatorname{area}(\Phi) = \iint\limits_T ||\tfrac{\partial\varphi}{\partial u}(\xi) \times \tfrac{\partial\varphi}{\partial v}(\xi)||\ \mathrm{d}u\ \mathrm{d}v}$$

Note that this formula is similar to that for a line integral in that we have to put in a scaling factor which measures how the parametrization changes the (infinitesimal) length of the curve (resp. area of the surface). Note also that unlike the case of curves this formula only covers the case of surfaces in $\mathbb{R}^3$ rather than in a general $\mathbb{R}^n$.

Example: Find the area of the standard sphere $S = S_0(a)$ in $\mathbb{R}^3$ given by $x^2 + y^2 + z^3 = a^2$, with $a > 0$. Recall the parametrization of S from above given by

$$\varphi : T \to \mathbb{R}^3\ ,\ \varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$$T = \{(u, v) \in \mathbb{R}^2 | 0 \le u < 2\pi, -\frac{\pi}{2} \le v < \frac{\pi}{2}\},$$

$$x(u, v) = a\cos u \cos v,\ \ y(u, v) = a \sin u \cos v,\ \ z(u, v) = a \sin v.$$

So we have

$$\frac{\partial\varphi}{\partial u} = (-a \sin u \cos v, a \cos u \cos v, 0) \text{ and } \frac{\partial\varphi}{\partial v} = (-a \cos u \sin v, -a \sin u \sin v, a \cos v).$$

$$\Rightarrow \frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u\cos v & a\cos u\cos v & 0 \\ -a\cos u\sin v & -a\sin u\sin v & a\cos v \end{pmatrix} =$$

$$= \det\begin{pmatrix} a\cos u\cos v & 0 \\ -a\sin u\sin v & a\cos v \end{pmatrix}\mathbf{i} - \det\begin{pmatrix} -a\sin u\cos v & 0 \\ -a\cos u\sin v & a\cos v \end{pmatrix}\mathbf{j} +$$

$$+ \det\begin{pmatrix} -a\sin u\cos v & a\cos u\cos v \\ -a\cos u\sin v & -a\sin u\sin v \end{pmatrix}\mathbf{k} =$$

$$= a^2 \cos u \cos^2 v\mathbf{i} + a \sin u \cos^2 v\mathbf{j} + a^2(\sin^2 u \sin v \cos v + \cos^2 u \sin v \cos v)\mathbf{k} =$$

$$= a^2 \cos u \cos^2 v\mathbf{i} + a \sin u \cos^2 v\mathbf{j} + a^2 \sin v \cos v\mathbf{k}.$$

$$\Rightarrow ||\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}|| = a^2(\cos^2 u \cos^4 v + \sin^2 u \cos^4 v + \sin^2 v \cos^v)^{1/2} =$$

$$= a^2(\cos^4 v + \sin^2 v \cos^2 v)^{1/2} = a^2|\cos v|.$$

$$\Rightarrow \text{area}(S) = a^2 \int_0^{2\pi} \mathrm{d}u \int_{-\pi/2}^{\pi/2} |\cos v| \ \mathrm{d}v = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos v \ \mathrm{d}v$$

$$\Rightarrow \text{area}(S) = 4\pi a^2.$$

Here is a useful result:

Proposition: *Let Φ be a surface in $\mathbb{R}^3$ parametrized by a $\mathcal{C}^1$, 1-1 function*

$$\varphi : T \to \mathbb{R}^3 \ \textit{of the form} \ \varphi(u,v) = (u, v, h(u,v)).$$

In other words, Φ is the graph of $z = h(x,y)$. Then for any integrable scalar field f on Φ, we have

$$\iint_\Phi f \ dS = \iint_T f(u,v,h(u,v)) \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1} \ du \ dv.$$

Proof.

$$\frac{\partial \varphi}{\partial u} = (1, 0, \frac{\partial h}{\partial u}) \text{ and } \frac{\partial \varphi}{\partial v} = (0, 1, \frac{\partial h}{\partial v}).$$

$$\Rightarrow \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial h}{\partial u} \\ 0 & 1 & \frac{\partial h}{\partial v} \end{pmatrix} = -\frac{\partial h}{\partial u}\mathbf{i} - \frac{\partial h}{\partial v}\mathbf{j} + \mathbf{k}.$$

$$\Rightarrow ||\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}|| = \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1}.$$

Now the assertion follows by the definition of $\iint_\Phi f \ \mathrm{d}S$.

Example. Let Φ be the surface in $\mathbb{R}^3$ bounded by the triangle with vertices (1,0,0), (0,1,0) and (0,0,1). Evaluate the surface integral $\iint_\Phi x \ \mathrm{d}S$.

Note that Φ is a triangular piece of the plane $x + y + z = 1$. Hence Φ is parametrized by

$$\varphi : T \to \mathbb{R}^3 \ , \ \varphi(u,v) = (u, v, h(u,v)),$$

where $h(u,v) = 1 - u - v$, and $T = \{0 \leq v \leq 1 - u, 0 \leq u \leq 1\}$.

$$\frac{\partial h}{\partial u} = -1 \ , \ \frac{\partial h}{\partial v} = -1, \text{ and } \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1} = \sqrt{1+1+1} = \sqrt{3}.$$

By the Proposition above, we have:

$$\iint_{\Phi} x \, \mathrm{d}S = \sqrt{3} \int_0^1 \int_0^{1-u} u \, \mathrm{d}u \, \mathrm{d}v = \sqrt{3} \int_0^1 u(1-u) \, \mathrm{d}u = \frac{\sqrt{3}}{6}.$$

There is also a notion of an integral of a vector field over a surface. As in the case of line integrals this is in fact obtained by integrating a suitable projection of the vector field (which is then a scalar field) over the surface. Whereas for curves the natural direction to project on is the tangent direction, for a surface in $\mathbb{R}^3$ one uses the **normal** direction to the surface.

Note that a **unit normal vector** to Φ at $\xi = \varphi(u, v)$ is given by

$$\mathbf{n} = \frac{\frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi)}{||\frac{\partial \varphi}{\partial u}(\xi) \times \frac{\partial \varphi}{\partial v}(\xi)||}$$

and that $\mathbf{n} = \mathbf{n}(u, v)$ varies with $(u, v) \in T$. This defines a unit vector field on Φ called the **unit normal field.**

Definition: Let F be a vector field on Φ. Then the **surface integral** of F over Φ, denoted $\iint_{\Phi} F \cdot \mathbf{n} \, \mathrm{d}s$, is defined by

$$\iint_{\Phi} F \cdot \mathbf{n} \, \mathrm{d}S = \iint_{T} F(\varphi(u, v)) \cdot \mathbf{n}(u, v) ||\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}|| \, \mathrm{d}u \, \mathrm{d}v.$$

Again, we say that F is integrable over Φ if this integral converges.

By the definition of $\mathbf{n}$, we have:

$$\iint_{\Phi} F \cdot \mathbf{n} \, \mathrm{d}S = \iint_{T} F(\varphi(u, v)) \cdot (\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}) \, \mathrm{d}u \, \mathrm{d}v.$$

As in the case of the line integral there is a notation for this integral which doesn't make explicit reference to the parametrization φ but only to the coordinates (x, y, z) of the ambient space. If $F = (P, Q, R)$ we write

$$\iint_{\Phi} F \cdot \mathbf{n} \, \mathrm{d}S = \iint_{\Phi} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy. \tag{1}$$

Here the notation $v \wedge w$ indicates a product of vectors which is bilinear (i.e. $(\lambda_1 v_1 + \lambda_2 v_2) \wedge w = \lambda_1 (v_1 \wedge w) + \lambda_2 (v_1 \wedge w)$ and similarly in the other variable) and antisymmetric (i.e. $v \wedge w = -w \wedge v$, in particular $v \wedge v = 0$). In this sense $\wedge$ is similar to $\times$ on $\mathbb{R}^3$ except that $v \wedge w$ does not lie in the same space where v and w lie. On the positive side $v \wedge w$ can be defined for vectors v, w in any vector space. After these lengthy remarks let's see how this formalism works out in practice. If the surface Φ is parametrized by a function $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ then $dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$ and by the properties of $\wedge$ outlined above we have

$$dy \wedge dz = (\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u})du \wedge dv$$

which is the x-coordinate of $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}$. So the equation (1) does indeed hold if we interpret an integral $\int_T f du \wedge dv$ (over some region T in $\mathbb{R}^2$) as an ordinary double integral $\int_T f du dv$ whereas $\int_T f dv \wedge du$ equals $-\int_T f du dv$. All of this suggests that one should give meaning to dx, dy, dz as elements of some vector space (instead of being pure formal as in the definition of multiple integrals). This can be done and is in fact necessary to a complete development of integration in higher dimensions and on spaces more general than $\mathbb{R}^n$.