Chaos

Poincaré section
A Poincaré section is obtained by cutting the phase space by a plane and plotting every instances when the trajectory crosses the plane in a specific direction. In a system with 2 degrees of freedom, trajectories near stable points are ellipses with each ellipses corresponding to winding number. Each of these ellipses is actually a cross section of an 2-torus.

Perturbation
As we increase the perturbation (ie. increase the driving force), N-tori with winding numbers closer to rational numbers start to break up. These tori will then form new fixed points where the original tori were. A torus with winding number of r/s will dissolve into an integer multiple of s fixed points surrounded by new tori. Between each of these new fixed points, there are hyperbolic fixed points: near these the motion is chaotic.

Rational Winding Number
Any real number can be decomposed using continued fraction expansion. This is done by first removing the integer portion of the number, then take the reciprocal of the remaining decimals. The integer portion from each step forms the continued fraction expansion for that number. A number with a small entries is more irrational. The most irrational number is the golden mean.

KAM theorem
For sufficiently small perturbation, most invariant tori are not destroyed but are only deformed.