Chapter 6

Green's Theorem in the Plane

0 Introduction

Recall the following special case of a general fact proved in the previous chapter. Let C be a piecewise $\mathcal{C}^1$ **plane curve**, i.e., a curve in $\mathbb{R}^2$ defined by a piecewise $\mathcal{C}^1$-function

$$\alpha : [a, b] \to \mathbb{R}^2$$

with **end points** $\alpha(a)$ and $\alpha(b)$. Then for any $\mathcal{C}^1$ scalar field φ defined on a connected open set U in $\mathbb{R}^2$ containing C, we have

$$\int_C \nabla\varphi \cdot d\alpha = \varphi(\alpha(b)) - \varphi(\alpha(a)).$$

In other words, the integral of the gradient of φ along C, in the direction stipulated by α, depends only on the difference, measured in the right order, of the values of φ on the end points. Note that the two-point set $\{a, b\}$ is the boundary of the interval $[a, b]$ in $\mathbb{R}$. A general question which arises is to know if similar things hold for integrals over surfaces and higher dimensional objects, i.e., if the integral of the analog of a gradient, sometimes called an **exact differential**, over a geometric shape M depends only on the integral of its **primitive** on the boundary ∂M.

Our object in this chapter is to establish the simplest instance of such a phenomenon for **plane regions**. First we need some preliminaries.

1 Jordan curves

Recall that a curve C parametrized by $\alpha : [a, b] \to \mathbb{R}^2$ is said to be **closed** iff $\alpha(a) = \alpha(b)$. It is called a **Jordan curve**, or a **simple closed curve**, iff α is piecewise $\mathcal{C}^1$ and 1-1 (injective) on the open interval (a, b). Geometrically, this means the curve doesn't cross itself. Examples of Jordan curves are **circles**, **ellipses**, and in fact all kinds of **ovals**. The **hyperbola** defined by $\alpha : [0, 1] \to \mathbb{R}^2,\ x \mapsto \frac{c}{x}$, is (for any $c \neq 0$) **simple**, i.e., it does not intersect itself, but it is **not closed**. On the other hand, the **clover** is **closed**, but **not simple**.

Here is a fantastic result, in some sense more elegant than the ones in Calculus we are trying to establish, due to the French mathematician Camille Jordan; whence the name **Jordan curve**.

Theorem. *Let C be a Jordan curve in $\mathbb{R}^2$. Then there exists connected open sets U, V in the plane such that*

(i) U, V, C are pairwise mutually disjoint,

and

(ii) $\mathbb{R}^2 = U \cup V \cup C$.

In other words, any Jordan curve C separates the plane into two disjoint, connected regions with C as the common boundary. Such an assertion is obvious for an oval but not (at all) in general. There is unfortunately no way we can prove this magnificent result in this course. But interested students can read a proof in Oswald Veblen's article, "Theory of plane curves in Non-metrical Analysis situs," Transactions of the American Math. Society, **6**, no. 1, 83-98 (1905).

The two regions U and V are called the **interior** or **inside** and **exterior** or **outside** of C. To distinguish which is which let's prove

Lemma: In the above situation exactly one of U and V is bounded. This is called the interior of C.

Proof. Since $[a, b]$ is compact and α is continuous the curve $C = \alpha([a, b])$ is compact, hence closed and bounded. Pick a disk $D(r)$ of some large radius r containing C. Then $S := \mathbb{R}^2 \setminus D(r) \subseteq U \cup V$. Clearly S is connected, so any two points $P, Q \in S$ can be joined by a continuous path $\beta : [0, 1] \to S$ with $P = \beta(0)$, $Q = \beta(1)$. We have $[0, 1] = \beta^{-1}(U) \cup \beta^{-1}(V)$ since S is covered by U and V. Since β is continuous the sets $\beta^{-1}(U)$ and $\beta^{-1}(V)$ are open subsets of $[0, 1]$. If $P \in U$, say, put $t_0 = \sup\{t \in [0, 1] | \beta(t) \in U\} \in [0, 1]$. If

$\beta(t_0) \in U$ then, since $\beta^{-1}(U)$ is open, we find points in a small neighborhood of t_0 mapped to U. If $t_0 < 1$ this would mean we'd find points bigger than t_0 mapped into U which contradicts the definition of t_0. So if $\beta(t_0) \in U$ then $t_0 = 1$ and $Q = \beta(1) \in U$. If, on the other hand, $\beta(t_0) \in V$, there is an interval of points around t_0 mapped to V which also contradicts the definition of t_0 (we'd find a smaller upper bound in this case). So the only conclusion is that if one point $P \in S$ lies in U so do all other points Q. Then $S \subseteq U$ and $V \subseteq D(r)$ so V is bounded.

Recall that a parametrized curve has an orientation. A Jordan curve can either be oriented counterclockwise or clockwise. We usually orient a Jordan curve C so that the **interior**, V say, lies to the **left** as we traverse the curve, i.e. we take the counterclockwise orientation. This is also called the **positive** orientation. In fact we could define the counterclockwise or positive orientation by saying that the interior lies to the left.

Note finally that the term "interior" is a bit confusing here as V is **not** the set of interior points of C; but it is the set of interior points of the union of V and C.

2 Simply connected regions

Let R be a region in the plane whose interior is connected. Then R is said to be **simply connected** (or s.c.) iff every Jordan curve C in R can be continuously deformed to a point without crossing itself in the process. Equivalently, for **any** Jordan curve $C \subset R$, the interior of C lies completely in the interior of R.

Examples of s.c. regions:

(1) $R = \mathbb{R}^2$

(2) R=interior of a Jordan curve C.

The **simplest case of a non-simply connected plane region** is the **annulus** given by $\{v \in \mathbb{R}^2 | c_1 < \|v\| < c_2\}$ with $0 < c_1 < c_2$. The reason is the *hole in the middle* which prevents certain Jordan curves from being collapsed all the way.

When a region is not simply connected, one often calls it **multiply connected** or m.c. The annulus is often said to be *doubly connected* because we can cut it into two subregions each of which is simply connected. In general, if a region has m holes, it is said to be $(m+1)$-connected, for the simple reason that we can cut it into $m+1$, but no smaller, number of simply connected subregions.

Sometimes we would need a similar, but more stringent notion. A region R is said to be **convex** iff for any pair of points P, Q the line joining P, Q lies entirely in R.

A star-shaped region is simply connected but not convex.

3 Green's theorem for s.c. plane regions

Recall that if f is a vector field with image in $\mathbb{R}^n$, we can analyze f by its coordinate fields $f_1, f_2, \ldots, f_n$, which are scalar fields. When $n = 2$ (resp. $n = 3$), it is customary notation to use P, Q (resp. P, Q, R) instead of f_1, f_2 (resp. f_1, f_2, f_3).

Theorem A (Green) *Let $f = (P, Q)$ be a $\mathcal{C}^1$ vector field on a connected open set Y in $\mathbb{R}^2$. Suppose C is a piecewise $\mathcal{C}^1$ Jordan curve with inside (or interior) U such that $\Phi := C \cup U$ lies entirely in Y. Then we have*

$$\iint_{\Phi} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \;=\; \oint_{C} f \cdot d\alpha$$

Here C is the boundary $\partial\Phi$ of Φ, and moreover, the integral over C is taken in the positive direction. Note that

$$f \cdot d\alpha = (P(x,y), Q(x,y)) \cdot (x'(t), y'(t))dt,$$

and so we are justified in writing

$$\oint_{C} f \cdot d\alpha = \oint_{C} (Pdx + Qdy).$$

Given any $\mathcal{C}^1$ Jordan curve C with $\Phi = C \cup U$ as above, we can try to finely subdivide the region using $\mathcal{C}^1$ arcs such that Φ is the union of subregions $\Phi_1, \ldots, \Phi_r$ with Jordan curves as boundaries and with non-intersecting insides/interiors, such that each Φ_j is simultaneously a region of type I and

II (see chapter 5). This can almost always be achieved. So we will content ourselves, mostly due to lack of time and preparation on Jordan curves, with proving the theorem only for regions which are of type I and II.

Theorem A follows from the following, seemingly more precise, identities:

(i) $\iint\limits_{\Phi} \frac{\partial P}{\partial x} dxdy = -\oint\limits_{C} Pdx$

(ii) $\iint\limits_{\Phi} \frac{\partial Q}{\partial y} dxdy = \oint\limits_{C} Qdy.$

Now suppose Φ is of type I, i.e., of the form $\{a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$, with $a < b$ and φ_1, φ_2 continuous on $[a, b]$. Then the boundary C has 4 components given by

$$\begin{aligned} C_1 : &\quad \text{graph of } \varphi_1(x),\ a \leq x \leq b \\ C_2 : &\quad x = b, \varphi_1(b) \leq y \leq \varphi_2(b) \\ C_3 : &\quad \text{graph of } \varphi_2(x),\ a \leq x \leq b \\ C_4 : &\quad x = a, \varphi_1(a) \leq y \leq \varphi_2(a). \end{aligned}$$

As C is positively oriented, each C_i is oriented as follows: In C_1, traverse from $x = a$ to $x = b$; in C_2, traverse from $y = \varphi_1(b)$ to $y = \varphi_2(b)$; in C_3, go from $x = b$ to $x = a$; an in C_4, go from $\varphi_2(a)$ to $\varphi_1(a)$.

It is easy to see that

$$\int\limits_{C_2} Pdx = \int\limits_{C_4} Pdx = 0,$$

since C_2 and C_4 are vertical segments allowing no variation in x. Hence we have

(1)

$$-\oint\limits_{C} Pdx = -\left(\int\limits_{C_1} Pdx + \int\limits_{C_3} Pdx\right) = \int\limits_{a}^{b} [P(x, \phi_2(x)) - P(x, \phi_1(x)]dx.$$

On the other hand, since f is a $\mathcal{C}^1$ vector field, $\frac{\partial P}{\partial y}$ is continuous, and since Φ is a region of type I, we may apply Fubini's theorem and get

(2)

$$\iint_{\Phi} \frac{\partial P}{\partial y} dx dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}.$$

Note that x is fixed in the inside integral on the right. We see, by the fundamental theorem of 1-variable Calculus, that

(3)

$$\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y} = P(x, \varphi_2(x)) - P(x, \varphi_1(x)).$$

Putting together (1), (2) and (3), we immediately obtain identity (i).

Similarly (ii) can be seen to hold by exploiting the fact that Φ is also of type II.

Hence Theorem A is now proved for a region Φ which is of type I and II.

To prove this result for a general region, the basic idea is to cut it into a finite number of subregions each of which is of type I or of type II. For a rigorous treatment of the general situation, read Apostol's *Mathematical Analysis* (chap. 10). For a more geometric point of view, look at Spivak's *Calculus on Manifolds*.

4 An area formula

The example below will illustrate a **very useful consequence of Green's theorem**, namely that **the area of the inside of a $\mathcal{C}^1$ Jordan curve** C can be computed as

$$A = \frac{1}{2} \oint_C (x dy - y dx). \qquad (*)$$

A proof of this fact is easily supplied by taking the vector field $f = (P, Q)$ in Theorem A to be given by $P(x, y) = -y$ and $Q(x, y) = x$. Clearly, f is $\mathcal{C}^1$ everywhere on $\mathbb{R}^2$, and so the theorem is applicable. The identity for A follows as $\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y} = 1$.

Example:

Fix any positive number r, and consider the **hypocycloid** C parametrized by

$$\alpha : [0, 2\pi] \to \mathbb{R}^2, \;\; t \mapsto (r\cos^3 t, r\sin^3 t).$$

Then C is easily seen to be a piecewise $\mathcal{C}^1$ Jordan curve. Note that it is also given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}}$. We have

$$A = \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\int_0^{2\pi} (xy'(t) - yx'(t))dt.$$

Now, $x'(t) = 3r\cos^2 t(-\sin t)$, and $y'(t) = 3r\sin^2 t\cos t$. Hence

$$xy'(t) - yx'(t) = (r\cos^3 t)(3r\sin^2 t\cos t) + (r\sin^3 t)(3r\cos^2 t\sin t)$$

which simplifies to $3r^2\sin^2 t\cos^2 t$, as $\sin^2 t + \cos^2 t = 1$. So we obtain

$$A = \frac{3r^2}{2}\int_0^{2\pi}\left(\frac{\sin 2t}{2}\right)^2 dt = \frac{3r^2}{8}\int_0^{2\pi}\left(\frac{1-\cos 4t}{2}\right) dt,$$

by using the trigonometric identities $\sin 2u = 2\sin u\cos u$ and $\cos 2u = 1 - 2\sin^2 u$. Finally, we then get

$$A = \frac{2r^2}{16}\left[\int_0^{2\pi}(1-\cos 4t)dt\right] = \frac{3r^2}{16}\left(t - \frac{\sin 4t}{4}\right)\Big]_0^{2\pi}$$

i.e.,

$$A = \frac{3\pi r^2}{8}.$$

5 Green's theorem for multiply connected regions

We mentioned earlier that the **annulus in the plane** is the simplest example of a non-simply connected region. But it is not hard to see that we can cut this region into two pieces, each of which is the interior of a $\mathcal{C}^1$ Jordan curve.

We may then apply Theorem A of §3 to each piece and deduce statement over the annulus as a consequence. To be precise, pick any point z in $\mathbb{R}^2$ and consider

$$\Phi = \bar{B}_z(r_2) - \bar{B}_z(r_1),$$

for some real numbers r_1, r_2 such that $0 < r_1 < r_2$. Here $\bar{B}_z(r_i)$ denotes the closed disk of radius r_i and center z.

Let C_1, resp. C_2, denote the positively oriented (circular) boundary of $\bar{B}_z(r_1)$, resp. $\bar{B}_z(r_2)$. Let D_i be the **flat diameter** of $\bar{B}_z(r_i)$, i.e., the set $\{x_0 - r_i \leq x \leq x_0 + r_i,\ y = y_0\}$, where x_0 (resp. y_0) denotes the x-coordinate (resp. y-coordinate) of z. Then $D_2 \cap \Phi$ splits as a disjoint union of two horizontal segments $D_+ = D_2 \cap \{x > x_0\}$ and $D_- = D_2 \cap \{x < x_0\}$. Denote by C_i^+ (resp. C_i^-) the upper (resp. lower) half of the circle C_i, for $i = 1, 2$. Then $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ (resp. Φ^-) is the region bounded by $C^+ = C_2^+ \cup D_- \cup C_1^+ \cup D_+$ (resp. $C^- = C_2^- \cup D_+ \cup D_1^- \cup D_-$). We can orient the piecewise $\mathcal{C}^1$ Jordan curves C^+ and C^- in the positive direction. Let U^+, U^- denote the interiors of C^+, C^-. Then $U^+ \cap U^- = \emptyset$, and $\Phi^\pm = C^\pm \cup U^\pm$.

Now let $f = (P, Q)$ be a $\mathcal{C}^1$-vector field on a connected open set containing $\bar{B}_z(r_2)$. Then, combining what we get by applying Green's theorem for s.c. regions to Φ^+ and Φ^-, we get:

$$\iint\limits_{\Phi} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint\limits_{C_2} (Pdx + Qdy) - \oint\limits_{C_1} (Pdx + Qdy), \qquad (*)$$

where both the line integrals on the right are taken in the positive (counterclockwise) direction. Note that the minus sign in front of the second line integral on the right comes from the need to orient $C^\pm$ positively.

In some sense, this is the key case to understand as any multiply connected region can be broken up into a finite union of shapes each of which can be continuously deformed to an annulus. Arguing as above, we get the following (slightly stronger) result:

Theorem B (Green's theorem for m.c. plane regions) *Let* $C_1, C_2, \ldots, C_r$ *be* **non-intersecting** *piecewise* $\mathcal{C}^1$ *Jordan curves in the plane with interiors* $U_1, U_2, \ldots, U_r$ *such that*

(i) $U_1 \supset C_i,\ \forall i \geq 2,$

and

(ii) $U_i \cap U_j = \emptyset,$ *for all* $i, j \geq 2.$

Put $\Phi = C_1 \cup U_1 - \cup_{i=2}^r U_i$, *which is* **not simply connected** *if* $r \geq 2$. *Also let* $f = (P, Q)$ *be a* $\mathcal{C}^1$ *vector field on a connected open set* S *containing* Φ. *Then we have*

$$\iint\limits_{\Phi} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint\limits_{C_1} (Pdx + Qdy) - \sum_{i=2}^{r} \oint\limits_{C_i} (Pdx + Qdy)$$

where each $C_j, j \geq 1$ *is positively oriented.*

Corollary: *Let* $C_1, \dots C_r$, f *be as in Theorem B. In addition, suppose that* $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ *everywhere on* S. *Then we have*

$$\oint\limits_{C_1} (Pdx + Qdy) = \sum_{i=2}^{r} \oint\limits_{C_i} (Pdx + Qdy).$$

6 The winding number

Let C be an oriented piecewise $\mathcal{C}^1$ curve in the plane, and let $z = (x_0, y_0)$ be a point not lying on C. Then the **winding number of** C **relative to** z is intuitively the number of times C wraps around z in the positive direction. (If we reverse the orientation of C, the winding number changes sign.) Mathematically, this number is defined to be

$$W(C, z) := \frac{1}{2\pi} \oint\limits_{C} \left(-\frac{y - y_0}{r^2} dx + \frac{x - x_0}{r^2} dy\right),$$

where $r = \|(x, y) - z\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

When C is parametrized by a piecewise $\mathcal{C}^1$ function $\alpha : [a, b] \to \mathbb{R}^2$, $\alpha(t) = (u(t), v(t))$, then it is easy to see that

$$W(C, z) = \frac{1}{2\pi} \int_a^b \frac{(u(t) - x_0)v'(t) - (v(t) - y_0)u'(t)}{(u(t) - x_0)^2 + (v(t) - y_0)^2} dt$$

Some write $W(\alpha, z)$ instead of $W(C, z)$.

We note the following **useful result** without proof:

Proposition: *Let* C *be a piecewise* $\mathcal{C}^1$ *closed curve in* $\mathbb{R}^2$, *and let* $z \in \mathbb{R}^2$ *be a point not meeting* C.

(a) $W(C, z) \in \mathbb{Z}$.

(b) C *Jordan curve* $\Rightarrow$ $W(C, z) \in \{0, 1, -1\}$.

More precisely, in the case of a Jordan curve, $W(C, z)$ *is 0 if* z *is outside* C*, and it equals* ± 1 *if* z *is inside* C*.*

The reader is advised to do the easy verification of this Proposition for the unit circle C. In this case, when z is outside C, the winding number is zero. When it is inside, $W(C, z)$ is 1 or -1 depending on whether or not C is oriented in the positive (counter-clockwise) direction.

Another fun exercise will be to exhibit, for each integer n, a piecewise $\mathcal{C}^1$, closed curve C and a point z not lying on it, such that $W(C, z) = n$. Of course this cannot happen if C is a Jordan curve if $n \neq 0, \pm 1$.

Note: If $\bar{C}$ denotes the curve obtained from C by reversing the orientation, then the mathematical definition does give $W(\bar{C}, z) = -W(C, z)$.