CHAPTER 4 – MULTIPLE INTEGRALS

The objects of this chapter are five-fold. They are:

(1) Discuss when scalar-valued functions f can be integrated over closed rectangular boxes R in $\mathbb{R}^n$; simply put, f is *integrable* over R iff there is a unique real number, to be denoted $I_R(f)$ or $\int_R f$ when it exists, which is caught between the *upper and lower sums* relative to *any* partition P of R.

(2) Show that any *continuous* function f can be integrated over R.

(3) Discuss Fubini's theorem, which when applicable, allows one to do multiple integrals as *iterated integrals*, i.e, integrate one variable at a time.

(4) Show that bounded functions with *negligible sets of discontinuities* can be integrated over R.

(5) Discuss integrals of continuous functions over general compact sets.

§4.1 Basic notions

We will first discuss the question of integrability of bounded functions on closed rectangular boxes, and then move on to integration over slightly more general regions.

Recall that in one variable calculus, the integral of a function over an interval $[a, b]$ was defined as the limit, when it exists, of certain sums over finite partitions P of $[a, b]$ as P becomes finer and finer. To try to transport this idea to higher dimensions, we need to generalize the notions of partition and refinement.

In this chapter, R will always denote a closed rectangular box in $\mathbb{R}^n$, written as $[a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n]$, where $a_j, b_j, \in \mathbb{R}$ for all j with $a_j < b_j$.

Definition. A **partition** of R is a finite collection P of subrectangular (closed) boxes $S_1, S_2, \ldots, S_r \subseteq R$ such that

(i) $R = \cup_{j=1}^r S_j$, and

(ii) the interiors of S_i and S_j have no intersection for all $i \neq j$.

Definition. A **refinement** of a partition $P = \{S_j\}_{j=1}^r$ of R is another partition $P' = \{S'_k\}_{k=1}^m$ with each S'_k contained in some S_j.

It is clear from the definition that given any two partitions P, P' of R, we can find a third partition P'' which is simultaneously a refinement of P and of P'.

Now let f be a bounded function on R, and let $P = \{S_j\}_{j=1}^r$ a partition of R. Then f is certainly bounded on each S_j, i.e., $f(S_j)$ is a bounded subset of $\mathbb{R}$. It was proved in Chapter 2 that every bounded subset of $\mathbb{R}$ admits a **sup** (lowest upper bound) and an **inf** (greatest lower bound).

Definition. The **upper** (resp. **lower**) **sum** of f over R relative to the partition $P = \{S_j\}_{j=1}^r$ is given by

$$U(f,P) = \sum_{j=1}^{r} \operatorname{vol}(S_j) \sup(f(S_j))$$
$$\left(\text{resp.} \quad L(f,P) = \sum_{j=1}^{r} \operatorname{vol}(S_j) \inf(f(S_j)).\right)$$

Here $\operatorname{vol}(S_j)$ denotes the volume of S_j. Of course, we have

$$L(f,P) \le U(f,P)$$

for all P.

More importantly, it is clear from the definition that if $P' = \{S'_k\}_{k=1}^m$ is a refinement of P, then

$$L(f,P) \le L(f,P') \quad \text{and} \quad U(f,P') \le U(f,P).$$

Put

$$\mathcal{L}(f) = \{L(f,P) \mid P \text{ partition of } R\} \subseteq \mathbb{R}$$

and

$$\mathcal{U}(f) = \{U(f,P) \mid P \text{ partition of } R\} \subseteq \mathbb{R}.$$

Lemma. *$\mathcal{L}(f)$ admits a* **sup**, *denoted $\underline{I}(f)$, and $\mathcal{U}(f)$ admits an* **inf**, *denoted $\overline{I}(f)$.*

Proof. Thanks to the discussion in Chapter 2, all we have to do is show that $\mathcal{L}(f)$ (resp. $\mathcal{U}(f)$) is bounded from above (resp. below). So we will be done if we show that given any two partitions P, P' of R, we have $L(f,P) \le U(f,P')$ as then $\mathcal{L}(f)$ will have $U(f,P')$ as an upper bound and $\mathcal{U}(f)$ will have $L(f,P)$ as a lower bound. Choose a third partition P'' which refines P and P' simultaneously. Then we have $L(f,P) \le L(f,P'') \le U(f,P'') \le U(f,P')$. Done. $\square$

Definition. A bounded function f on R is **integrable** iff $\underline{I}(f) = \overline{I}(f)$. When such an equality holds, we will simply write $I(f)$ (or $I_R(f)$ if the dependence on R needs to be stressed) for $\underline{I}(f)$ $(= \overline{I}(f))$, and call it the **integral of f over R**. Sometimes we will write

$$I(f) = \int_R f \quad \text{or} \quad \int \cdots \int_R f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Clearly, when $n = 1$, we get the integral we are familiar with, often written as $\int_{a_1}^{b_1} f(x_1)\, dx_1$.

This definition is hard to understand, and a useful criterion is given by the following

Lemma. *f is integrable over R iff for every $\varepsilon > 0$ we can find a partition P of R such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof. If f is integrable, $I_R(f)$ is arbitrarily close to the sets of upper and lower sums, and we can certainly find, given any $\varepsilon > 0$, some P such that $I_R(f) - L(f, P) < \varepsilon/2$ and $U(f, P) - I_R(f) < \varepsilon/2$. Done in this direction. In the converse direction, since

$$L(f, P) \leq \underline{I}(f) \leq \overline{I}(f) \leq U(f, P)$$

for any P, if $U(f, P) - L(f, P) < \varepsilon$, we must have

$$\overline{I}(f) - \underline{I}(f) < \varepsilon.$$

Since ε is an arbitrary positive number, $\underline{I}(f)$ must equal $\overline{I}(f)$, i.e., f is integrable over R.

§4.2 Step functions

The obvious question now is to ask if there are integrable functions. One such example is given by the **constant function** $f(x) = c$, for all $x \in R$. Then for any partition $P = \{S_j\}$, we have

$$L(f, P) = U(f, P) = c \sum_{j=1}^{r} \mathrm{vol}(S_j) = c \ \mathrm{vol}(R).$$

So $\underline{I}(f) = \overline{I}(f)$ and $\int_R f = c \ \mathrm{vol}(R)$.

This can be jazzed up as follows.

Definition. A **step function on *R*** is a function f on R which is constant on each of the subrectangular boxes S_j of **some** partition P.

Lemma. *Every step function f on R is integrable.*

Proof. By definition, there exists a partition $P = \{S_j\}_{j=1}^{r}$ of R and scalars $\{c_j\}$ such that $f(x) = c_j$, if $x \in S_j$. Then, arguing as above, it is clear that for **any** refinement P' of P, we have

$$L(f, P') = U(f, P') = \sum_{j=1}^{r} c_j \ \mathrm{vol}(S_j).$$

Hence, $\underline{I}(f) = \overline{I}(f)$. □

§4.3 Integrability of continuous functions

The most important bounded functions on R are continuous functions. (Recall from Chapter 1 that every continuous function on a compact set is bounded, and that R is compact.) The first result of this chapter is given by the following

Theorem. *Every continuous function f on a closed rectangular box R is integrable.*

Proof. Let S be any closed rectangular box contained in R. Define the **span of f on S** to be

$$\mathrm{span}_f(S) = \sup(f(S)) - \inf(f(S)).$$

A basic result about the span of continuous functions is given by the following:

The Small Span Theorem. *For every $\varepsilon > 0$, there exists a partition $P = \{S_j\}_{j=1}^r$ of R such that $\mathrm{span}_f(S_j) < \varepsilon$, for each $j \leq r$.*

Let us first see how this implies the integrability of f over R. Recall that, by a Lemma of section 4.1, it suffices to show that, given any $\varepsilon > 0$, there is a partition P of R such that $U(f,P) - L(f,P) < \varepsilon$. Now by the small span theorem, we can find a partition $P = \{S_j\}$ such that $\mathrm{span}_f(S_j) < \varepsilon'$, for all j, where $\varepsilon' = \varepsilon/\operatorname{vol}(R)$. Then clearly,

$$U(f,P) - L(f,P) < \varepsilon' \operatorname{vol}(R) = \varepsilon.$$

Done. $\square$

It now remains to supply a **proof of the small span theorem**. We will prove this by contradiction. Suppose the theorem is false. Then there exists $\varepsilon_0 > 0$ such that, for **every** partition $P = \{S_j\}$ of R, $\mathrm{span}_f(S_j) \geq \varepsilon_0$ for some j. For simplicity of exposition, we will only treat the case of a rectangle $R = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$. The general case is very similar, and can be easily carried out along the same lines with a bit of book-keeping. Divide R into four rectangles by subdividing along the bisectors of $[a_1, b_1]$ and $[a_2, b_2]$. Then for one of these four rectangles, call it R_1, we must have that for every partition $\{S_j\}$ of R_1 there is a j so that $\mathrm{span}_f(S_j) \geq \varepsilon_0$. Do this again and again, and we finally end up with an infinite sequence of **nested** closed rectangles $R = R_0, R_1, R_2, \ldots$, such that, for every $m \geq 0$, the span of f is at least ε_0 for **any** partition of $P_m = \{S_{j,m}\}$ of R_m on some $S_{j,m}$. Let $z_m = (x_m, y_m)$ denote the southwestern corner of R_m, for each $m \geq 0$. Then the sequence $\{z_m\}_{m \geq 0}$ is bounded, and so we may find the least upper bound (**sup**) α (resp. β) of x_m (resp. y_m). Put $\gamma = [\alpha, \beta]$. Then $\gamma \in R$ as the northeastern corner of R is clearly an upper bound of the z_m. Since f is continuous at γ, we can find a non-empty closed rectangular subbox S of R containing γ such that $\mathrm{span}_f(S) < \varepsilon_0$. But by construction R_m will have to lie inside S if m is large enough, say for $m \geq m_0$. This gives a contradiction to the span of f being $\geq \varepsilon_0$ on some open set of every partition of R_{m_0}. Thus the small span theorem holds for (f, R).

§4.4 Bounded functions with negligible discontinuities

One is very often interested in being able to integrate bounded functions over R which are continuous except on a very "small" subset. To be precise, we say that a subset Y of $\mathbb{R}^n$ has **content zero** if, for every $\varepsilon > 0$, we can find closed rectangular boxes $Q_1, \ldots, Q_m$ such that

(i) $Y \subseteq \cup_{i=1}^m Q_i$, and
(ii) $\sum_{i=1}^m \operatorname{vol}(Q_i) < \varepsilon$.

Examples. (1) A finite set of points in $\mathbb{R}^n$ has content zero. (Proof is obvious!)

(2) Any subset Y of $\mathbb{R}$ which contains a non-empty open interval (a, b) does **not** have content zero.

Proof. It suffices to prove that (a, b) has non-zero content for $a < b$ in $\mathbb{R}$. Suppose (a, b) is covered by a finite union of closed intervals I_i, $1 \leq i \leq m$ in $\mathbb{R}$. Then clearly, $S := \sum_{i=1}^{m} \text{length}(I_i) \geq \text{length}(a, b) = b - a$. So we can never make S less than $b - a$.

(3) The line segment $L = \{x, 0) \mid a_1 < x < b_1\}$ in $\mathbb{R}^2$ has content zero. (Comparing with (2), we see that the notion of content is very much dependent on what the ambient space is, and not just on the set.)

Proof. For any $\varepsilon > 0$, cover L by the single closed rectangle

$$R = \left\{ (x, y) \;\middle|\; a_1 \leq x \leq b_1,\ -\frac{\varepsilon}{4(b_1 - a_1)} \leq y \leq \frac{\varepsilon}{4(b_1 - a_1)} \right\}.$$

Then $\text{vol}(R) = (b_1 - a_1)\frac{\varepsilon}{2(b_1 - a_1)} = \frac{\varepsilon}{2} < \varepsilon$, and we are done.

The third example leads one to ask if any bounded curve in the plane has content zero. The best result we can prove here is the following

Proposition. *Let $\varphi : [a, b] \to \mathbb{R}$ be a* **continuous** *function. Then the graph Γ of φ has content zero.*

Proof. Note that $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y = \varphi(x)\}$. Let $\varepsilon > 0$. By the small span theorem, we can find a partition $a = t_0 < t_1 < \cdots < t_r = b$ of $[a, b]$ such that $\text{span}_\varphi([t_{i-1}, t_i]) < \frac{\varepsilon}{(b-a)}$, for every $i = 1, \ldots, r$. Thus the piece of Γ lying between $x = t_{i-1}$ and $x = t_i$ can be enclosed in a closed rectangle S_i of area less than $\frac{\varepsilon(t_i - t_{i-1})}{(b-a)}$.

Now consider the collection $\{S_i\}_{1 \leq i \leq r}$ which covers Γ. Then we have

$$\sum_{j=1}^{r} \text{area}(S_j) < \frac{\varepsilon}{(b-a)} \sum_{i=1}^{r} (t_i - t_{i-1}) = \varepsilon. \quad \square$$

Theorem. *Let f be a bounded function on R which is continuous except on a subset Y of content zero. Then f is integrable on R.*

Proof. Let $M > 0$ be such that $|f(x)| \leq M$, for all $x \in R$. Since Y has content zero, we can find closed subrectangular boxes $S_1, \ldots, S_m$ of R such that

(i) $Y \subseteq \cup_{i=1}^{m} S_i$, and
(ii) $\sum_{i=1}^{m} \text{vol}(S_i) < \frac{\varepsilon}{4M}$.

Extend $\{S_1, \ldots, S_m\}$ to a partition $P = \{S_1, \ldots, S_r\}$, $m < r$, of R. Applying the small span theorem, we may suppose that $S_{m+1} \ldots, S_r$ are so chosen that (for each $i \geq m+1$) $\operatorname{span}_f(S_i) < \frac{\varepsilon}{2\operatorname{vol}(R)}$. (We can apply this theorem because f is continuous outside the union of $S_1, \ldots, S_m$.) So we have

$$\begin{aligned} U(f,P) - L(f,P) &\leq 2M \sum_{i=1}^{m} \operatorname{vol}(S_i) + \sum_{i=m+1}^{r} \operatorname{span}_f(S_i)\operatorname{vol}(S_i) \\ &< (2M)\left(\frac{\varepsilon}{2M}\right) + \frac{\varepsilon}{2\operatorname{vol}(R)} \sum_{i=m+1}^{r} \operatorname{vol}(S_i). \end{aligned}$$

But the right hand side is $\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, because $\sum_{i=m+1}^{r} \operatorname{vol}(S_i) \leq \operatorname{vol}(R)$. $\square$

Example. Let $R = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$, and $f : R \to \mathbb{R}$ the function defined by $f(x,y) = x + y$ if $x \leq y$ and $x - y$ if $x \geq y$. Show that f is integrable on R.

Let $D = \{(x,x) \mid 0 \leq x \leq 1\}$ be the "diagonal" in R. Then D has content zero as it is the graph of the continuous function $\varphi(x) = x$, $0 \leq x \leq 1$. Moreover, f is discontinuous only on D. So f is continuous on $R - D$ with D of content zero, and consequently f is integrable on R.

Remark. We can use this theorem to define the integral of a **continuous function f on any closed bounded set** B in $\mathbb{R}^n$ **if the boundary** of B **has content zero**. Indeed, in such a case, we may enclose B in a closed rectangular box R and define a function $\tilde{f}$ on R by making it equal f on B and 0 on $R - B$. Then $\tilde{f}$ will be continuous on all of R except for the boundary of B, which has content zero. So $\tilde{f}$ is integrable on R. Since $\tilde{f}$ is 0 outside B, it is reasonable to set

$$\int_B f = \int_R \tilde{f}.$$

It is often useful to consider a finer notion than content, called *measure*. Before giving this definition recall that a set X is countable iff there is a bijection (or as some would say, *one-to-one correspondence*, between X and a subset of the set $\mathbb{N}$ of natural numbers. Check that $\mathbb{Z}$ and $\mathbb{Q}$ are countable, while $\mathbb{R}$ is not.

A subset Y of $\mathbb{R}^n$ is said to have **measure zero** if, for every $\varepsilon > 0$, we can find a countable collection of closed rectangular boxes $Q_1, Q_2, \ldots, Q_m, \ldots$ such that

(i) $Y \subseteq \cup_{i \geq 1} Q_i$, and
(ii) $\sum_{i \geq 1} \operatorname{vol}(Q_i) < \varepsilon$.

One can use open rectangular boxes instead of closed ones, and the resulting definition will be equivalent.

Examples. (1) A countable set of points in $\mathbb{R}^n$ has measure zero.

(2) Any subset Y of $\mathbb{R}$ which contains a non-empty open interval (a,b) does **not** have measure zero.

(3) A countable union of lines in $[0,1] \times [0,1] \subset \mathbb{R}^2$ has measure zero.

We will state the following result without proof:

Theorem. *Let R be a closed rectangular box in $\mathbb{R}^n$, and f a bounded function on R which is continuous except on a subset Y of measure zero. Then f is integrable on R.*

§4.5 Fubini's theorem

So far we have been meticulous in figuring out when a given bounded function f is integrable on R. But if f is integrable, we have developed no method whatsoever to actually find a way to integrate it except in the really easy case of a step function. We propose to ameliorate the situation now by describing a very reasonable and computationally helpful result. We will state it in the plane, but there is a natural analog in higher dimensions as well. In any case, many of the intricacies of multiple integration are present already for $n = 2$, and it is a wise idea to understand this case completely at first.

Theorem (Fubini). *Let f be a bounded, integrable function on $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$. For x in $[a_1, b_1]$, put $A(x) = \int_{a_2}^{b_2} f(x,y)\,dy$ and assume the following*

(i) *$A(x)$ exists for each $x \in [a_1, b_1]$, i.e., the function $y \mapsto f(x,y)$ is integrable on $[a_2, b_2]$ for any fixed x in $[a_1, b_1]$;*
(ii) *$A(x)$ is integrable on $[a_1, b_1]$.*

Then

$$\iint_R f(x,y)\,dxdy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y)\,dy \right) dx.$$

In other words, once the hypotheses (i) and (ii) are satisfied, we can compute $\int_R f$ by performing two 1-dimensional integrals in order. One cannot always reverse the order of integration, however, and if one wants to integrate over x first, one needs to assume the obvious analog of the conditions (i), (ii).

Proof. Let $P_1 = \{B_i \mid 1 \le i \le \ell\}$ (resp. $P_2 = \{C_j \mid 1 \le j \le m\}$) be a partition of $[a_1, b_1]$ (resp. $[a_2, b_2]$), with B_i, C_j closed intervals in $\mathbb{R}$. Then $P = P_1 \times P_2 = \{B_i \times C_j\}$ is a partition of R. By hypothesis (i), we have

$$L(f_x, P_2) \le A(x) \le U(f_x, P_2),$$

where f_x is the one-dimensional function $y \mapsto f(x,y)$. Then applying hypothesis (ii), we get

$$L(L(f_x, P_2), P_1) \le \int_{a_1}^{b_1} A(x)\,dx \le U(U(f_x, P_2), P_1).$$

But we have

$$
\begin{aligned}
L(L(f_x, P_2), P_1) &= \sum_{i=1}^{\ell} \operatorname{length}(B_i) \inf(L(f_x, P_2)(B_i)) \\
&= \sum_{i=1}^{\ell} \operatorname{length}(B_i) \sum_{j=1}^{m} \operatorname{length}(C_j) \inf(f(B_i \times C_j)) = L(f, P).
\end{aligned}
$$

Similarly for the upper sum. Hence $L(f,P) \leq \int_{a_1}^{b_1} A(x)\,dx \leq U(f,P)$. Given any partition Q of R, we can find a partition P of the form $P_1 \times P_2$ which refines Q. Thus $L(f,Q) \leq \int_{a_1}^{b_1} A(x)\,dx \leq U(f,Q)$, for **every** partition Q of R. Then by the uniqueness of $\int_R f$, which exists because f is integrable, $\int_{a_1}^{b_1} A(x)\,dx$ is forced to be $\int_R f$. $\square$

Remark. The reason we denote $\int_{a_2}^{b_2} f(x,y)\,dy$ by $A(x)$ is the following. The double integral $\iint_R f(x,y)\,dxdy$ is the **volume** subtended by the graph $\Gamma = \{(x,y,f(x,y)) \in \mathbb{R}^3\}$ over the rectangle R. (Note that Γ is a "surface" since f is a function of two variables.) When we fix x at the same point x_0 in $[a_1,b_1]$, the intersection of the plane $\{x = x_0\}$ with Γ in $\mathbb{R}^3$ is a curve, which is none other than the graph Γ_{x_0} of f_{x_0} in the (y,z)-plane shifted to $x = x_0$. The **area** under Γ_{x_0} over the interval $[a_2,b_2]$ is just $\int_{a_2}^{b_2} f_{x_0}(y)\,dy$; whence the name $A(x_0)$. Note also that as x_0 goes from a_1 to b_1, the whole volume is swept by the slice of area $A(x_0)$.

A natural question to ask at this point is whether the hypotheses (i), (ii) of Fubini's theorem are satisfied by many functions. The answer is yes, and the prime examples are continuous functions.

Theorem. *Let f be a continuous function on $R = [a_1,b_1] \times [a_2,b_2] \subseteq \mathbb{R}^2$. Then $\int_R f$ can be computed as an iterated integral in either order. To be precise, we have*

$$
\iint_R f(x,y)\,dxdy = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} f(x,y)\,dy \right] dx = \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x,y)\,dx \right] dy.
$$

Proof. Since f is continuous on (the compact set) R, it is certainly bounded. Let $M > 0$ be such that $|f(x,y)| \leq M$. We have also seen that it is integrable. For each x, the function $y \mapsto f(x,y)$ is integrable on $[a_2,b_2]$ because of continuity on $[a_2,b_2]$. So we get hypothesis (i) of Fubini. To get hypothesis (ii), it suffices to show that $A(x) = \int_{a_2}^{b_2} f(x,y)\,dy$ is continuous in x. For h small, we have

$$
|A(x+h) - A(x)| = \left| \int_{a_2}^{b_2} (f(x+h,y) - f(x,y))\,dy \right| \leq \int_{a_2}^{b_2} |f(x+h,y) - f(x,y)|\,dy.
$$

By the small span theorem we can find a partition $\{S_j\}$ of R with $\operatorname{span}_f(S_j) < \varepsilon/(b_2 - a_2)$. If h is small enough so that $(x+h,y)$ and (x,y) lie in the same box for all y

(which we can achieve since x is fixed and there are only finitely many boxes) we have $|f(x+h,y)-f(x,y)| < \mathrm{span}_f(S_j) < \varepsilon/(b_2-a_2)$ where S_j is a box containing both points. Note that this argument also works if (x,y) lies on the vertical boundary between two boxes: for positive h we land in one box and for negative h in the other. Hence

$$\int_{a_2}^{b_2} |f(x+h,y)-f(x,y)|\,dy < \varepsilon$$

for h sufficiently small. This shows that $A(x)$ is continuous and hence integrable on $[a_1,b_1]$. We have now verified both hypotheses of Fubini, and hence

$$\iint_R f(x,y)\,dxdy = \int_{a_1}^{b_1}\left[\int_{a_2}^{b_2} f(x,y)\,dy\right]dx.$$

To prove that $\int_R f$ is also computable using the iteration in reverse order, all we have to do is note that by a symmetrical argument, the integral $\int_{a_1}^{b_1} f(x,y)\,dx$ makes sense and is continuous in y, hence integrable on $[a_2,b_2]$. The Fubini argument then goes through. □

Remark. We will note the following extension of the theorem above without proof.

Let f be a continuous function on a closed rectangular box $R = [a_1,b_1]\times\cdots\times[a_n,b_n]$. Then the integral of f over R is computable as an iterated integral

$$\int_{a_1}^{b_1}\left[\cdots\left[\int_{a_{n-1}}^{b_{n-1}}\left[\int_{a_n}^{b_n} f(x_1,\ldots,x_n)\,dx_n\right]dx_{n-1}\right]\ldots\right]dx_1.$$

Moreover, we can compute this in any order we want, e.g., integrate over x_2 first, then over x_5, then over x_1, etc. Note that there are $n!$ possible ways here of permuting the order of integration.

§4.6 Integration over special regions

Let Z be a **compact set in** $\mathbb{R}^n$. Since it is bounded, we may enclose it in a closed rectangular box R. If f is a bounded function on Z, we may define an extension $\tilde{f}$ to R by setting $\tilde{f}(x)$ to be $f(x)$ (resp. 0) for x in Z (resp. in $R-Z$).

Let us say that f **is integrable over** Z if $\tilde{f}$ is integrable over R, and put

$$\int_Z f = \int_R \tilde{f}.$$

It is clear that this definition is independent of the choice of R. In §4.4, where we introduced the notion of content, we remarked that if the boundary of Z had content zero and if f is continuous, then $\tilde{f}$ would be integrable on R. The same idea easily proves the following

Theorem. *Let Z be a compact subset of $\mathbb{R}^n$ such that the* **boundary of Z has content zero**. *Then* **any** *function f on Z which is* **continuous** *on Z is integrable over Z.*

In fact, one can replace content by measure in this Theorem.
Now we will analyze the simplest cases of this phenomenon in $\mathbb{R}^2$.

Definition. A **region of type I** in $\mathbb{R}^2$ is a set of the form $\{a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, where φ_1, φ_2 are continuous functions on $[a, b]$.

A **region of type II** in $\mathbb{R}^2$ is a set of the form $\{c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$, where ψ_1, ψ_2 are continuous functions on $[c, d]$.

A **region of type III** in $\mathbb{R}^2$ is a subset which is simultaneously of type I and type II.

Remark. Note that a circular region is of type III.

Theorem. *Let f be a continuous function on a subset S of $\mathbb{R}^2$.*

(a) *Suppose S is a region of type I defined by $a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$, with φ_1, φ_2 continuous. Then f is integrable on S and*

$$\int_S f = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx.$$

(b) *Suppose S is a region of type II defined by $c \leq y \leq d$, $\psi_1(y) \leq x \leq \psi_2(y)$, with ψ_1, ψ_2 continuous. Then f is integrable on S and*

$$\int_S f = \int_c^d \left(\int_{\psi_1(x)}^{\psi_2(y)} f(x, y)\, dx \right) dy.$$

Proof. We will prove (a) and leave the symmetrical case (b) to the reader.

(a) Let $R = [a, b] \times [c, d]$, where c, d are chosen so that R contains S. Define $\tilde{f}$ on R as above (by extension of f by zero outside S). By the Proposition of §4.4, we know that the graphs of φ_1 and φ_2 are of content zero, since φ_1, φ_2 are continuous. Thus the main theorem of §4.4 implies that $\tilde{f}$ is integrable on R as its set of discontinuities is contained in the boundary ∂S of S. It remains to prove that $\int_S f$ $(= \int_R \tilde{f})$ is given by the iterated integral $\int_a^b (\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy)\, dx$. For each $x \in (a, b)$, the integral $\int_c^d \tilde{f}(x, y)\, dy$ exists as the set of discontinuities in $[c, d]$ has at most two points. Moreover, the function $x \mapsto \int_c^d \tilde{f}(x, y), dy$ is integrable on $[a, b]$. Hence (the proof of) Fubini's theorem applies in this context and gives

$$\int_R f = \int_a^b \left(\int_c^d \tilde{f}(x, y)\, dy \right) dx.$$

Since the inside integral (over y) is none other than $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy$, the assertion of the theorem follows. $\square$

§4.7 Examples

(1) Compute $\int_R f$, where R is the closed rectangle $[-1,1] \times [2,3]$ and f the function $(x,y) \mapsto x^2y - x\cos\pi y$. Since f is continuous on R, we may apply Fubini's theorem and compute $\int_R I$ as the iterated integral

$$I = \int_{-1}^{1}\left(\int_2^3 (x^2y - x\cos\pi y)\,dy\right)dx.$$

Recall that in $\int_2^3 (x^2y - x\cos\pi y)\,dy$, x is treated like a constant, hence equals

$$x^2\int_2^3 y\,dy - x\int_2^3 \cos\pi y\,dy = x^2\left(\frac{3^2}{2} - \frac{2^2}{2}\right) - x\left(\frac{\sin\pi y}{\pi}\right)\Big]_2^3 = \frac{5}{2}\,x^2.$$

$$\Rightarrow I = \frac{5}{2}\int_{-1}^{1} x^2\,dx = \frac{5}{2}\left(\frac{x^3}{3}\right)\Big]_{-1}^{1} = \frac{5}{3}\,.$$

We could also have computed it in the opposite order to get

$$\begin{aligned} I &= \int_2^3\left[\int_{-1}^{1}(x^2y - x\cos\pi y)\,dx\right]dy \\ &= \int_2^3\left(y\left(\frac{x^3}{3}\right)\Big]_{-1}^{1} - \cos\pi y\left(\frac{x^2}{2}\right)\Big]_{-1}^{1}\right)dy \\ &= \int_2^3\left(\frac{2y}{3}\right)dy = \frac{y^2}{3}\Big]_2^3 = \frac{5}{3}\,. \end{aligned}$$

(2) Find the volume of the tetrahedron T in $\mathbb{R}^3$ bounded by the planes $x=0$, $y=0$, $z=0$ and $x-y-z=-1$.

Note first that the base of T is a triangle $\triangle$ defined by $-1 \le x \le 0$, $0 \le y \le x+1$. Given any (x,y) in $\triangle$, the height of T above it is simply given by $z = x-y+1$. Hence we get by the Theorem of §4.6,

$$\begin{aligned} \mathrm{vol}(T) &= \iint_\triangle (x-y+1)\,dxdy = \int_{-1}^{0}\left(\int_0^{x+1}(x-y+1)\,dy\right)dx \\ &= \int_{-1}^{0}\left(xy - \frac{y^2}{2} + y\right)\Big]_0^{x+1} dx \\ &= \int_{-1}^{0}\frac{(x+1)^2}{x}\,dx = \int_0^1 \frac{u^2}{2}\,du = \frac{1}{6}\,. \end{aligned}$$

(3) Fix $a,b>0$, and consider the region S inside the ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in $\mathbb{R}^2$. Compute $I = \iint_S \sqrt{a^2-x^2}\,dxdy$.

Note that S is a region of type I as we may write it as

$$\left\{-a \le x \le a,\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}\right\}.$$

Since the function $(x, y) \mapsto \sqrt{a^2 - x^2}$ is continuous, we can apply the main theorem of §4.6. We obtain

$$\begin{aligned} I &= \int_{-a}^{a} \sqrt{a^2-x^2}\left(\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b-\sqrt{1-\frac{x^2}{a^2}}} dy\right) dx \\ &= \frac{2b}{a}\int_{-a}^{a}(a^2-x^2)\,dx = \frac{2b}{a}\left(a^2x - \frac{x^3}{3}\right)\Bigg]_{-a}^{a} \\ &= 4a^2b - \frac{4a^2b}{3} = \frac{8a^2b}{3}. \end{aligned}$$

§4.8 Applications

Let S be a thin plate in $\mathbb{R}^2$ with matter distributed with density $f(x, y)$ (= mass/unit area). The **mass of S** is given by

$$m(S) = \iint_S f(x, y)\,dxdy.$$

The **average density** is

$$\frac{m(S)}{\text{area}} = \frac{\iint_S f(x, y)\,dxdy}{\iint_S dxdy}.$$

The **center of mass** of S is given by $\bar{z} = (\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m(S)}\iint_S x\,f(x, y)\,dxdy$$

and

$$\bar{y} = \frac{1}{m(S)}\iint_S y\,f(x, y)\,dxdy.$$

When the **density is constant**, the center of mass is called the **centroid** of S.

Suppose L is a fixed line. For any point (x, y) on S, let $\delta = \delta(x, y)$ denote the (perpendicular) distance from (x, y) to L. The **moment of inertia about L** is given by

$$I_L = \iint_S \delta^2(x, y)\,f(x, y)\,dxdy.$$

When L is the x-axis (resp. y-axis), it is customary to write I_x (resp. I_y).

Note that the center of mass is a linear invariant, while the moment of inertia is quadratic.

An interesting use of the centroid occurs in the computation of volumes of revolutions. To be precise we have the following

Theorem (Pappus). *Let S be a region of type I, i.e., given as $\{a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, with φ_1, φ_2 continuous. Suppose that $\min_x \varphi_1(x) > 0$, so that S lies above the x-axis. Denote by V the volume of the solid M obtained by revolving S about the x-axis, and by $\bar{z} = (\bar{x}, \bar{y})$ the centroid of S. Then*

$$V = 2\pi\bar{y}\, a(S),$$

where $a(S)$ is the area of S.

Proof. Let V_i denote the volume of the solid obtained by revolving $\{(x, \varphi_1(x) \mid a \leq x \leq b\}$ about the x-axis. Then

$$V_i = \pi \int_a^b \varphi_i(x)^2\, dx.$$

(This is a result from one-variable calculus.) But clearly, $V = V_2 - V_1$. So we have

$$V = \pi \int_a^b [\varphi_2(x)^2 - \varphi_1(x)^2]\, dx.$$

On the other hand, we have by the definition of the centroid,

$$\bar{y} = \frac{1}{a(S)} \iint_S y\, dxdy.$$

Since y is continuous and S a region of type I, we have

$$\begin{aligned}
\bar{y} &= \frac{1}{a(S)} \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} y\, dy \right) dx \\
&= \frac{1}{a(S)} \int_a^b \tfrac{1}{2} [\varphi_2(x)^2 - \varphi_1(x)^2]\, dx.
\end{aligned}$$

The theorem now follows immediately. $\square$

Examples. (1) Let S be the semi-circular region $\{-1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$. Compute the centroid of S.

Since S is of type I, we have

$$\begin{aligned}
a(S) &= \iint_S dxdy = \int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} dy \\
&= \int_{-1}^1 \sqrt{1-x^2}\, dx = 2\int_0^1 \sqrt{1-x^2}\, dx.
\end{aligned}$$

Put $x = \sin t$, $0 \leq t \leq \frac{\pi}{2}$. Then $dx = \cos t\, dt$ and $\sqrt{1-x^2} = \cos t$. So we get

$$\begin{aligned}
a(S) &= 2\int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 2\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos t}{2} \right) dt \\
&= 2\left[\frac{\pi}{4} + \left.\frac{\sin 2t}{4} \right]_0^{\frac{\pi}{2}} \right] = \frac{\pi}{2}\,.
\end{aligned}$$

Of course, we could have directly reasoned by geometry that the area of a semi-circular region of radius 1 is $\frac{\pi}{2}$.

Let $\bar{z} = (\bar{x}, \bar{y})$ be the centroid.

$$\begin{aligned}\bar{x} &= \frac{1}{a(S)} \iint_S x\,dxdy = \frac{2}{\pi}\int_{-1}^{1}\Big(\int_0^{\sqrt{1-x^2}} dy\Big)x\,dx \\ &= \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}\,dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\cos^2 t\,dt = 0,\end{aligned}$$

since the integrand is an odd function.

Again, the fact that $\bar{x} = 0$ can be directly seen by geometry. The key thing is to compute $\bar{y}$. We have

$$\begin{aligned}\bar{y} &= \frac{2}{\pi}\int_{-1}^{1} dx\Big(\int_0^{\sqrt{1-x^2}} y\,dy\Big) = \frac{1}{\pi}\int_{-1}^{1}(1-x^2)\,dx \\ &= \frac{1}{\pi}\Big(x - \frac{x^3}{3}\Big)\Big]_{-1}^{1} = \frac{2}{\pi} - \frac{2}{3\pi} = \frac{4}{3\pi}\,.\end{aligned}$$

So the centroid of S is $(0, \frac{4}{3\pi})$.

(2) Find the volume V of the **torus** π obtained by revolving about the x-axis a circular region S of radius r (lying above the x-axis).

The area $a(S)$ is πr^2, and the centroid $(\bar{x}, \bar{y})$ is located at the center of S (easy check!). Let b be the distance from the center of S to the x-axis. Then by Pappus' theorem,

$$V = 2\pi\bar{y}\,a(S) = 2\pi b(\pi r^2) = 2\pi^2 r^2 b.$$