Chaos

When the trajectory in phase space is a closed curve, it is called a limit cycle. When the system is dissipative, then the area (volume) in phase space gets smaller over time. The set of points the trajectory traces out after a long time (after transients) in a dissipative system is called an attractor. In a chaotic system, the attractor is very sensitive to the initial condition and is called a strange attractor.

In a dissipative system, by changing a parameter, we can often get the period of the trajectory in phase space to double. This is an example of a bifurcation. Chaos may develop as the result of an infinite number of bifurcations.

Poincaré Section

Since it is difficult for us to visualize the phase trajectory of more than 1 variable, it helps to use a Poincaré section. A Poincaré section is obtained by setting one of the phase space variables to a constant and plotting the other phase space variables.

Often, the Poincaré section for a dissipative system exhibits wave like structures that resemble lines. But as one zooms into these regions of lines, one will discover structure to be formed by multiple lines and further zooming in results in even more of the same structure. The dimension of this structure is decimal number between 1 (line) and 2 (area) and is called a fractal.

1-D Map

Given, the function $F(x)$, we can plot a 1-D map, $x_{n+1} = F(x_n)$ against $x_n$. In this map we have stable behavior when $|F'(x_n)| \leq 1$ and unstable behavior when $|F'(x_n)| > 1$. The probability of visiting any regional of points on this plot, $\rho(x) dx$, is invariant under iteration after the transient response has died out.

Lyapunov Exponent

We can calculate the Lyapunov exponent, $\lambda(x_0)$ for the function $F(x_n) = x_{n+1}$,

$$\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{d(x_0)} \right| = \lim_{n \to \infty} \frac{1}{n} \log |F'(x_{n-1})F'(x_{n-2}) \cdots F'(x_1)F'(x_0)|$$

In higher dimensions, the Lyapunov exponents are the eigenvalues of the matrix $M$, stability matrix, which maps the initial conditions to the next time step. Chaotic behavior is found when any $\lambda > 0$ and non-chaotic behavior is found when all $\lambda < 0$.

Dimensions

One way to find the dimension of the system is the box method. Given a Poincaré section, cover the plot with boxes of length $\varepsilon$ and determine the number of boxes, $N$, needed in order to contain all points on the Poincaré section. The Capacity Dimension, $D_0$, of the system is given by $D_0 = D_C = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}$
Another way of determining the dimensions of a fractal structure is by using the Lyapunov exponents. First, order the Lyapunov exponents such that $\lambda_1 > \lambda_2 > \lambda_3 \ldots$, then plot $s_j = \sum_{i=1}^{j} \lambda_i$ against $i$. The intersection of this plot and the horizontal axis is the Lyapunov Dimension, $D_1$ (Information Dimension), of the system.

**Problem 1:** Lorenz Model: The Lorenz model is a set of evolution equation for $(X(t), Y(t), Z(t))$

\[
\begin{align*}
\dot{X} &= -\sigma(X - Y) \\
\dot{Y} &= rX - Y - XZ \\
\dot{Z} &= XY - bZ
\end{align*}
\]

We will use the values $\sigma = 10, b = \frac{8}{3}$ and vary $r$.

Derive the evolution equation for an infinitesimal perturbation $(\delta X(t), \delta Y(t), \delta Z(t))$ from a solution $(X(t), Y(t), Z(t))$ Now use this to:

(a) Find at what value $r_c$ of $r$, increasing from $r = 0$, does the solution $(0, 0, 0)$ become unstable?

(b) Show that for $r > r_c$ there are three steady state solutions.

(c) At what value of $r$ do all three steady state solutions go unstable? Is the new instability stationary (real eigenvalues) or oscillatory (complex eigenvalues)?

(d) Show how you would calculate the Lyapunov exponents.

**Problem 2: Lyapunov exponents in higher dimensions.** The Henon map is given by:

\[
\begin{align*}
x_{n+1} &= 1 - ax_n^2 + y_n \\
y_{n+1} &= bx_n
\end{align*}
\]

Common parameters used are $a = 1.4$ and $b = 0.3$. Set up a scheme for calculating the two Lyapunov eigenvalues of the map, i.e. do all the algebra to make explicit what you would iterate and how you would then calculate the Lyapunov exponents.