1 Tuesday, September 28, 2010

1.1 Introduction

The class is called combinatorial analysis, but it really will be about graph theory, which includes matchings, colorings, and extremal G-T. Grades will be assigned on the basis of the homework, of which there will be 7-8.

Our notion of a graph is a finite set with a collection of distinguished two-element subsets. We could spend an entire day arguing over definitions, but we won’t.

1.2 Eulerian circuits

The motivation for graph theory comes from the Königsberg bridge problem, a recreational curiosity. It motivates the following definitions:

- A walk is a sequence of the form \( v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n \) where edge \( e_i \) connects \( v_i \) and \( v_{i+1} \).
- A trail is a walk with no edge occurring more than once.
- An Eulerian trail is a trail containing all the edges of the graph.
- An Eulerian circuit is a trail with \( v_0 = v_n \).
- The neighborhood \( N(v) \) of a vertex \( v \) is the set of vertices which it shares edges with.
- The degree of a vertex \( v \) is the number of edges incident at that vertex.
- A graph \( G \) is disconnected if there is a partition of the vertex set \( V = V_1 \cup V_2 \) such that for any \( v_1 \in V_1 \) and \( v_2 \in V_2 \), there is no edge between \( v_1 \) and \( v_2 \). Say \( G \) is connected otherwise.

Of course, the solution to the problem examines the parity of the vertex degrees.

Proposition. There is an Eulerian circuit if and only if \( G \) is connected and every vertex has even degree.

Proof. The idea of the proof is that we just make a circuit, then add more edges to it.

We induct on the number of edges, and the base case is trivial. Start with a vertex \( v_0 \). Let \( C \) be the longest Eulerian trail starting at \( v_0 \). If \( C = v_0, e_1, \ldots, v_n \), then since each vertex degree in this trail is even, we must have \( v_n = v_0 \), i.e. it must be a circuit.

Now if \( C \) has all the edges of \( G \), we are done. Otherwise, there is some vertex \( u \) in \( C \) with unused edges (if not, \( G \) is disconnected). Remove the edges of \( C \) from \( V \) to make graph \( G' \), which retains the property of all edges having even degree. Now the connected component of \( u \) has fewer edges than \( G \), so by the induction hypothesis gives us an Eulerian circuit. We can splice this circuit into \( C \), violating its maximality, as desired.
1.3 Hamiltonian circuits

A common feature of graph theory is that when you ask a question in which the protagonist is a vertex or an edge, you can often ask the other question. What is the vertex analog of an Eulerian circuit?

**Definition.** A walk with no repeated vertex is called a *path.*

**Definition.** A path \( P = v_0, v_1, \ldots, v_n \), where \( v_n, v_0 \) is an edge, is called a *cycle.*

**Definition.** A *Hamiltonian path* in a graph \( G \) is a path containing all the vertices of \( G \).

**Definition.** A *Hamiltonian cycle* in a graph \( G \) is a cycle containing all the vertices of \( G \).

When does a graph \( G \) have a Hamiltonian cycle? Well, multiple edges are irrelevant, because we won’t be retracing them. Also notice that if a graph has a Hamiltonian circuit, adding extra edges doesn’t change the answer.

We would think as a heuristic that more edges would help, but there are some extreme counterexamples, e.g. \( K_{n-1} \) with one vertex attached by only one edge. Let’s consider some examples of the borderline: graphs with no Hamiltonian cycle such that if any edge is added, a Hamiltonian cycle is formed.

Let \( |V| = n \), and \( V = \{1, 2, \ldots, n\} \). Assume 1 and \( n \) are not adjacent (there must be some nonadjacent edges). Then adding this edge creates a Hamiltonian cycle which must go through that edge. Renumber the intermediate vertices so that the cycle goes \( 1, 2, \ldots, n \), i.e. \( k \leftrightarrow k + 1 \) for \( 1 \leq k \leq n - 1 \).

Now consider the neighbors of 1 and \( n \). If \( 1 \leftrightarrow i + 1 \) and \( n \leftrightarrow i \), then there’s a clear Hamiltonian cycle. Let \( N_n = \{j : n \leftrightarrow j\} \) and \( N_i = \{i : 1 \leftrightarrow i\} \). Then \( N_n \) and \( N_i' = N_i - 1 \) must be disjoint. Since \( N_1', N_n \subseteq \{1, 2, \ldots, n - 1\} \), \( d(n) + d(1) = |N_n| + |N_i'| \leq n - 1 \). So we conclude that if \( G \) is such a maximal non-Hamiltonian graph, then pairs of vertices without an edge between them have total degree at most \( n - 1 \). In particular, we have proved the following theorem.

**Theorem** (Dirac). If \( n \geq 3 \) and \( d(v) \geq n/2 \) for every vertex \( v \in V(G) \), then every \( G \) has a Hamiltonian cycle.

Of course, we have also arrived there with this weaker condition.

**Theorem** (Ore). If \( n \geq 3 \) and \( d(u) + d(v) \geq n \) whenever \( u \not\sim v \), \( G \) is Hamiltonian.

We can even generalize this a little bit further. We have actually shown that if \( u \not\sim v \) and \( d(u) + d(v) \geq n \), then \( G + \{u, v\} \) is Hamiltonian if and only if \( G \) is Hamiltonian. We can iterate this until we get to a point where every pair of vertices with vertex degrees summing to at least \( n \) is adjacent. This reduces the cases to check to a smaller set, but it’s still very difficult.

One classical example graph which is difficult to see is not Hamiltonian is the Peterson graph. Draw a pentagon and a pentagram, align vertices and connect them. Of course, all vertices have degree 3 so we can’t use the previous argumentation. Try to prove that the Hamiltonian graph is not Hamiltonian.

One note on pictures for graphs: They can be very misleading. For instance, the Peterson graph can also be drawn with a hexagon with four vertices in the middle.

2 Thursday, September 30, 2010

2.1 Hamiltonicity Continued

Recall last time that we discussed Eulerian circuits and Hamiltonian circuits. For Eulerian circuits, we found a necessary and sufficient condition: All vertices have even degree. For Hamiltonian circuits, we only came up with partial results, Dirac’s and Ore’s theorems. Let’s revisit those.

**Definition.** The *Hamiltonian closure* \( C_H(G) \) of a graph \( G \) with \( n \) vertices is obtained as follows: For every \( x \neq y \) with \( x \not\sim y \) and \( d(x) + d(y) \geq n \), add the edge \( x \leftrightarrow y \). Continue this process
Why is this closure unique? We have different choices we could make. But if \(d(x) + d(y) \geq n\), this statement remains true when we add other edges. We showed last time that \(G\) is Hamiltonian iff \(C_H(G)\) is Hamiltonian.

**Theorem** (Chvátal). Let the vertices of \(G\) be \(\{1, \ldots, n\}\), and let the degrees be \(d_1 \leq d_2 \leq \cdots \leq d_n\). For \(i < \frac{n}{2}\), if \(d_i > i\) or \(d_{n-i} \geq n-i\), then \(G\) is Hamiltonian.

This theorem is not very elegant, but the proof is.

**Proof.** Since these degree conditions hold for \(C_H(G)\) if they hold for \(G\), assume that \(G = C_H(G)\). We claim that \(G = K_n\), the complete graph on \(n\) vertices.

Suppose otherwise, and we will produce \(i < n/2\) such that \(d_i \leq i\) and \(d_{n-i} < n-i\). To do so, pick \(x \neq y\) such that \(x \leftrightarrow y\) and such that \(d(x) + d(y)\) is maximal among all nonadjacent pairs. Since \(G\) is closed, we must have \(d(x) + d(y) < n\). WOLOG, \(d(y) \geq d(x) = i\). Then \(2i = 2d(x) \leq d(x) + d(y) < n\) so \(i < n/2\).

Consider the vertices not adjacent to \(x\). Not counting \(x\) itself, there are \(n-i-1\) of them, and by the maximality of the pair \((x, y)\), each of these must have degree \(\leq d(y)\), so we must have \(d_{n-i} \leq d(y) < n-i\).

For the other side, turn the argument around. We get that \(x\) and the \(n - d(y) - 1\) nonneighbors of \(y\) all have degrees \(\geq i\), so we must have \(d_x \leq d_n - d(y) \leq i\), as desired. \(\square\)

Of course, this theorem is still very clunky. We only need the list of degrees, but this will often tell us very little about the graph. If the degree set is \(\{2, 2, 2, 2, 2, 2, 2\}\), the graph could be an 8-cycle, or two 4-cycles, or a 5-cycle and a triangle. One has a Hamiltonian cycle and the others aren’t even connected.

### 2.2 The Chvátal-Erdős Theorem for Hamiltonicity

First, we need some definitions.

**Definition.** To *delete a vertex* is to remove that vertex and all edges incident on it. To *delete an edge* is to remove the edge from the edge set of the graph.

**Definition.** The connectivity of \(G\), denoted \(\kappa(G)\), is the minimum number of vertices that must be deleted in order to obtain a disconnected graph.

Note that the singleton graph with one vertex is not considered connected, so \(\kappa(K_n) = n - 1\).

It is easy to see that Hamiltonian graphs \(G\) must have \(\kappa(G) > 1\), since the Hamiltonian cycle minus any deleted vertex connects the rest of the vertices. This gives rise to a vague heuristic that if \(\kappa(G)\) is large (we’ll see precisely what we need later), then \(G\) is more likely to be Hamiltonian.

Let \(\kappa(G) = k > 1\). First notice that every degree must be at least \(k\), since otherwise we could remove the neighbors of the given vertex.

Let \(C\) be the longest cycle in \(G\); we want to find when \(C\) contains all the vertices. First, \(C\) has at least \(k+1\) vertices, by the following argument: Add vertices \(v_1, \ldots, v_r\) to a path until it is no longer possible, i.e. all \(\geq k\) neighbors of \(v_r\) are previous \(v_i\). Pick the earliest and connect the cycle; it contains \(v_r\) and all its neighbors, at least \(k+1\) vertices.

Now delete the vertices of \(C\), yielding \(G \setminus C\), and if \(G\) is not Hamiltonian, this will not be empty. Let \(H\) be a component of \(G \setminus C\), and \(u_1, \ldots, u_j\) be the neighbors of vertices of \(H\) in \(C\), and by connectivity, \(j \geq k\). First, \(u_1\) and \(u_{i+1}\) cannot be adjacent, because otherwise we can obtain a larger cycle through \(H\).

Let the clockwise neighbors of \(u_i\) be \(v_i\) in \(C\). Then note that there is no edge between the \(v_i\) and \(H\) for all \(i\). Therefore, taking the \(v_i\) and at least one vertex in \(H\), we have an independent set of at least \(k+1\) vertices. This implies the following result.

**Definition.** A set of vertices with no edges between any pair is called *independent*.

**Definition.** The independence number \(\alpha(G)\) of a graph \(G\) is the size of the largest independent set in \(G\).

**Theorem.** For graphs \(G\) with at least 3 vertices, if \(\kappa(G) \geq \alpha(G)\), then \(G\) is Hamiltonian.
Again, this isn’t too useful because $\alpha(G)$ is hard to compute. But if we knew that the graph is not Hamiltonian, we could use this proof to generate independent sets. Actually, this is only possible if we can generate a cycle.

In conclusion, Hamiltonicity is a very difficult condition, because there are very many different ways to approach it. Every little neat criterion has a theorem lurking behind it.

What about necessary conditions for Hamiltonicity?

**Proposition.** If $\exists S \subseteq V(G)$ such that $G \setminus S$ has more than $|S|$ components, then $G$ is not Hamiltonian.

Indeed, a Hamiltonian cycle must visit at least one vertex of $S$ between visits to the connected components of $G \setminus S$.

However, this is not sufficient. A counterexample is the following: Take $K_4$, with vertices $\{a, b, c, d\}$, and add vertices $a \leftrightarrow e \leftrightarrow b, a \leftrightarrow f \leftrightarrow c$, and $a \leftrightarrow g \leftrightarrow d$. You can check that this has no Hamiltonian cycle by satisfies the above property.

The bad news is that Hamiltonicity is hard. The good news is that it’s a field of active research. Any new method you have for coming up with cycles gives you a new theorem. With that, we’ll move on.

### 2.3 Matchings

Let $G$ be a finite group and $H < G$ (a strict) subgroup. Let the left cosets of $H$ be $\{g_1H, g_2H, \ldots, g_nH\}$. Are the right cosets of $H$ given by $\{Hg_1, Hg_2, \ldots, Hg_n\}$?

Let’s start by letting the right cosets by $\{Hg'_1, Hg'_2, \ldots, Hg'_n\}$. Of course, if any left and right coset have an element in common, we can use that element as a representative for each.

We can imagine this as a graph theory problem: Connect the cosets which share an element, making a bipartite graph. We want to know if there is a matching.

Next time, we’ll prove Hall’s marriage theorem in three different ways.

### 3 Tuesday, October 5, 2010

#### 3.1 Systems of Distinct Representatives

**Definition.** Let $A = \{A_1, \ldots, A_n\}$, with each $A_i \subseteq X$, for some finite set $X$. We call $Y \subseteq X$ a system of distinct representatives for $Y = \{x_1, \ldots, x_n\}$ if for each $A_i \in A$, we have $x_i \in A_i$.

Recall the marriage problem: $n$ men have a preference set of women in a population. When can we marry off all the men to women of their preferences?

Suppose for some $1 \leq k \leq n$, there are $i_1, \ldots, i_k$ such that $|\bigcup_{j=1}^k A_{i_j}| < k$. Then such a pairing is clearly impossible. The opposite is Hall’s condition: For any $I \subseteq \{1, 2, \ldots, n\} =: [n]$, $|\bigcup_{i \in I} A_i| \geq |I|$. For notation, we will let $A_I := \bigcup_{i \in I} A_i$.

**Theorem (Hall’s Theorem for SDR).** If $\forall I \subseteq [n]$, Hall’s condition is satisfied, then there is an SDR for $A$.

This is the type of theorem which can be proven however you try. We’ll prove this in three ways.

**Hall’s Proof.** The idea: We want to remove extraneous edges until we have just our matching.

We induct on $\sum_{i=1}^n |A_i|$. Since each $|A_i| \geq 1$, the base case is $|A_i| = 1$ for all $i$, in which case we are clearly done.

Otherwise, suppose that $x \neq y \in A_1$. If either $\{A_1 \setminus \{x\}, A_2, \ldots, A_n\}$ or $\{A_1 \setminus \{y\}, A_2, \ldots, A_n\}$ satisfies Hall, we are done. So we conclude otherwise. Therefore, there exist $I, J \in \{2, 3, \ldots, n\}$ such that $|(A_1 \setminus \{x\}) \cup A_j| < |J| + 1$ and $|(A_1 \setminus \{y\}) \cup A_J| < |J| + 1$. Let $S = (A_1 \setminus \{x\}) \cup A_I$ and $T = (A_1 \setminus \{y\}) \cup A_J$. Then we have $|S| \leq |I|$ and $|T| \leq |J|$. We have $|S \cup T| = |A_1 \cup J| \geq 1 + |I \cup J|$ by Hall’s condition, and $|S \cap T| \geq |A_I \cap J| \geq |I \cap J|$. Adding, $|S| + |T| = |S \cup T| + |S \cap T| \geq 1 + |I \cup J| + |I \cap J| = 1 + |I| + |J|$.  \(\square\)
Second Proof. Suppose for each \( I \subseteq [n], |A_I| > |I| \). Then pick \( x \in A_I \) arbitrarily and check that \((A_2 \setminus \{x\}, A_3 \setminus \{x\}, \ldots, A_n \setminus \{x\})\) satisfy Hall’s condition, so we can induct on \( n \).

Otherwise, there exists \( I \subseteq [n] \) such that \(|A_I| = |I|\). By induction we can find a matching between \( A_I \) and \( I \). Let \( J = [n] \setminus I \). Does \( \{A_j \setminus A_I : j \in J\} \) still satisfy Hall’s condition? If not, there exists some \( J' \subset J \) such that \( \left|\bigcup_{j \in J'} (A_j \setminus A_I)\right| = |A_{J'} \setminus A_I| < |J'|\). But \(|A_{J \cup J'}| \geq |J'| + |I|\) as \( A \) satisfies Hall’s condition.

Meanwhile, \(|A_{J \cup J'}| \leq |A_I| + |A_{J'}| < |I| + |J'|\). \( \square \)

Remark. A simple consequence of this proof is this: If \(|A_i| = m_i\) for \( i = 0, \ldots, n-1 \) and if \( m_0 \leq m_1 \leq \cdots \leq m_{n-1} \), then the number of distinct matchings is at least \( \prod_{i=0}^{n-1} \max\{m_i - i, 1\} \).

Remark. We started this discussion from an algebra problem: We had a finite group \( G \) and subgroup \( H \). Translating to this, we write the left cosets as \( H_1, \ldots, H_n \) and the right cosets as \( H'_1, \ldots, H'_n \). Connect \( H_i \) and \( H'_j \) if they intersect. We want to know if there is an SDR.

Of course, the answer is yes. We prove this by noting that \(|H_I| = \left|\bigcap_{i \in I} H_i\right| = |H||I|\), and therefore this cannot be contained in a smaller number of right cosets of \( H \), as desired.

Now we continue with our third proof.

Third Proof. Recall the definition of a bipartite graph:

**Definition.** \( G \) is bipartite if \( V(G) = A \cup B \), \( A \cap B = \emptyset \), and all edges of \( G \) contain one vertex of \( A \) and one vertex of \( B \).

**Definition.** A matching in a graph \( G \) is a subgraph \( M \) in which every vertex of \( G \) has degree at most 1 in \( M \).

The problem of finding an SDR for \( A \) is equivalent to finding a matching in which every vertex in \( A \) has degree 1. For \( I \subseteq A \), let \( N_I = \{b \in B : \exists a \in I, a \leftrightarrow b\} \), and Hall’s condition says \(|N_I| \geq |I|\).

Now let \( M \) be a matching in \( G \). If every \( a \in A \) has degree 1, we are done. Otherwise, there is a vertex \( a_0 \in A \) without an edge. Construct paths \( a_0, b_1, a_1, \ldots, b_n, a_n \) such that \( b_i \leftrightarrow a_i \) in \( M \), and \( a_{i-1} \leftrightarrow b_i \) in \( G \), and therefore, not in \( M \). We can always continue these chains from somewhere, since \(|N(\{a_0, a_1, \ldots, a_n\})| > n\). So we can extend this indefinitely, contradicting the finiteness of \( G \). \( \square \)

4 Thursday, October 7, 2010

4.1 Stochastic Applications

**Definition.** A real matrix \( A = ((a_{ij}))_{i,j=1,\ldots,n} \) is called doubly stochastic if (i) \( a_{ij} \geq 0 \), and (ii) \( \sum_i a_{ij} = \sum_j a_{ij} = 1 \).

We imagine \( a_{ij} \) as the probability of moving to state \( j \) from state \( i \). Therefore, every permutation matrix is a deterministic stochastic process.

Let \( D \subseteq \mathbb{R}^n \) be the set of all doubly stochastic matrices. This isn’t a vector space since we can’t add, but for any \( 0 \leq \lambda \leq 1 \), if \( A, B \in D \), \( \lambda A + (1 - \lambda)B \in D \). That is, any point along the line connecting \( A \) and \( B \) is in the set, making \( D \) convex. But the permutation matrices must be the endpoints of such lines, so they’re on the boundary in some sense. The next theorem clarifies this.

**Theorem** (Birkhoff-McLane-von Neumann). \( D \) is the convex hull of all Permutation matrices. That is, for any \( A \in D \), \( \exists \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0} \) and permutation matrices \( P_1, \ldots, P_k \), such that (i) \( \sum_{i=1}^k \lambda_i = 1 \), and (ii) \( A = \sum_{i=1}^k \lambda_i P_i \).

Here’s the proof idea: Find some \( P \) such that \( A - \lambda P \) has more zeros, and induct on the number of nonzero entries. But we need to pick \( \lambda \) and \( P \). \( \lambda \) can be the smallest nonzero entry. \( P \) can be thought of as a matching between rows and columns, so we need to find that matching.
Proof. We induct on the number of nonzero entries. Since there is at least one in each column, the base case is to arrange a matching matrix with at least one nonzero entry, in which case we must have a permutation matrix as desired. Now otherwise there is some entry that is strictly between 0 and 1. Create a bipartite graph $G$ with the vertices corresponding to the rows and columns, connected if they share a nonzero entry. We claim this graph has a matching. Indeed, any $k$ rows have sum $k$, so if their nonzero entries are in only $m$ columns with $m < k$, the sum of those columns is $k$, greater than $m$, a contradiction. So any set of $k$ vertices has at least $k$ neighbors, satisfying Hall’s condition.

Now a matching corresponds to a permutation matrix $P$ such that where $P$ has a 1, the entry in $A$ is nonzero. Let $λ$ be the minimum of these entries. $λ < 1$ since $A$ is not a permutation matrix. Then $A−λP$ is a stochastic matrix with at least one fewer zero entry, so by the induction hypothesis, it can be written as $∑λiPi$. Therefore, $A = λP + ∑(1−λ)λiPi$, written in the from we want, as desired.

Here’s an example from more of mainstream graph theory.

Definition. A vertex cover of a graph is a subset $U ⊆ V$ such that every edge of $G$ is incident with some $x ∈ U$.

Proposition. If $U$ is a vertex cover and $M$ is a matching, then $|U| ≥ |M|$ for any matching $M$.

Proof. Each edge of $M$ must contain one vertex in $U$. That is, $\min_{\text{vertex covers } U} |U| ≥ \max_{\text{matchings } M} |M|$. König’s lemma says that equality is achieved for bipartite graphs.

Theorem. If $G$ is bipartite, $\min_{\text{vertex covers } U} |U| = \max_{\text{matchings } M} |M|$.

Proof. Let $M$ be bipartite with parts $A$ and $B$, and $U$ be a minimum vertex cover of $G$. Write $X = U ∩ A$ and $Y = U ∩ B$, $A \setminus X = X$, and $B \setminus Y = Y$. We want to produce a complete matching between $X$ and $Y$, and symmetrically, $Y$ and $X$, so combining these yields a matching between $X$ and $Y$. So we need to verify Hall’s condition. Let $I ⊆ X$ and suppose that $|N(I) ∩ Y| ≤ |I|$. Then $(X \setminus I) ∪ N(I) ∪ Y$ is a smaller vertex cover, contradiction. So it satisfies Hall’s condition and we have a matching of the same size as $U$.

Recall the last proof of Hall’s theorem, where we found augmenting paths. Note that we didn’t use bipartiteness at any point. In fact, any graph with an augmenting path can be improved in that sense. What about the other way? Does every matching which is not maximal have an augmenting path?

Suppose $N$ is a larger matching. Consider a new graph with the edges which either $M$ or $N$ have. Each vertex has degree at most 2, so the graph consists of paths and cycles. Moreover, the edges alternate between edges from $M$ and edges from $N$. Any cycle is even, so it has the same number of edges from $M$ and from $N$. Since there are more edges in $N$, there must be a path with one more $N$ edge than $M$ edge, so this is our augmented path. We have shown

Proposition. If $M$ is a matching with no augmenting path, then $M$ is maximal.

A few words about this algorithm: Finding an augmenting path in the bipartite case corresponds to finding a path between two vertices in a directed graph. Without going into it, this is very easy using the usual algorithms. However, the same method doesn’t work in a non-bipartite graph, since you can go on either direction along an edge, and odd cycles can cause you to revisit a vertex without it corresponding to an augmenting path.

Definition. A matching $M$ in a graph $G$ is a perfect matching (PM) if it contains all the vertices of $G$.

Suppose $G$ has a perfect matching $M$. Pick $U ⊆ V(G)$ and consider $G \setminus U$. Let $o(H)$ be the number of odd components of a graph $H$. Then since each odd component of $G \setminus U$ must have an edge in $M$ with a vertex in $U$, we must have $o(G \setminus U) ≤ |U|$. This was Tutte’s observation. For instance, a graph consisting of a vertex connected to a vertex in each of three $K_3$s has no perfect matching.
Let’s do this in general for any matching. Take out the vertices \( U \), and let \( \{ E_i \}_{i=1}^r \) be the even components and \( \{ O_i \}_{i=1}^s \) the odd components of \( G \setminus U \). Note that \( |M| \leq |U| + \sum_{i=1}^r |E_i| + \sum_{j=1}^s |O_j| - \frac{1}{2}o(G \setminus U) = |U| + \frac{1}{2}(\sum |E_i| + \sum |O_j|) - \frac{1}{2}o(G \setminus U) = |U| + \frac{1}{2}(\sum |E_i| + |O_j| - \frac{1}{2}o(G \setminus U)) \). This gives us an upper bound on the size of \( M \).

Is this tight? To spoil the surprise, yes. This is called the Tutte-Berge formula, and we’ll prove it next time.

5 Tuesday, October 12, 2010

Homework 2 was just sent out 10 minutes ago, and is due next Tuesday.

5.1 Matchings in Graphs

In bipartite graphs, we showed that a matching \( M \) in \( G \) is maximum if it has no augmenting path. What about in general? Tutte observed: Let \( U \subseteq V \) and \( M \) a matching in \( G \). Then \( |M| \leq \frac{1}{2}(|V| + |U| - o(G \setminus U)) \), where \( o(G \setminus U) \) is the number of odd components of \( G \setminus U \). Therefore, we can write: \( \max |M| \leq \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - o(G \setminus U)) \).

Theorem (Tutte-Berge Formula). Equality holds, that is, \( \max |M| = \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - o(G \setminus U)) \).

Proof. To prove this, we will use an algorithm known as Edmonds’ Blossom Algorithm (1965). This appeared in a paper called “Trees, Paths, and Flowers.”

Definition. Suppose \( M \) is a matching in a graph \( G \). A flower is an alternating path of non-\( M \) and \( M \) edges, respectively, terminating in a vertex of an odd cycle, all of whose vertices except the end of the path are \( M \)-saturated within the cycle.

We’re trying to find an augmenting path in this case, and flowers are one nuisance that can prevent us from being able to do so. We wonder, are there other nuisances? And how do we deal with flowers?

The part of the flower before you reach the odd cycle is called the stem, and the cycle is called the blossom. We can switch the edges in the stem from non-\( M \) to \( M \) and vice versa, and then there’s a unique unmatched vertex.

Now we shrink the blossom, replacing all vertices in the cycle with a single vertex, and make this vertex adjacent to all of the original neighbors of elements in the cycle. Of course, none of these edges will be in \( M \) since all the vertices in the blossom have \( M \)-neighbors in the blossom. This preserves maximality, since augmenting paths translate easily.

Here’s the algorithm: Let \( M \) be a matching. Label all unsaturated vertices even. Suppose \( u \) is even and \( v \leftarrow u \). If \( v \) is even, append \( uv \) to \( M \). Otherwise label \( v \) as odd, and \( v \) has an \( M \) partner, so label that vertex even, and continue in this fashion, where neighbors of odd vertices are labeled even, and the \( M \)-partners the even vertices are labeled odd. Also label all vertices with the original even vertex from which it got its label. So \( v \) would be labeled (odd, \( u \)).

Now if there are two even vertices which are adjacent, we have two cases. If they have the same second label, we get a blossom, so shrink it. If they have different second labels, we get an augmenting path, so switch it to get a larger matching.

The graph is finite, so this process must terminate. Then we separate the vertices into the odd vertices \( O \), the even vertices \( E \), and the unlabelled vertices \( U \). Now all edges from even vertices go to odd vertices. Moreover, all odd vertices are matched with an even vertex, and the unlabelled vertices are all matched with each other. Therefore, \( |M| \geq |O| + \frac{1}{2}|U| = \frac{1}{2}(|V| + |O| - |E|) \).

Now notice that if you remove the odd vertices, each even vertex is isolated, i.e. an odd component. Therefore, in the Tutte-Berge formula, we take \( O = U \), so the formula holds.

When a blossom unfurls, it adds \( |B| - 1 \) matching edges, and \( |B| - 1 \) vertices. The blossom is then an odd component, with its only outside edges being to odd vertices.
This algorithm has complexity $O(n^2m)$ where $n$ is the number of vertices and $m$ is the number of edges.

5.2 Stable Matchings

This problem arose out of the question of marriages. But the reality of such a situation is more complicated; out of the possible partners, there are preferences.

Let $(M, W, \pi)$ be a bipartite graph with components $M$ and $W$, and for each $v \in M \cup W$, $\pi_v$ is a total order on $N(v)$. Let the vertices of $M$ be $a, b, c$ and of $W$ be $A, B, C$, where in a matching $M$ we pair up $a$ and $A$, etc.

**Definition.** A matching $M$ is stable if for any $a, B \notin M$, then either $\pi_a(A) > \pi(a)B$ or $\pi_B(b) > \pi_B(a)$.

Note that by definition, a stable matching is maximal, i.e. there is no $M'$ such that $M' \supseteq M$. If two unmarried people know each other, they’ll get married. We wonder if there is always a stable matching.

(This question also applies to students applying to schools, but for some reason this is not as interesting of a problem. :P)

In every Jane Austen novel, there’s a particular algorithm they use. The men ask the first lady on their list, and the women weigh their options. They reject anyone worse than their best current option, and sometimes tell the man they’ll get back to them. Then the men who were rejected move on, and eventually all the men have outstanding proposals, and they all get married.

This time-honored tradition does produce a stable matching. Suppose there’s a better marriage for both sides to be had. If the man proposed to the woman, she should have accepted, because her options only get better. If he didn’t, then that means he found a more preferable woman, so he’s happy.

Maybe there are mathematical reasons behind some other time-honored social traditions...

6 Thursday, October 14, 2010

**Definition.** Let $G$ be a graph. Suppose $f : V \to \mathbb{N} \cup \{0\}$. A subgraph $H$ of $G$ is called a $f$-factor if $\forall v \in V(G), d_H(v) = f(v)$.

**Example.** A perfect matching is an $f$-factor, where $f \equiv 1$. We call this a 1-factor.

Tutte realized that the problem of finding an $f$-factor actually reduces to finding a 1-factor for a slightly modified graph:

Replace each vertex $v$ with $d_G(v)$ vertices, and attach one former edge involving $v$ to each. Then the problem of finding an $f$-factor reduces to choosing $f(v)$ vertices from each cluster, or not choosing $d_G(v) - f(v)$ from each. For each $v$, add $d_G(v) - f(v)$ vertices and make a complete bipartite graph between these and the $d_G(v)$ vertices that $v$ split into. Then there’s a natural correspondence between $f$-factors and 1-factors of this modified graph.

6.1 Networks

**Definition.** A directed graph $\vec{D}$ is a graph $G$ and an orientation $\pi$ for each edge, i.e. for each edge $e$, $\pi(e) \in e$ (i.e. we pick one of the vertices as a sink).

**Definition.** A directed graph $\vec{D}$ with two distinguished vertices $s, t$ (called source and sink, respectively), and a nonnegative function $c : E(\vec{D}) \to \mathbb{R}^+ \cup \{0\}$ called the capacity of the network.

**Definition.** A flow on a network is a function $f : E(\vec{D}) \to \mathbb{R}^+ \cup \{0\}$ such that for all $v \neq s, t$, “Kirchoff’s Law” is satisfied, i.e. $\sum_{e \in e, \pi(e) \neq v} f(e) = \sum_{\pi(e) = v} f(v)$. (We write $e$ “leaves” $v$ for the first sum and $e$ “enters” $v$ for the second. Then whatever enters $v$ also leaves $v$.) Moreover, $f(e) \leq c(e)$ for all $e \in E(\vec{D})$.

**Definition.** If $f$ is a flow, let the volume be $v(f) := \sum_e$ leaving $s f(e) - \sum_e$ entering $s f(e)$. 

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It is straightforward to see that \( v(f) = \sum_{e \text{ entering} \, t} f(e) - \sum_{e \text{ leaving} \, t} f(e) \). Note that the volume can be negative; in fact, the setup is symmetric about switching \( s \) and \( t \).

In general, the following is true:

**Definition.** Suppose \( X \subseteq V(\bar{D}) \) such that \( s \in X, t \notin X \). We shall call the pair \((X, \bar{X})\) a cut in the network.

For cuts \((X, \bar{X})\), define \( f(X, \bar{X}) = \sum_{e \text{ from} \, X \text{ to} \, \bar{X}} f(e) - \sum_{e \text{ from} \, \bar{X} \text{ to} \, X} f(e) \). Then \( f(X, \bar{X}) = v(f) \).

We can write \( \phi(v, e) = 1 \) if \( \pi(e) = v \), \(-1\) if \( v \notin e \) but \( \pi(e) \neq v \), and \( 0 \) if \( v \notin e \). Then \( f(X, \bar{X}) = \sum_{v \in X} \sum_{e \in E} f(v) \phi(v, e) \). Then for \( v \neq s \), \( \sum_{e \in E} f(v) \phi(v, e) = 0 \), so for any cut \((X, \bar{X})\), \( v(f) = f(X, \bar{X}) \).

**Question:** Given a network with capacity function \( c \), how large can a flow be, i.e. what is \( \sup_f v(f) \)?

First, our supremum can be replaced with a maximum. Consider a sequence of flows \( f_n \) with \( v(f_n) \to v \) as \( n \to \infty \). Look at one edge \( e \), and there is a convergent subsequence on \( f_n(e) \). Repeat the process on this subsequence and another edge, until you have a sequence that converges for each edge, and take the limit. Thus, there is a maximum flow.

Note that for any cut \( c \), we have \( v(f) \leq \sum_{e \text{ from} \, X \to \, \bar{X}} c(e) =: C(X, \bar{X}) \). That is, \( \max_f v(f) \leq \min_X C(X, \bar{X}) \). Equality holds here too.

**Theorem** (Ford-Fulkerson Theorem, Max Flow-Min Cut Theorem). \( \max_f v(f) = \min_X c(X, \bar{X}) \).

**Proof.** Suppose we have a path in \( D \) \( s \to x_1 \to x_2 \to \cdots \to x_n \to t \), let \( f_i \) be the flow on the edge from \( x_i \) to \( x_{i+1} \), and let the capacity on that edge be \( c_i \). Then suppose that \( f_i < c_i \) for all correctly oriented edges, and \( f_i > 0 \) for all oppositely oriented edges. Then we can increase all of the correct flows by \( \epsilon \) and decrease all the opposite edges by \( \epsilon \) for some small \( \epsilon \), increasing the total flow. Call a path like this an \( f \)-augmenting path.

For the maximal flow, therefore, there must be no such path. We want to construct a cut that matches this maximal flow. Generalize the notion of augmenting paths to end at vertices other than the sink. Then let \( X \) contain \( s \) and any vertex which can be reached by an \( f \)-augmenting path. Consider an edge from \( y \in X \) to \( z \in \bar{X} \). If the flow does not match the capacity, then extend the augmenting path from \( s \) to \( y \) to go to \( z \), contradicting \( z \in \bar{X} \), so the flow must match the capacity. Similarly, edges from \( \bar{X} \) to \( X \) must have zero flow, so the cut matches the flow, as desired.

\[ \square \]

7 Tuesday, October 19, 2010

Homework 3 was sent out and is due next Tuesday, October 26.

7.1 Ford-Fulkerson Max Flow-Min Cut Theorem

The theorem states: Suppose \((\bar{D}, c)\) is a network, with (finite) capacity function \( c \). Then \( \max_f \text{ flow } v(f) = \min_X \text{ cut } c(X, \bar{X}) \).

If \( c \) is integral, i.e. if \( c(e) \in \mathbb{N} \cup \{0\} \) for all \( e \in E(\bar{D}) \), then the augmenting path algorithm gives a terminating procedure for obtaining a flow with maximum value, starting from the 0 flow. In particular, there is a maximum integral flow. Indeed, every time we apply the augmenting procedure, we increase the flow by a positive integer, so it can only do so a finite number of times, as the min-cut is finite.

**Remark.** If \( c \) is not integral (in particular, irrational), it is possible that we pick augmenting paths that do not increment the flow sufficiently. Hence this algorithm may not terminate, and in fact, the flow values can convert to a number strictly less than the value of a maximum flow. There is an example involving a directed \( K_{4,4} \) between the source and sink, with capacities involving \( \sqrt{\pi - 1} \). By iterating particular choices of augmenting paths, you can get the cost to increase by exponentially smaller amounts at each step.

There is a better algorithm that always terminates, but we won’t go there.
Theorem. Suppose $A$ is a $b \times v$ 0-1 matrix, with all row sums equal to $k$ and all column sums equal to $r$. (In particular, $rv = kb$). Suppose $\alpha \in \mathbb{Q}$ with $0 < \alpha < 1$ such that $k' = \alpha k$ and $r' = \alpha r$ are both integers. Then one can change some of the 1’s of $A$ into 0’s such that in the resulting matrix $A'$, every row sums to $k'$ and every column sums to $r'$.

Proof. The matrix $A$ associates a bipartite graph $G_A$ with a 1 in the $(i, j)$ cell corresponding to an edge from the $i$th vertex of $R$ to the $j$th vertex of $C$. Thus the degree of all elements in $R$ is $k$ and the degree of all elements in $C$ is $r$.

Add a source $s$ connected to $R$ and a sink $t$ connected to $C$, and add capacities $c((s, r)) = k$, $c((c, t)) = r$, and $c((r, c)) = 1$, and let the flows match these capacities. Then this is clearly a maximal flow.

Now consider a new network with $c((s, r)) = \alpha k$ and $c((c, t)) = \alpha r$, both of which are integers. Keep $c((r, c)) = 1$. Then $f' = \alpha f$ is a flow, and again a maximum flow since \{s\} is a cut with the same value. But we have shown by our algorithm that if there is also a maximal integral flow $f''$ which achieves the same flow. Now the edges from $R$ to $C$ must have flow either 0 or 1 in $f''$. Translating this back to the matrix, we are done. \qed

Generally, this typo of argument works: Take some maximal gadget, then tweak it by a rational factor, and you can find an integral maximal gadget.

7.2 Vertex Analogue of Max Flow Min Cut Theorem

Here’s a little bit of a different problem: Suppose we have a network, but our capacities are not forced on networks (like pipe cross-sections) but instead on vertices. There’s an analogous theorem.

Let $c_v : V(D) \rightarrow \mathbb{R}^+ \cup \{0\}$ be a capacity function, and $f_v : E(D) \rightarrow \mathbb{R}^+ \cup \{0\}$ where for each $v \neq s, t, \sum_{w \rightarrow v} f(w, v) = \sum_{v \rightarrow y} f(v, y) \leq c_v(v)$. We wonder if there is any max flow min cut analog here.

For this graph, construct another network where every vertex $x$ is replaced with two, $x_-$ and $x_+$, with an edge from $x_-$ and $x_+$ with capacity $c_v(x)$. Add edges with infinite capacity from $x_+$ to $y_-$ if $x \rightarrow y$ is a directed edge in the original graph. Then you can see that flows correspond, and carrying the original max flow min cut theorem over, we have $\max f \nu(f) = \min_{(X, \bar{X})} c(X, \bar{X})$ as desired.

7.3 Menger’s Theorem

For $k \geq 1$, recall that a graph $G$ is said to be $k$-connected if no set of $k - 1$ vertices upon deletion results in a disconnected graph. In particular, 2-connected graphs need 2 vertices to be deleted to disconnect $G$.

A sufficient condition: Every two vertices lie on a cycle. Indeed, if a vertex $v$ disconnects the graph, it would not disconnect a cycle. Menger’s theorem says that this is sufficient.

Theorem (Menger’s Theorem, Vertex Version). A graph $G$ is $k$-connected iff any two distinct vertices have at least $k$ mutually internally disjoint paths joining them.

Proof. If it is $k$-connected, let $s$ and $t$ be two vertices. Make this into a network by replacing every edge with two directed edges, one in each direction. Now the minimum cut in the vertex sense must be at least $k$, so there is a maximum flow of at least $k$, which is therefore a set of $k$ disjoint paths, as desired. \qed

The edge analog has a similar statement and proof.

7.4 Covering $K_n$ with Matchings

Try to construct a covering of the edges of $K_n$ into matchings. Of course, $n$ has to be even.

There’s a nice general solution. Label your set $\{\infty, 1, 2, \ldots, 2n + 1\}$. For the $i$th matching $M_i$, pair $(\infty, i)$ and $\{(x, y) | x + y \equiv 2i \pmod{2n + 1}\}$. Each edge appears exactly once and each vertex is paired up exactly once in the matching, as desired.
Let’s generalize this. We have partitioned the set of all 2-element subsets of \{1, 2, \ldots, 2n\}. Similarly, can we partition the \(k\)-element subsets of \{1, 2, \ldots, kn\}? This has been done for \(k = 3\) with great effort, and \(k = 4\) with a really long argument that hasn’t been published. Then in 1973 Baranyai proved it in general:

**Theorem** (Baranyai’s Theorem). For all \(k, n\) one can partition the \(k\)-element subsets of \{1, 2, \ldots, kn\}.

It’s not at all clear how this can be proven using our ideas of flows and networks, but it’s a beautiful proof that appears a lot. We’ll cover it next time. (Wow, that’s a great cliffhanger.)

### 8 Thursday, October 21, 2010

**Definition.** Let \(k \leq n\) be positive integers. By a *simple\(^\prime\) \(k\)-uniform hypergraph on \(n\) vertices, we shall mean a pair \((V, \mathcal{E})\) with \(|V| = n\) and \(E \subseteq \binom{V}{k}\), i.e. \(\mathcal{E}\) is a collection of \(k\)-element subsets of the set \(V\). The complete \(k\)-uniform hypergraph is the hypergraph \(H = \binom{V}{k}\).

**Definition.** A 1-factor of \(X\) is a collection \(\mathcal{E}' \subseteq \mathcal{E}(X)\) such that \(E \neq E'\) in \(\mathcal{E}'\), we have \(E \cap E' = \emptyset\), and \(\mathcal{E} = \bigcup_{E \in \mathcal{E}'} E = V\). A 1-factorization of \(X\) is a collection \(\{\mathcal{E}_\lambda : \lambda \in \Lambda\}\) of 1-factors such that \(\mathcal{E}_{\lambda} \cap \mathcal{E}_{\lambda'} = \emptyset\).

**Theorem** (Baranyai’s Theorem). The complete \(k\)-uniform hypergraph on \(n\) vertices admits a 1-factorization iff \(k | n\).

The proof is by induction. First, let’s motivate it. Suppose you had a solution, and write it down. Then you leave your room and your annoying phony friend decides to erase every appearance of \(n\). Well, you can fix that. What if he erases all the \(n\)'s and \(n - 1\)'s? Still, you’d be fine. Okay, let’s prove that you can add an element one at a time and not do any damage.

**Proof.** Each 1-factor adds \(\frac{n}{k}\) \(k\)-cycles, so there must be \(\binom{n}{k} \frac{k}{n} = \binom{n-1}{k-1}\).

Suppose our ‘friend’ deletes all the elements \(l + 1, l + 2, \ldots, n\) for some integer \(l\). So, we are looking at a collection of \(M\) hypergraphs \(A_1, \ldots, A_M\), each of which partitions \{1, 2, \ldots, \(l\)\}. (There my be copies of \(\emptyset\) in each of these \(A_i\).

Note that each \(S \subseteq \{1, 2, \ldots, l\}\) appears exactly \(\binom{n-l}{k-|S|}\) times. If we wish to ‘add back’ \(l + 1\), then after a successful addition, we will see that partitions of \{1, 2, \ldots, \(l + 1\)\} such that each \(S \subseteq \{1, 2, \ldots, l + 1\}\) occurs \(\binom{n-l}{k-1}\) times. Therefore, if we can add \(l + 1\) into one set of each \(A_i\) such that after the addition, each \(S \subseteq \{1, 2, \ldots, l + 1\}\) occurs \(\binom{n-l}{k-1}\) times, then induction will take us through.

So for each \(S \subseteq \{1, 2, \ldots, l + 1\}\), it must occur \(\binom{n-l}{k-1}\) times. So we need to pair up the \(A_i\) with such subsets \(S\), so as to add \(n + 1\) to each of them. Set up a network: Put a source \(s\) connected to \(A_1, \ldots, A_M\) with capacity 1. Connect each of these to their elements \(S\) with capacity 1, and connect each \(S\) to the sink with capacity \(\binom{n-l}{k-|S|-1}\).

Now consider the following rational flow: Let \(f(s, A_i) = 1\) and \(f(S, t) = \binom{n-l}{k-|S|-1}\). To get the other one, we need \(\sum_{S \in A_i} f(A_i, S) = 1\), and \(\sum_{n-l} f(A_i, S) = \binom{n-l}{k-|S|-1}\). We guess \(f(A_i, S) = \frac{k-|S|}{n-1}\). Thus \(\sum_{S \in A_i} f(A_i, S) = \sum_{S \in A_i} \frac{k-|S|}{n-1} = \sum_{S \in A_i} \frac{k}{n-1} - \frac{1}{n-1} \sum_{S \in A_i} |S| = \frac{n}{n-1} - \frac{l}{n-1} = 1\), so it satisfies Kirchhoff’s law at the \(A_i\). Then at \(S\), \(\sum_{n-l} f(A_i, S) = \sum_{n-l} \frac{k-|S|}{n-1} = \frac{k-|S|}{n-1} (\binom{n-l}{k-1}) = (\binom{n-l}{k-1})\), as desired.

Now we perform the induction and add \(l + 1\) to copies of the sets that have been chosen. That is, we make copies of the \(S\) and modify \(\binom{n-l}{k-1}\) copies of it into \(S \cup \{l + 1\}\). This leaves \(\binom{n-l}{k-1}\) copies of \(S\), as desired.

Can we find particulars in our 1-factorization? In the matching case, can we find a covering with 1-factors, any pairs of whom form a Hamiltonian cycle. The answer is yes, but we’re done with this topic now.
8.1 Graph Colorings

**Definition.** Let $G$ be a graph and $k \in \mathbb{N}$. A vertex $R$-coloring (or proper coloring) is a map $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that whenever $u \leftrightarrow v$ in $G$, $c(u) \neq c(v)$.

Why think of this? There's a famous problem, but let's take another real-life example. Suppose there is a conference, and some members of your audience want to attend multiple talks. How can we schedule the conference so everyone is able to attend everything? We make the speakers or talks the vertices, and connect them with an edge if some audience member wants to attend both. Then a coloring is a schedule, where schedule same-color talks at the same time. This is an optimization problem: We want to minimize the number of colors needed.

**Definition.** The smallest $k$ such that there is a $k$-coloring on a graph $G$ is called the chromatic number of $G$, denoted $\chi(G)$.

Clearly, we have $\chi(G) \leq |V(G)|$, with equality iff $G$ is a complete graph. We can also imagine it locally: Each vertex causes no problems if it is colored different from its neighbors. Thus,

**Proposition.** If $\Delta(G)$ is the maximum degree of $G$, then $\chi(G) \leq \Delta(G) + 1$.

**Proof.** Order the vertices and pick colors for them sequentially. Each vertex has at most $\Delta(G)$ neighbors before it with colors already chosen, so there is at least one available color for each vertex. \qed

Is this bound tight? What are the equality cases? The complete graph works again. Also, an odd cycle. (Bipartite graphs are those with chromatic number 1 or 2.) But these are the only equality cases, as we’ll start with next time.

9 Tuesday, October 26, 2010

**Theorem** (Brooks’ Theorem). Suppose connected $G \neq K_n, C_{2n+1}$ for any $n$. Then $\chi(G) \leq \Delta(G)$, the maximum degree of $G$.

**Proof (Lovász).** We shall demonstrate a coloring scheme that properly colors $G$ using $\Delta$ colors. Consider the greedy coloring algorithm: List the vertices $v_1, v_2, \ldots, v_n$ and color them one by one to avoid the others. If every vertex has a neighbor after it in the sequence, there will only be $\Delta - 1$ colors taken before getting to a particular vertex, so we can color with $\Delta$ colors, except for the last vertex.

Now, if there is some vertex with degree less than $\Delta$, we can make this our last vertex, and since $G$ is connected, we can make our list as before, so every other vertex has a neighbor after it. So we may suppose that all vertices have degree $\Delta$. (We could also imbed every graph with maximum degree $\Delta$ in a $\Delta$-regular graph.) Moreover, if $\Delta = 1$ or $\Delta = 2$, the theorem is clearly true, so take $\Delta > 2$.

We first handle the case when $G$ is not 2-connected. Let the maximal subgraphs $H \subset G$ which are 2-connected be called blocks. It is simple to check that for any two distinct blocks $B_1$ and $B_2$, $|V(B_1) \cap V(B_2)| \leq 1$. In particular, one can define what is called the block decomposition graph of $G$ ($b(G)$) whose vertices are blocks of $G$ and 2 blocks are adjacent if they share a vertex. It is easy to see that $b(G)$ is a tree, i.e. acyclic and connected. Then we can color the blocks one at a time, so that after the first one, every later block will have a vertex of degree smaller than $\Delta$, so it can be colored. Therefore, it suffices to consider $G$ 2-connected.

If $G$ is regular and 2-connected, then we claim that there exist distinct vertices $v_1, v_2, v_n$ with $v_1 \neq v_2$, $v_1 \leftrightarrow v_n$ $\leftrightarrow v_2$ and $G \setminus \{v_1, v_2\}$ is connected. Then we can write the vertices of $G \setminus \{v_1, v_2\}$ in a list with each vertex having a later neighbor except $v_n$, which is last. Then append $v_1$ and $v_2$ to the beginning, assign them the same color, and you will have an option for $v_n$, as desired.

So we need to show that such vertices exist. First suppose that $G$ is 3-connected, and pick arbitrary $v = v_n$. Not all of its neighbors can be neighbors, since that would imply $G = K_{\Delta -1}$, so there are some unconnected neighbors of $v_n$, as desired.

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Finally, suppose that $G$ is 2-connected but not 3-connected. Then pick $v_n$ such that $G \setminus \{v_n\}$ is not 2-connected. It must be 1-connected, so it has the block structure we discussed earlier. Any leaves in this tree must have vertices connected to $v_n$, so pick two, and make the vertices $v_1$ and $v_2$. Then $v_n$ is connected to another block, so removing $v_1$ and $v_2$ leaves a connected graph, as desired.

Let’s review what we’ve shown so far.
(i) $\chi(G) \leq n(G)$, with equality iff $G = K_n$.
(ii) $\chi(G) \leq \Delta + 1$, with equality iff $G = K_n, C_{2n+1}$.

What about lower bounds for $\chi$? We can figure out the following:
(i) If there is a triangle in $G$, then $\chi(G) \geq 3$.
(ii) Generally, if $K_r \subset G$, then $\chi(G) \geq r$.

So it looks like the maximal clique size (largest imbedded $K_r$) is relevant. If we consider a graph with $K_r$ and an odd cycle, connecting all pairs of vertices between them, this graph has maximal clique $k + 2$ and chromatic number $k + 3$, so they do not need to be equal.

Question: Does there exist a graph $G$ such that $G$ has no triangle but $G$ has $\chi(G) = 4$?

Note that this is a little stronger: The maximum clique size is 2, but the chromatic number is 4, two greater.

Mycielski’s graph solves this problem. Consider a pentagon, and create a dummy vertex inside each of the outer vertices. Connect the dummy to each of the neighbors of its clone, and connect all the dummies to a central vertex. Check that this has no triangles, but has chromatic number 4. Let’s check the chromatic number. We can color the pentagon with 3 colors, and copy the colors to the dummies, then assign the fourth color to the central vertex. To show that the chromatic number is at most 4, give the central vertex color 3, then the dummies must be colored with colors 1 and 2. Whatever color we assign to the dummy must also work for its clone, if we restrict our attention to the original pentagon. Then we have a 2-coloring of the pentagon, a contradiction.

This works inductively. Let the pentagon be $M_3$, this graph be $M_4$, and in general, make $M_{n+1}$ by making dummy vertices for each vertex of $M_n$, and connecting the dummies to a vertex we call $\infty$. Check that, by the exact same argument, $\chi(M_n) = n$, but $M_n$ contains no triangles.

So we’ve proven the following proposition:

**Proposition.** There exist graphs which are triangle-free, but with arbitrarily large chromatic number.

So what about the smallest cycle length? This example introduces a lot of 4-cycles. Tutte proved that it’s possible with a smallest cycle size 6, before...

**Theorem (Erdős).** Given any integers $k,l$, there exist graphs with $\chi(G) > k$ and smallest cycle length $l$.

We’ll prove this next time. Surprisingly, the proof is not constructive. This gets rid of all hope that the local structure will tell us anything about the overall chromatic number.

10 Thursday, October 28, 2010

Recap: For $k \geq 3$, we gave examples of $G_k$ such that $\chi(G_k) = k$ and $G_k$ is triangle-free. Tutte proved that there exists $G_k$ such that $\chi(G_k) = k$ and the girth of $G_k = 6$. *Girth* is the size of a smallest cycle in the graph.

Erdős (1960?) in his paper *Graph Theory and Probability*: Given $k,l \in \mathbb{N}$ there exists $G_{k,l}$ such that $\chi(G_{k,l}) > k$ and the girth of $G_{k,l} > l$.

If there are $n$ vertices, there are $2^\binom{n}{2}$ graphs. Let’s review some basic probability.
10.1 Probability

Let $\Omega$ be a finite set, $P$ a probability function on $\Omega$, i.e. $P : \Omega \to [0, 1]$ such that $\sum_{x \in \Omega} P(x) = 1$.

**Definition.** A (real) random variable is a function from $\Omega \to \mathbb{R}$.

For instance, a fair coin tossed 10 times has $|\Omega| = 2^{10}$.

**Definition.** The expectation of $X$, $E(X) = \sum_{x \in \mathbb{R}} x P(X = x)$.

**Proposition.** If $E(X) = \lambda$, then $P(X > 2n) < \frac{1}{2}$.

**Proof.** We’re assuming $x \geq 0$ for all $x$. Then $E(X) = \sum x \in \mathbb{R} x P(X = x) > (2\lambda)^{1/2} = \lambda$. □

Let’s try to tease out what Erdős was thinking.

**Proof.** We want $G$ such that the girth of $G$ is $> l$, and $\chi(G) > k$. Note that if the maximum independent set is $\alpha$, then we must have $\chi(G) \geq \frac{n}{\lambda}$, since colorings are partitions into independent sets.

Think of this process of ‘selecting’ a graph from the sample space. But make the probability of $e \in G$ a real $p$, not necessarily equal to $1/2$ (weight it differently). We’ll try to fix $p$ such that the probability that the girth is above $l$ and $\chi(G) > k$ is as large as possible. We’ll try to calculate this probability, or equivalently, the probability that the girth is bounded by $l$ or $\alpha(G) > n/k$, since $\chi(G) \geq \frac{n}{\chi(G)}$.

For any integer $x$, let’s calculate $P(\alpha(G) \geq x)$, i.e. the probability that some $x$ vertices are independent. This is $(\frac{n}{x})^x (1-p)^{(\frac{n}{x}+1)} \leq n^x e^{-p (\frac{x-1}{x})} = (e^{-p(x-1)/x})^x = (e^{-p(x-1)/2 + log n})^x$. So we want $\frac{p(x-1)}{2} > log n$, or equivalently, $p(x-1) > 2\log n$.

Now let’s examine the probability that the girth is at most $l$, i.e. the probability there is some $i$-length cycle with $3 \leq i \leq l$. Let $N_{<l}$ be the number of cycles of size $\leq l$ in this randomly generated graph. What is the expected value of $N_{<l}$? We have $E(X) = \sum x P(X = x)$. Note that $E(\cdot)$ is linear, a very useful fact.

[Aside: The linearity of expected value also appears in the problem of n people getting n scrambled letters. The expected number of people to get their own mail is $\sum_{i=1}^{n} 1/n = 1$.]

Let’s calculate this expected value. For any cyclic arrangement of vertices $C$, $N_{<l} = \sum C_{\text{cycle}} 1_C$, where $E(1_C) = p^l$. Now how many cycles of length $i$ are there? We must regard the order, up to a different starting point and flipping the cycle around. Therefore, there are $\frac{n(n-1)\cdots(n-i+1)}{2i}$ cyclic sets of size $i$, and we are calculating

$$\sum_{i=3}^{n} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i \leq \frac{1}{6} \sum_{i=3}^{l} (np)^i.$$  

Now we want to take $np \leq 1$, which would make sense anyways, because if $p \leq 1/n$, so the graph would be quite sparse. Sparse graphs tend to have small chromatic number, so it’s not a good idea anyways. So then we bound our sum: $E(N_{<l}) < \frac{(np)^l}{6}$. Unfortunately, we can’t get this less than 1 easily.

Let’s write $p = \frac{f(n)}{n}$. Observe that

(i) As we showed above, if $\frac{f(n)}{n}(x-1) > 2\log n$, $\alpha(G) < x$ with high probability as $n$ becomes large.

(ii) If $E(N_{<l}) < \frac{1}{6} (f(n))^l < \frac{n}{4}$, then we have $P(N_{<l} > \frac{n}{4}) < \frac{1}{2}$. Now these conditions hold if $f(n) = n^{1/(l+1)}$. Indeed, this makes $f(n)$ grow faster than $\log n$, but $f(n)^l$ grow slower than $n$.

Now the probability of both of these events is greater than one half, so there is some graph that satisfies both. That is, there exists $G$ such that $G$ has $\frac{n}{4}$ small cycles, and $\alpha(G) < 2n^{1-1/(l+1)}\log n$. Now from each small cycle of $G$, throw away a vertex. The resulting graph $G^*$ has no small cycles and at least $\frac{n}{4}$ vertices, and $\alpha(G^*) \leq \alpha(G)$. Therefore, $\chi(G^*) \geq \frac{n/2}{\chi(G^*)} \geq \frac{n}{2\alpha(G)} > \frac{n^{1/(l+1)}}{2(2n \log \pi)} = \frac{n^{1/(l+1)}}{4\log \pi}$, which grows without bound for large $n$, so $G^*$ is a graph of what we’re looking for. □
This non-constructive proof is rather astounding. Later, Lovasz gave a constructive proof, but this one is a bit more transparent.

This idea of using probabilistic methods in combinatorics will be the topic of Ma121c this year.

11 Tuesday, November 2, 2010

Recall what we have proven:

1. For $G$ connected, $\chi(G) \leq \Delta(G)$ if $G \neq K_n, C_{2n+1}$ (Brooks).

2. (Erdős 1959) $\exists$ graphs of arbitrarily large girth and chromatic number.

Before these problems arose in terms of scheduling, there was a famous historical problem. In 1852, a student of De Morgan declared: “I can color the countries of maps with 4 colors.” Of course, the motivation for doing this is so that we can distinguish between neighboring countries. De Morgan thought about this problem for a while, and eventually asked Hamilton the same question in 1859. Hamilton told him he wasn’t interested, but the problem eventually made it to Cayley, who tried to solve it but couldn’t. He spread it widely, until Kempe claimed to prove it in 1879. Heawood found an error in his proof in 1890, but was able to repair it so that it showed 5 colors would do.

Until this time, graph theory was called the “slums of topology”. It was a research field that no one was really interested in until the advent of modern computer science. Once Kempe’s proof was found to be erroneous, it generated a lot more buzz in this topic. In the 1970s, Appel and Haken reduced it to a hundred thousand cases, and used a computer to check them. In 1995-6, Robertson et al. reduced it to a few thousand cases and checked them all by hand. This raises some philosophical questions: Do we consider things true if they’ve only been checked by a computer?

Each country is represented by a vertex and there are edges between adjacent countries. Then the statement is saying that certain types of graphs can be 4-colored. Which types?

Definition. A planar graph is a graph that can be embedded on the plane, such that there are no crossing edges. That is, any two edges, as subsets of $\mathbb{R}^2$, can only intersect at their endpoints, the vertices.

Note that the graph corresponding to any map is a planar graph.

Theorem (The Four-Color Theorem). Any planar graph can be 4-colored.

We’ll be able to prove the following weaker result:

Theorem (Heawood). was able to prove that any planar graph can be 5-colored.

Before going there, let’s talk about planar graphs some more.

To properly talk about embeddings into the plane, we’d have to use the full Jordan curve theorem. We’ll wave our hands and say that we can suppose that the edges are piecewise linear. This statement can be fully justified, but we won’t do it here. Now these piecewise linear curves separate the plane into a bunch of polygons which we call faces. Now each edge occurs in 2 faces of a planar graph.

Theorem (Euler’s polyhedral formula). If $G$ is planar and connected, $v = |V(G)|$, $e = |E(G)|$ and $f$ is the number of faces of $f$, then $v - e + f = 2$.

Proof. Anything you try to induct on will work. We’ll induct on the number of edges. Suppose first that $G$ is a tree, then note that $f = 1$ and $e = v - 1$, so $v - e + f = 2$, as desired. If there is some cycle in $G$, remove an edge from some cycle. Then the new graph has $e' = e - 1$, $v' = v$, and $f' = f - 1$, so $v - e + f = v' - e' + f' = 2$ by the induction hypothesis.

Now let’s prove that $K_5$ is not planar.
Proof. Suppose \( G \) is a planar graph. Add edges to \( G \) until you can add no more, keeping planarity intact. Then a graph with the maximal number of edges has all faces triangles. (If there were a larger number of vertices in a face, we could connect opposite vertices to add more edges.) Now count the number of edge-face pairs \((e, F)\) such that the edge \( e \) is on the face \( F \). This is both \( 2e \) and \( 3f \), so \( 3f = 2e \). Now we also have \( v - e + f = 2 \), so we get \( v - e + \frac{2}{3}e = 2 \), so \( e = 3v - 6 \). Now \( K_5 \) has \( v = 5 \) so the maximal planar graph on 5 vertices has \( 3 \cdot (5 - 6) = 9 \) edges. But \( K_5 \) has 10 edges, so it cannot be planar. \( \square \)

Note other consequences of this. \( 2e \) is the total sum of the degrees, so the average degree is \( 6 - \frac{1}{2}v < 6 \). This means in particular that there exists some vertex with degree less than 6. We claim that this implies that any planar graph can be 6-colored. Indeed, suppose there is a minimal counterexample; it has a vertex with degree less than 6. Color the rest of the graph with 6 colors and add the vertex back in, and there will be a possible color for that vertex.

Now let’s proof Heawood’s stronger claim, that any planar graph can be 5-colored.

Proof. Suppose not, and pick a minimal counterexample. If there is a vertex with at most 4 neighbors, we can remove it and the 5-coloring of the rest will extend, so we suppose that every vertex has degree at least 5. Pick some vertex \( v \) with degree 5. By hypothesis, removing \( v \) yields a 5-colorable graph, and so 5-color it, and we can suppose that the neighbors of \( v \), call them \( v_1, v_2, v_3, v_4, v_5 \), with \( v_i \) colored \( i \).

Kempe’s argument went like this: Consider the induced subgraph by the vertices colored 1 and 3. If \( v_1 \) and \( v_2 \) are in different connected components, then swap the colors in one of those components between 1 and 3, and similarly, there must be a path between \( v_3 \) and \( v_5 \) colored the same, so we can color \( v \). So there must be a path of vertices between \( v_1 \) and \( v_3 \) through vertices colored 1 and 3. Similarly, there must be a path between \( v_2 \) and \( v_4 \) through vertices colored 2 and 4. But these paths must then intersect, a contradiction, because an intersection must be in a vertex which must simultaneously be colored either 1 or 3 or 2 or 4.

The proof of the 4-color theorem involves irreducible configurations, i.e. configurations that cannot be inducted easily upon, and it eventually reaches contradictions in all of them. There are a lot of cases, on the order of 700 or 7000, in the nicest proof.

### 11.1 Edge Colorings

We can of course turn this question around to color the edges rather than the vertices, so that two edges that meet at a common vertex are considered adjacent and must be colored differently. The chromatic index of the graph \( \chi'(G) \) is the minimum number of colors needed to color the edges.

Clearly, we have \( \chi'(G) \geq \Delta(G) \). On Homework 3, Exercise 2, we showed that if \( G \) is bipartite, then \( E(G) \) can be partitioned into \( \Delta(G) \) matchings, i.e. \( \chi'(G) = \Delta(G) \). This is not true in general, since an odd cycle needs 3 colors but has maximal degree 2.

**Theorem** (Vizing, 1960s). \( \chi'(G) \leq \Delta(G) + 1 \).

This sandwiches \( \chi'(G) \) between \( \Delta(G) \) and \( \Delta(G) + 1 \). But it is computationally NP-complete to find out which one it is for arbitrary graphs. Then again, this is not too far off; if we think of scheduling problems, this isn’t too much of a loss.

**Proof Idea.** We add edges one at a time, or induct on the number of edges. We want to add an edge between \( u \) and \( v \). Let \( a_1 \) be a color that does not occur at \( v \). If it does not occur at \( u \), we can add an edge with color \( a_1 \) between the two, so there must be an edge on \( u \) with color \( a_1 \). Then if \( v_1 \) is missing color \( a_2 \), examine whether \( u \) is missing \( a_2 \) as well. If it is, we can recolor that edge \( a_2 \) and color \( u \leftrightarrow a_1 \) and we’re done. So we must assume that there is another edge at \( u \) with color \( a_2 \). Continue this process around to generate a list of edges, and eventually you’ll get some way to recolor them. We’ll complete this proof on Thursday. \( \square \)
Proof. We’ll color the graph edge by edge using $\Delta + 1$ colors, modifying the coloring so that it is proper at each step.

Note that some color is missing at each vertex. We want to color the edge $u \leftrightarrow v$. Suppose $a_0$ is missing at $u$ and $a_1$ is missing at $v$. If $a_1$ were missing at $u$ as well, then we could color this edge $a_1$. So there is an edge $u \leftrightarrow v_1$ with color $a_1$. Then $v_1$ must be missing a color $a_2$. If this color were missing at $u$, we could recolor $u \leftrightarrow v_1$ with color $a_2$ and color $u \leftrightarrow v$ with color $a_1$. We can continue until one of the neighbors of $u$ misses an earlier color.

If that color is $a_0$, label that edge $a_0$ and carry it backwards to color $u \leftrightarrow v$ with $a_1$. Otherwise suppose that $v_1$ is missing $a_k$ for $k < l$. Then if $a_0$ is not present at $v_l$, recolor that edge $a_0$ and push all the colors backwards. Otherwise, consider an alternating path starting from $v_l$ alternating between the colors $a_0$ and $a_k$. If this terminates somewhere, switch the colors and then push all the colors in the other $v_l$ backwards. Check that this works in every case. \qed

Definition. For a graph $G$, define $L(G)$, the line graph of $G$, as follows: $V(L(G)) = E(G)$, and $e \neq f \in E(G)$ are adjacent iff $e \cap f \neq \emptyset$.

So $\chi'(G) = \chi(L(G))$. This means that every edge coloring problem is at most as hard as a vertex-coloring problem.

If $G$ is bipartite, $\chi(L(G)) = \omega(L(G))$, where $\omega$ denotes the size of a maximal clique, i.e. a maximum complete subgraph. Indeed, every vertex in $G$ becomes a clique, and every clique must come from a vertex. In fact, this is true in general, except for triangles, which could come from a degree 3 vertex or from another triangle.

Let’s ask the question in general: For which graphs $G$ is it true that $\omega(G) = \chi(G)$?

This is clearly true for $K_n$, and for bipartite graphs (both are two). We found that it was true for line graphs of bipartite graphs.

But this question is a little too general, because whatever graph we have, we can just append a $K_{\chi(G)}$ and it will still work. So a more appropriate question is the following:

Definition (C. Berge). A graph $G$ is called perfect if for all induced subgraphs $H$ of $G$, we have $\chi(H) = \omega(H)$.

Proposition. If $G$ is bipartite, its complement is also perfect.

We actually haven’t defined complements yet.

Definition. We say $\bar{G}$ is a complement of $G$ if $V(\bar{G}) = V(G)$ and for every pair of distinct vertices $u, v$ in $V(G)$ are adjacent in precisely one of $G$ and $\bar{G}$.

Now let’s prove this proposition.

Proof. Cliques in $G$ correspond to independent sets in $\bar{G}$. Therefore, $\omega(\bar{G}) = \alpha(G)$.

Now let the parts of $G$ be $A$ and $B$, with $a$ and $b$ vertices respectively, and let $M$ be a maximal matching. Then in $\bar{G}$, we can color each of the pairs in the matching with the same color, and each of the other $(a - |M|) + (b - |M|) = a + b - 2|M|$ vertices their own color, so we’ve shown $\chi(\bar{G}) \leq a + b - |M|$. Now by König’s theorem, $|M|$ is the size of the minimum vertex cover $|C|$ of $G$. So $\chi(\bar{G}) \leq |V| - |C|$.

Now if we have a vertex cover, that means that the complement is an independent set, i.e. a clique in the complement graph. Therefore, $\chi(\bar{G}) \leq \omega(\bar{G})$, as desired. Now the same is true on any subgraph, as desired. \qed
**Example.** If $G$ is bipartite, $\overline{L(G)}$ is also perfect.

This isn’t true in general: In fact, the only graphs whose line graphs are perfect are bipartite graphs. What about complements?

**Proposition** (Berge’s Conjecture). $G$ is perfect iff $\overline{G}$ is perfect.

A consequence of this conjecture: If $G$ contains the complement of an induced odd cycle, then $G$ is not perfect. A stronger conjecture:

**Proposition.** A graph $G$ is perfect iff $G$ has no induced odd cycles of size $\geq 5$ or complements of induced odd cycles of size $\geq 5$.

This one became known as the strong Perfect Graph Conjecture, and the first one as the Weak Perfect Graph Conjecture. Lovasz proved Weak PGC in the 1970s at the age of 21 or 22. The Strong PGC was proved in 2006 by Chudnovsky, Robertson, Seymour and Thomas in 2006, published in 2009. The proof is around 150 pages long.

We’ll show Lovasz’s proof of the Weak PGC. Suppose $G$ is perfect. Color the graph into $\omega$ independent sets. Remove one of these sets $I$, and it can be colored with $\omega - 1$ colors. Then as $G$ is perfect, this is the size of the maximal clique in $G \setminus I$. This every maximal clique of the original graph must meet one vertex of $I$. In fact, this is equivalent to perfectness:

**Proposition.** $G$ is perfect iff for every induced subgraph $H$ of $G$, there is an independent set $I$ in $H$ such that $\omega(H \setminus I) < \omega(H)$.

**Proof.** We’ve shown one side; for the other, induct on $|G|$. Then take $H = G$, so there is some independent set $I$ with $\omega(G - I) < \omega(G)$. By the induction hypothesis, $\chi(G - I) = \omega(G - I)$. Therefore, $\chi(G) \leq \chi(G - I) + 1 = \omega(G - I) + 1 \leq \omega(G)$, as desired.

We’ll finish the proof of the weak PGC next time.

## 13 Tuesday, November 9, 2010

### 13.1 Perfect Graphs

Recall: $G$ is perfect if for all induced $H \subseteq G$, $\chi(H) = \omega(H)$. Equivalently, there exists some independent set $I$ such that $\omega(G - I) < \omega(G)$. $\overline{G}$ is the complement of $G$.

**Theorem** (Lovász, Fulkerson, 1972). $G$ is perfect iff $\overline{G}$ is perfect.

To prove this, we need another definition:

**Definition.** Let $G$ and $H$ be graphs and $v \in V(G)$. The substitution graph $G(v; H)$ is defined as follows: $V(G(v; H)) = (V(G) \setminus \{v\}) \cup V(H)$ and $x \leftrightarrow y$ in $G(v; H)$ iff $x \leftrightarrow y$ in $G$ if $x, y \neq v$ and $w \leftrightarrow x$ in $G(v; H)$ if $x \leftrightarrow v$ in $G$ and $w \in V(H)$. That is, replace $v$ in $G$ with a copy of $H$ and connect every neighbor of $v$ to every vertex in the copy of $H$.

**Proposition** (Lovász). If $G$ is perfect, then $G^* = G(v; K_r)$ is perfect for any $r \geq 0$.

Fulkerson thought that this statement was too good to be true. Realize that for $r = 0$ we are deleting $v$, which is trivial since every induced subgraph of $G$ is also perfect. $r = 1$ gives back $G$, so the first relevant value is $r = 2$.

**Proof.** Let us consider a coloring of $G$ with $\chi(G) = \omega(G)$ colors. Let $C$ denote the color class of the vertex $v$, the independent set that meets every maximal clique, and let $K$ be any maximal clique of $G^*$. Let $I^* = C \cup \{y\}$ for any $y \in H$. If $K$ contains any vertex of $H$, it contains all of them, so it meets $I^*$. Otherwise it is also a maximal clique of $G$ so it must intersect $C \subseteq I^*$, as desired. Every induced subgraph of $G$ has the same structure, so the same proof works for any induced subgraph of $G^*$.
Remark. In fact, the same proof shows that if $G, H$ are perfect, then $G^* = G(v; H)$ is also perfect.

Now let’s prove the theorem.

Proof of Weak PGT. Suppose not. Let $G$ be a minimal counterexample, with $G$ perfect but $\bar{G}$ not perfect. Therefore, for every independent set $I \subseteq \bar{G}$, there is some disjoint maximal clique $C$ of $\bar{G}$. Translating into $G$ and switching our names out, for every clique $K$ in $G$, there is some disjoint maximal independent set $I_K$.

By the proposition, we can blow up every vertex into a complete graph. Let $G^* = G[K_{f_1}, \ldots, K_{f_n}]$ where $f_i = f(v_i)$ are integers TBD. Then $G^*$ will also be perfect. Now the maximal size of an independent set $\alpha(G^*) = \alpha(G)$. Since $G^*$ is perfect,

$$\omega(G^*) = \chi(G^*) \geq \frac{|V(G^*)|}{\alpha(G^*)} = \frac{1}{\alpha(G)} \sum_{v \in V(G)} f(v).$$

If we consider a maximal clique of $G^*$, it must include each of the $K_{f_i}$, i.e. $\omega(G^*) = \sum_{v \in K} f(v)$ for some clique $K$ in $G$. Therefore, we have $\sum_{v \in K} f(v) \geq \frac{|V(G)|}{\alpha(G)}$ for some clique $K$ in $G$. That is, if we show that $\frac{n}{\alpha(G)} > \sum_{v \in K} f(v)$ for every clique $K$ in $G$, we will have a contradiction.

Now what should we pick for $f(v)$? We’d like to use what we know. Well, we know that every clique $K$ has a disjoint maximal independent set $I_K$. So list all of the cliques $K_1, K_2, \ldots, K_N$ and let their associated independent sets be $I_1, I_2, \ldots, I_N$. We have two natural possibilities for $f(v)$: $|\{j : v \in I_j\}|$ and $|\{j : v \in K_j\}|$. The first of these gives us

$$\sum_{v \in V} f(v) = \sum_{j=1}^{n} |I_j| = \alpha(G)N,$$

since all of the $I_j$ are maximal. Dividing by $\alpha(G)$, we get $\frac{n}{\alpha(G)} = N$.

Now what about the other side? We have $\sum_{v \in K} f(v) \sum_{j=1}^{n} |K \cap I_j|$. Since $K$ is a clique and $I_j$ is an independent set, $|K \cap I_j| \leq 1$, so the sum is at most $N$. Can we do better? Well, $|K \cap I_K| = 0$, so it’s a strict bound, and we’re done.

Lovász originally proved the problem for hypergraphs. We’ll look at his proof, but remember that in general, difficult graph problems are much easier to solve than hypergraph problems.

Definition. For a hypergraph $H$, we say $H$ is $(\Delta, \chi')$-normal if for every induced $H' \subseteq H$, $\Delta(H') = \chi'(H')$, that is, the maximal degree equals the chromatic index. We also say that $H$ is $(\nu, \tau)$-normal if for all $H' \subseteq H$, $\nu(H') = \tau(H')$, the maximal matching size equals the minimum vertex cover.

Proposition (Lovász). $H$ is $(\Delta, \chi')$-normal iff $H$ is $(\nu, \tau)$-normal.

How is this equivalent? For a graph $G$, consider the hypergraph $H$ with vertices $C$, the set of cliques (of size $\geq 2$) of $G$, and edge set $V(G)$, i.e. two cliques are adjacent if they share a vertex.

- $\Delta(H)$ is the size of the largest clique in $G$, $\omega(G)$.
- $\chi'(H)$ is the chromatic number of $G$, $\chi(G)$.
- $\nu(H)$ is the independence number of $G$, $\alpha(G) = \omega(G')$.
- $\tau(H)$ is the number of cliques needed to partition $V(G)$, i.e. the chromatic number of $\bar{G}$, $\chi(\bar{G})$.

Therefore, it’s equivalent. This makes the proof a little more natural.
13.2 List Colorings (Vizing)

Consider an application: A bunch of radio stations want to broadcast, but they can’t pick the same frequency. Some pairs of them will overlap, which would be bad if they picked the same frequency. This is the same as graph coloring, but the colors available to each vertex are (possibly different) sets.

Note that if there are enough colors available to every vertex, then we can always color it. Let \( \chi_l(G) \), the list chromatic number of \( G \), be defined as the smallest \( n \) such that if each list has \( n \) elements, there is a coloring of \( G \). The greedy coloring guarantees that \( \chi_l(G) \leq \Delta(G) + 1 \).

Note that \( \chi_l(G) \geq \chi(G) \) because if we can do it with any choices of colors, we can do it in the special case of when all lists are the same. However, \( \chi_l(G) > \chi(G) \) is possible! Consider \( K_{3,3} \) with the three vertices in each part colored \( \{1, 2\}, \{2, 3\}, \) and \( \{1, 3\} \). Then each part cannot be colored with the same color, so two colors have to be taken on each side, contradiction.

We can generalize this. Pick the complete bipartite graph with \( (2^{k-1}) \) vertices on each side, and for each of them let the colors be one of the \( k \)-subsets of \( (2^{k-1}) \). Then if one side were \( k - 1 \)-colorable, exactly those \( k - 1 \) colors would be unavailable to one vertex, a contradiction. So each side requires \( k \) colors, and therefore, \( \chi_l(G) \geq 2k \). The list chromatic number can be arbitrarily large even for bipartite (chromatic number 2) graphs!

14 November 11, 2010

14.1 List Colorings

Recall: We have a graph \( G \) with a set of lists \( \{L_v\}_{v \in V(G)} \). We want to give every vertex a color such that no adjacent vertices have the same color.

In normal colorings, we can give a large independent set the same color, but that’s no longer possible here.

In a paper by Erdös, Rubin and Taylor, “Choosability in Graphs”, the problem was first fully addressed (Vizing had seen it as what you’re left with after a partial coloring). The first thing they investigated was how to get arbitrarily large list chromatic number. As we saw last time, they found bipartite graphs \( G \) with \( \chi_l(G) > k \). This was \( K_{m,m} \) with \( m = \binom{2^{k-1}}{k} = \frac{1}{2} \binom{2k}{k} \approx \frac{2^k}{\sqrt{2k}} \) (Stirling’s approximation).

Let’s consider things from the other perspective, as ERT did. We’re going to compute which graphs have \( \chi_l(G) \). We’ll consider minimal \( G \). If there is a vertex of degree \( < k \), then remove it and list-color the rest, so we must have degree at least \( k \). We can prove some things with this...

Proposition. 1. Any tree can be 2-list-colored. Indeed, there’s always a leaf.

2. (Exercise) Even cycles can be 2-list-colored.

3. Planar graphs can be 6-list-colored. (The average degree is less than 6.)

Let’s investigate planar graphs some more. In the ERT paper, they conjectured that if \( G \) is planar,

1. \( \chi_l(G) \leq 5 \).

2. This is the best possible, i.e. there exist \( G \) for which \( \chi_l(G) > 4 \).

Thomasson (1990-92?) proved 1. We’ll see his proof. For 2, M. Voigt (1994) found an example with 239 vertices. Later this was improved by someone to 130 vertices, and then in 1996, Mirzakhani found an example with 69 vertices. We’ll look at that later. First, let’s see Thomasson’s proof.

Theorem. \( \chi_l(G) \leq 5 \) for all planar \( G \).

Proof. Without loss of generality, assume \( G \) is almost triangulated, i.e. every face except the outer face is a triangle. Induct on the number of edges, proving the stronger statement that if the outer face is \( v_1, v_2, \ldots, v_n \),
(a) \(|L(v_i)| \geq 3\).

(b) \(|L(x)| \geq 5\) for all \(x\) not on the outer face.

(c) \(v_1\) and \(v_2\) must be colored arbitrarily (and differently) from their lists.

First suppose that two nonconsecutive vertices on the boundary are connected by an edge. This resulting graph can be split in two. Pick two vertices in one part on the boundary, and color that part using the inductive hypothesis. This pre-colors the two vertices on the dividing edge, so we can color the rest of the graph as well.

Otherwise, there are no cross-boundary edges. Then consider the neighbors of \(v_k\): 
\[v_{k-1}, u_1, u_2, \ldots, u_r, v_1.\]
Since it is almost triangulated, these must be neighbors in this order. Remove \(v_k\) and color the smaller graph with \(v_1 \rightarrow 1\) and \(v_2 \rightarrow 2\). \(v_k\) has at least two other colors, 3 and 4 (we could have one of them being 2). Remove 3 and 4 from the lists of the \(u_i\), so they have at least 3 color choices before doing the coloring. Now \(v_{k-1}\) might be colored 3 or 4, but there will always be a choice of a color for \(v_k\), as desired.

Now let’s see Mirzakiani’s example graph. First consider the square with a vertex in the middle connected to all. Give the middle vertex the list \((1, 2, 3, 4)\), and remove one of those from each of the other lists.

First notice that if we color this, one of the pairs of opposite vertices must be colored the same. Now stick two of these together along an edge, and remove the same color as the diagonally opposite vertex (so it’s not symmetric). Do the same thing on each edge. By casing it out, you can find that it cannot be colored with these colors.

Now add the color 5 to the outer vertices of this graph, and make three more copies of it, adding the colors 6, 7 and 8 to those graphs. Then connect one monster vertex to all of the outer vertices and give it the list \((5, 6, 7, 8)\). Then this cannot be 4-list colored.

14.2 The Dinitz Conjecture

J. Dinitz was a student of Rick Wilson. He investigated extending Latin squares. Recall their definition: it’s a coloring of a \(n \times n\) square such that each row and column does not have a repeated color. Of course, we have several examples of these. There’s the cyclic version. Any multiplication table of a group will work, but of course, there are going to be other tables.

He was investigating whether or not a \(2n \times 2n\) grid could be completed to a Latin square if three of the four quadrants had been colored. Now the list of possible colors on the fourth quadrant is not the same for every vertex.

So we can generalize to list colorings on the Latin square. Latin squares correspond to edge colorings of \(K_{n,n}\). Dinitz conjectured from small cases that we can always do this. That is, Dinitz conjectured that \(\chi_l(L(K_{n,n})) = n\).

Janssen showed with a complicated argument that \(\chi_l(L(K_{n,n})) \leq n + 1\). Then in 1995, Frank Galvin proved the conjecture. This is related to the Gallai-Roy theorem on our problem set, which looks at digraphs.

14.3 Colorings and Orientations

Suppose \(G\) is a graph with an orientation \(\sigma\). Fix a color \(c\), and let \(G_c \subseteq G\) be the induced subgraph with the vertices containing \(c\). This gives us some sort of inductive structure.

**Definition.** A kernel in a digraph is a set of vertices \(K\) such that

(i) \(K\) is independent.

(ii) \(\forall v \notin K\) there exists \(w \in K\) such that \(v \rightarrow w\).

The motivation for kernels comes from games, where a kernel would be a list of winning positions in a two player turn-based game.
**Proposition.** Suppose $\bar{G}$ is a digraph such that every induced sub-digraph of $G$ has a kernel. Then $\chi_l(G) \leq \Delta^+(G) + 1$, where $\Delta^+$ is the maximum outdegree of $G$.

**Proof.** Induce on $|V(G)|$. We’ll prove that we can list color each $v$ provided $|L_v| \geq d^+(v) + 1$. Pick a color $c$, and let $G_c$ be as before. By induction, since $G_c$ has a kernel $K_c$, so color $K_c$ using $c$ and remove $c$ from every vertex list of $G_c$. But note that in $G \setminus K_c$, $|L_v| \geq d^+(c) + 1$ still holds, so we are done.

Can we use this result? Well, in a general graph we can make around half the edges orienting outwards from each vertex. So what is the degree of in our line graph? Each cell is adjacent to all in the same row or column, or 2$n$ – 2. If half orient outwards at every vertex, we’ll have $\Delta^+(G) = n - 1$, so the proposition will give us $\chi_l(L(K_{n,n})) \leq n - 1 + 1 = n$, which is what we want.

How can we get half our edges to orient outwards? Consider an $n \times n$ Latin square. In the rows, orient edges in increasing order, and in the columns, orient the edges in decreasing order. Then each will have outdegree $n - 1$, and we’ll be done.

We just need to verify that there is a kernel in any induced subgraph. An induced subgraph $H$ corresponds to a subgraph of the complete bipartite graph. An independent set is a matching, so a kernel corresponds to a matching satifying a certain condition.

To understand this condition, think of the parts of the bipartite graph as men and women, and numbers in the Latin square as preference lists for each man and woman. We claim that a stable matching corresponds to a kernel. If there is any other edge/cell/marriage, then either there is a directed edge going to a cell in our independent set, or our matching is unstable.

Next time, we’ll move onto the last big topic for the course, extremal graph theory.

### 15 Tuesday, November 16, 2010

We polished up the proof of the previous theorem, then moved on.

#### 15.1 Extremal Graph Theory

We’ve encountered a lot of individual examples of problems from extremal graph theory. What makes it theory, rather than a bunch of clever examples?

Let’s take a simple question: What is the maximum number of degrees in a graph with $n$ vertices? Of course, the answer is $\binom{n}{2}$, but that’s just a symbol; we really mean $\frac{n(n-1)}{2}$, which is satisfactory because it is a polynomial in $n$. But formulas like $n!$ are a little less well-behaved.

Take another combinatorics question: How many partitions $p(n)$ are there of a positive integer $n$? That is, we are counting sums of the form $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k > 0$. We want to try to express $p(n)$ in another way.

Some of these kinds of questions have good answers. That’s where our theory comes from.

Consider the following question: Suppose $G$ is a graph with $n$ vertices and no triangles. How large can $e(G)$ be?

One natural family to look at are bipartite graphs. If $G$ is bipartite with $a$ and $b$ vertices in each part, then there are $ab$ edges, which is maximized under $a + b = n$ with $a = b = \frac{n}{2}$ if $n$ is even or \{a, b\} = \{(n-1)/2, (n+1)/2\}.

Is this the best possible bound? Suppose we have a graph with no triangles. Consider an edge $x \leftrightarrow y$. Then since there are no triangles, $N(x) \cap N(y) = \emptyset$. Therefore, $n \geq |N(x) \cup N(y)| = |N(x)| + |N(y)| = d(x) + d(y)$ for all edges $x \leftrightarrow y$. Adding all of these, we have $\sum_{x,y} (d(x) + d(y)) \leq ne(G)$. Now $d(x)$ shows up $d(x)$ times in this sum, therefore, so $\sum_{x} (d(x))^2 \leq ne(G)$. Now by Jensen’s inequality, as $x^2$ is convex, $\frac{1}{n} \sum d(x) \leq \sqrt{\frac{1}{n} \sum d(x)^2} \leq \sqrt{e(G)}$. Since $\sum_{x} d(x) = 2e(G)$, we get $\frac{2}{n} e(G) \leq \sqrt{e(G)}$, so $e(G) \leq \sqrt{\frac{n^2}{4}}$, so it is an upper bound.

Are there other equality cases? We’d have to have equality at all steps, and Jensen’s inequality only has equality if all degrees are the same: $d(x) = \frac{n}{2}$. Now taking a vertex $x$, $N(x)$ can have no edges within it, and each $y \in N(x)$ must have $n/2$ neighbors, so these must be $V(G) \setminus N(x)$.
The odd case is a little more work, but a similar argument shows that the only equality case is the almost-equal bipartite case.

This example illustrates some of the common features in extremal graph theory. We ended up using inequalities on real numbers that didn’t make any explicit reference to graph theory. We had a natural example and showed that it was maximal.

Turan asked: Suppose $G$ has $n$ vertices but no $K_r$. What is the maximal number of edges?

Taking a cue from the triangle-free case, we consider first $(k-1)$-partite graphs. Suppose the parts have size $a_1 \leq a_2 \leq \ldots \leq a_{r-1}$. Then $e(G) = \sum_{i<j} a_i a_j = \frac{1}{2} \left( (a_1 + \cdots + a_{r-1})^2 - \sum_{i=1}^{r-1} a_i^2 \right)$. This is maximized iff $a_{r-1} - a_1 \leq 1$, i.e. the parts are ‘equitable.’ This graph is called the Turán graph. That is, $T_r(n)$ is a $r$-partite graph on $n$ vertices which is as equitable as possible. Can we improve this bound? Let’s formally state the theorem.

**Theorem.** If $|V(G)| = n$ and $G \not\supset K_{r+1}$, then $e(G) \leq t_r(n)$, where $t_r(n)$ is the number of edges of the Turán graph.

In the $r \mid n$ case we have a nice formula. Each part has $\frac{n}{r}$ vertices, so there are $\binom{\frac{n}{r}}{2} = \frac{\frac{n}{r} - 1}{2} \frac{n^2}{r^2}$ edges. This is a theorem, unlike the trianglefree case, where it is very hard not to find a proof. Induction will carry you through.

**First Proof.** First consider $T_r(n)$. The min and max degrees are $\delta(T_r(n)) = n - \lceil n/r \rceil$ and $\Delta(T_r(n)) = n - \lfloor n/r \rfloor$. Therefore, in general, if $e(G) = t_r(n)$, then $e(G) \leq \delta(T_r(n)) \leq \Delta(T_r(n)) \leq \Delta(G)$. Now $t_r(n) - t_r(n-1) = n - \lfloor n/r \rfloor$. So we can see how the argument works: If there is a vertex $x$ with degree less than $n - \lfloor n/r \rfloor$, removing this vertex produces a graph $G \setminus x$ with $n-1$ vertices and more than $t_r(n-1)$ edges.

That is, this is our argument: We induct on $n$ to prove that if $G \not\supset K_{r+1}$, then $e(G) \leq t_r(n)$, with equality iff $G \cong T_r(n)$. We suppose there is a counterexample graph $G$ with $e(G) > t_r(n)$. Remove edges until you get $t_r(n)$ edges. If we show that this implies $G \cong T_r(n)$, then obviously we can’t add any edges to $T_r(n)$ so this implies the claim. So we must show that if $e(G) = t_r(n)$, then $G \cong T_r(n)$.

Now as before, we have $\delta(G) \leq n - \lceil n/r \rceil$. If this is not equality, removing the vertex of smallest degree yields a counterexample with $n-1$ vertices, so we must have $\delta(G) = n - \lfloor n/r \rfloor$. Then we must have $G \setminus x \cong T_r(n-1)$. An easy exercise shows that this implies $G \cong T_r(n)$.

**Second Proof (Erdős).** We’ll show that for any graph $G$ with $K_{r+1} \not\subseteq G$, there exists an $r$-partite graph $H$ on the same vertex set such that $d_H(x) \geq d_G(x)$ if $x \in V$. Of course, this will imply that the Turán graph is the best possible, since it was the best possible of the $r$-partite graphs.

He also proves this claim by induction, but this time on $r$. Let $x$ be a vertex of maximum degree in $G$. If $G' = G \setminus N(x)$ then $G'$ has no $K_r$. So by induction there is a $(r-1)$-partite graph $H'$ with $V(H') = N(x)$ and $d_{H'}(y) \geq d_G(y)$ for all $y \in N(x)$. Define $H$ as $H'$ along with edges from every $y \in H'$ to every $z \in V(G) \setminus N(x)$. Then the degree of any $y \in N(x)$ is $d_H(y) + |G \setminus N(x)| \geq d_G(y) + (n - d(x)) \geq d_G(y)$ as desired. Next, $x$ has not seen any change, so $d_H(x) = d_G(x)$. For any other $x \neq z \in G \setminus N(x)$, $z$ now has the same degree as $x$, so $d_H(z) = d_H(x) = d_G(z) \geq d_G(z)$ as desired.

**Third Proof.** Note that $t_r(n) > t_{r-1}(n)$. So we may suppose that there is some $K_r \subseteq G$. Call those vertices $K$. By induction on $n$, the number of edges within $G' = G \setminus K$ is at most $t_r(n-r)$, with equality if it is a Turán graph. Now the maximum number of edges is $t_r(n-r) + \binom{r}{2} + (r-1)(n-r) = t_r(n)$, as desired. Check that the only way to put these together in the equality case is a Turán graph.

A similar, but unsolved, problem: What is the maximum number of edges in a bipartite graph with $m$ and $n$ vertices in each part, such that there is no $K_{s,t}$ subgraph? This is known as the Zarankiewicz problem.
Then version for Jensen’s inequality: If $\bar{d} = d_{\bar{x}} \leq (\bar{d} - (t - 1))^t \leq \frac{m - 1}{m} n^t$, so $\bar{d} \leq \left(\frac{m - 1}{m}\right)^{1/t} n + (t - 1)$. Therefore, the number of edges is bounded by

$$z(m, n; s, t) \leq m\bar{d} \leq n(s - 1)^{1/t} m^{1 - 1/t} + m(t - 1).$$

Can we improve our argument at all? Well, that Jensen bashing was a little crude. There’s a discretized version for Jensen’s inequality: If $e(B) = m\bar{d} = km + r$ with $0 \leq r < m$ for integers $k, r$. Then instead of $m\binom{n}{t}$, we have $(m - r)\binom{k}{t} + r\binom{k + 1}{t} \leq (s - 1)\binom{n}{t}$.

Does this help? It does in small cases. Let’s consider $z(n, n; 2, 2)$. Our argument produced $m\binom{\tilde{d}}{2} \leq \binom{n}{2}$. Then $m = n$ so this simplifies to $\tilde{d}(\tilde{d} - 1) \leq n - 1$. Solving this quadratic yields $\tilde{d} \leq \frac{1 + \sqrt{4n - 3}}{2}$, so

$$z(n, n; 2, 2) \leq \frac{n}{2}(1 + \sqrt{4n - 3}).$$

Consider the case $n = 7$. Then this upper bound is exactly 21. Can we do this? Many of you have seen this setup before. Draw an equilateral triangle with the altitudes, and draw the incircle. We have seven points: the vertices, the feet of the altitudes, and the center. Each altitude, side, and the incircle passes through three of them, and no two intersect in more than two. Make the points into one side of the subgraph of $K_{7, 7}$, and the lines/circle into the other side, connecting them if the points intersect.

This setup comes from projective geometry. Projective geometry captures the notion of intersection as being crucial to geometry. You have points and lines, which are sets of points. You specify that any two points lie on a unique line and any two lines intersect in a unique point.

**Definition.** If $X$ is a finite set and $B$ is a collection of subsets of $X$. Then $(X, B)$ is called a finite projective plane of order $n$ if

1. Every point $x \in B$ lies on $n + 1$ lines in $B$.
2. Every line contains $n + 1$ points.
3. Any two points lie on a unique line.
4. Any two lines intersect at a unique point.

Notice that the construction is dual to itself, in that you can flip the notions of lines and points you get the same thing. You can calculate that there are $n^2 + n + 1$ points and lines in a finite projective plane of order $n$. 

16 Wednesday, November 17, 2010: Special Make-Up Class

16.1 Zarankiewicz Problem

This is an analogous bipartite problem. Consider a bipartite graph $B$ with parts $U$ and $V$, with $|U| = m$ and $|V| = n$. Suppose $K_{s, t} \not\subseteq B$, with the $s$-part in $U$ and $t$-part in $V$. What is the maximum value for $e(B)$? This maximum is denoted $z(m, n; s, t)$.

First consider an $x \in U$ along with $t$ neighbors in $V$. How many such arrangements of this are there? Counting from the perspective of $x$, we have $\sum_{x \in U} \binom{d(x)}{t}$. Counting from the perspective of the $t$-element subsets of $V$, we get $\sum_{T \subseteq V} n(T)$, where $n(T) = \# \{ x \in U : x \leftrightarrow y \in T \}$. If $K_{s, t}$ is not contained in $B$, then $n(T) \leq s - 1$ for all $T \subseteq V$. Therefore, $\sum_{x \in U} \binom{d(x)}{t} \leq (s - 1)\binom{n}{t}$.

Now apply Jensen’s inequality, since $\binom{\tilde{d}}{t}$ is convex. This gives us $\binom{\tilde{d}}{t} \leq \frac{1}{m} \sum_{x \in U} \binom{d(x)}{t}$. We want to ‘solve’ for $\tilde{d}$, so write $\binom{\tilde{d}}{t} = \frac{\tilde{d}^{d_1 - 1} \cdots (\tilde{d} - (t - 1))}{t!} \geq \frac{(d - (t - 1))^t}{t!}$. We also bound $\binom{\tilde{d}}{t} \leq \frac{n^t}{t!}$. Cancelling the $t!$’s, we get $(\tilde{d} - (t - 1))^t \leq \frac{m - 1}{m} n^t$, so $\tilde{d} \leq \left(\frac{m - 1}{m}\right)^{1/t} n + (t - 1)$. Therefore, the number of edges is bounded by

$$z(m, n; s, t) \leq m\tilde{d} \leq n(s - 1)^{1/t} m^{1 - 1/t} + m(t - 1).$$

This is an old result, not the best known. A better result by Furedi (1996) gives

$$z(m, n; s, t) \leq (s - t + 1)^{1/t} nm^{1 - 1/t} + tn + tm^{2 - 2/t}.$$
Now if we want to achieve equality in our bound above, we need \( \sqrt{4n - 3} = 2k + 1 \), or \( n = k^2 + k + 1 \). You can check that anything achieving equality will then satisfy the conditions of a projective plane, so we’ve shown that equality is achieved if and only if there is an appropriate projective plane. Now it’s a deep but unsolved problem that there is a projective plane of order \( k \) if and only if \( k \) is a prime power.

### 16.2 The Turán graph again

Recall that \( t_r(n) \geq (1 - \frac{1}{r}) \binom{n}{2} \). Really, we had \( \lim_{n \to \infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r} \). This is saying that for every \( \epsilon \) there is a sufficiently large \( n \) such that every graph \( G \) with \( e(G) \geq (1 - \frac{1}{r} + \epsilon) \binom{n}{2} \) has a \( K_{r+1} \). We can also ask, how many different \( K_{r+1} \)’s can we find? Well, a \( K_{r+2} \) wouldn’t be very interesting, so we can ask for edge-disjoint copies.

Taking \( r = 1 \) yields the simple case where we’re counting edges, and \( e(G) \geq \epsilon n^2 \). Create the corresponding bipartite graph to \( G \), \( G' \), where we make a copy of \( G \) and connect \( i \) and \( j \) if \( i \leftrightarrow j \) is an edge in \( G \). Now \( G' \) will have \( 2\epsilon n^2 \) edges. Now our bound for the Zarankiewicz number had exponent \( n^{2-1/t} \) for large \( n \). Since this is a strictly smaller exponent, we can find any \( K_{s,t} \) for large enough \( n \). Note that these pull back into \( K_{s,t} \) in \( G' \).

Let’s look at what we’ve shown. We’ve said that a nonzero density of single edges produces a \( K_{s,t} \). Erdos asked, if we have a nonzero density above the minimum for a \( K_{r+1} \), can we find a corresponding large enough multipartite graph? The answer is yes, and we can get these subsets to have size logarithmic in \( n \). We’ll prove this next time.

### 17 Thursday, November 18, 2010

#### 17.1 The Erdős-Stone Theorem

We already know that if \( e(G_n) \geq (1 - \frac{1}{r} + \epsilon) \binom{n}{2} \) for some \( \epsilon > 0 \), then \( G_n \supseteq K_{r+1} \), since the Turán graph has around \( (1 - \frac{1}{r})(\frac{n}{2})^2 \) edges. In fact, we can do better.

**Theorem.** If \( n \gg 0 \), suppose \( e(G_n) \geq (1 - \frac{1}{r} + \epsilon) \binom{n}{2} \). Then \( G_n \supseteq K_{r+1}(t) \), where \( K_{r+1}(t) \) is a complete \( (r + 1) \)-partite graph with each part containing \( t \) vertices. Moreover, we can take \( t = \lceil c(r, \epsilon) \log n \rceil \), some constant multiple of \( \log n \).

**Proof.** First, we realize that if we reduce \( n \) by a constant factor, the log will hardly be changed. So let’s get rid of our vertices of small degree. First, let’s show we can do this.

**Proposition.** Suppose \( e(G_n) \geq (c + \epsilon) \binom{n}{2} \). We claim that there is some \( H \subseteq G_n \) such that \( \delta(H) \geq c |V(H)| \) and \( |V(H)| \geq \delta n \) for some constant \( \delta(c) > 0 \).

**Proof.** If \( \delta(G_n) \geq cn \), we are done. Otherwise, \( \exists x_n \) such that \( d(x_n) < cn \). Remove it, making \( G_{n-1} = G_n \setminus \{x_n\} \), so \( \exists x_{n-1} \) such that \( d(x_{n-1}) < c(n - 1) \). Go on until \( G_l = G_{l+1} \setminus \{x_{l+1}\} \), and \( d(x_{l+1}) < c(l + 1) \). Now

\[
\begin{align*}
e(G_l) &> (c + \epsilon) \binom{n}{2} - c(l + 1) + \cdots + n \\
\binom{l}{2} &> (c + \epsilon) \binom{n}{2} - c \left[ \binom{n+1}{2} - \binom{l+1}{2} \right] \\
(1 - c) \binom{l}{2} &\geq \epsilon \binom{n}{2} - c(n - l)
\end{align*}
\]

In particular, if \( l \approx \sqrt{\epsilon n} \), we have a contradiction, since the left side is smaller than the dominant term on the right side. \( \square \)
So we can suppose WLOG that $\delta(G) \geq (1 - \frac{1}{r} + \epsilon) n$. We want to show that $G \supseteq K_{r+1}(t)$ for $t = \lceil c(r, \epsilon) \log n \rceil$ for $n \gg 0$. Now we can take $\delta(G) > (1 - \frac{1}{r}) n = \left(1 - \frac{1}{r-1} + \frac{1}{r(r-1)}\right) n$. We are tempted to try induction on $r$.

By the induction hypothesis, we have $G \supseteq K_r(T)$ with $T = \lceil c(r-1, 1/(r(r-1))) \log n \rceil$. Let this consist of the $K_i$. We want to take a constant portion out of each lump and find an $(r+1)$th lump to connect all of them to. We naturally look in $V \setminus K$. Naturally we’re going to look in vertices with high proportional degree in $K$, i.e. each must have $d(x, K) \geq \delta |K|$ for some constant $\delta$ that we’ll pick later. Call such vertices good and other vertices bad. How many good vertices are there?

Say the good vertices are $G$. If $x$ is bad, $d(x, K) < \delta |K|$. Counting all the edges from the bad and good vertices to $K$ leaves

$$e(K, V \setminus K) < |K||G| + \delta |K|(n - |K| - |G|).$$

On the other hand, every $w \in K$ is adjacent to $(1 - 1/r + \epsilon)n - |K|$ vertices outside of $K$, so $e(K, V \setminus K) > |K|((1 - 1/r + \epsilon)n - |K|)$. Manipulating, we therefore find that

$$((1 - 1/r + \epsilon)n - |K|) < |G| + \delta(n - |K| - |G|)$$

$$|G| > \frac{1 - 1/r + \epsilon - \delta}{1 - \delta} n - |K|$$

So set $\delta = 1 - 1/r + \epsilon/2$. This makes $|G| \geq \frac{cn^2}{1/r - \epsilon/2} \geq cn$. These are our candidates for being adjacent to the $K_i$.

If $x$ is good, then $d(x, K) \geq (1 - 1/r + \epsilon/2)vT = (r - 1)T + \frac{\epsilon}{2}T$. In particular, this implies that $d(x, K_i) \geq \frac{\epsilon}{2}T$, which we will want to be greater than $t$. Then for each $x$, pick $t$ neighbors in each $K_i$. This gives $x \leftrightarrow (T_1, T_2, \ldots, T_r)$ with $T_i \subseteq K_i$ and $x \leftrightarrow w$ for all $w \in T_i$.

On the other hand, the number of $r$-tuples $(T_1, T_2, \ldots, T_r)$ each of size $t$ is $\binom{rT}{t}$. Therefore, by pigeonhole, for $n$ large, there are at least $\frac{en}{(\binom{rT}{t})}$ good vertices connected to the same $r$-tuple $(T_1^*, \ldots, T_r^*)$. If $\frac{en}{(\binom{rT}{t})} \geq t$, we are done.

We actually need to work out some of our estimates to get the logarithmic bound on $t$. Taking $c^* = c(r-1, r(r-1))$, we have

$$\left(\frac{T}{t}\right)^r \leq \left(\frac{cT}{t}\right)^rt = \left(\frac{ec^*}{c}\right)^C\log n = \left[\frac{c^*n}{c}\right]^{cr},$$

so if $cr < 1$, then we are done. Taking a step back, we need $T = \frac{\epsilon}{2}T$.

The constant $c(r, \epsilon) = \frac{c(r-1, r(r-1))}{2^{r-1}(r-1)}$ works, so the cleanest proof will start from that direction.

We’ve proven this theorem:

**Theorem** (Bollobas, Erdos, 1973-4). If $e(G_n) \geq (1 - 1/r + \epsilon)(n\choose 2)$, then $G \supseteq K_{r+1}(t)$ with $t = \lceil \frac{r\log n}{2r - 1} \rceil$ for $n \gg 0$. In particular, for $n \gg 0$, $G \supseteq K_{r+1}(t)$ for all constants $t$.

### 17.2 Excluded subgraphs

**Definition.** Fix a graph $G$. By $\text{ex}(n; G)$ we mean the maximum number of edges in a graph on $n$ vertices with no subgraph isomorphic to $G$.

Let’s take an example, the Peterson graph $P$. What can we do? Well, there’s an odd cycle, so a complete bipartite graph will not have a Peterson subgraph. The Turan graph gives us the bound $\text{ex}(n, P) < (1 - 1/2)\binom{n}{2}$.

Moreover, we just proved that if $e(G) \geq (1 - 1/2 + \epsilon)\binom{n}{2}$, then there exist arbitrarily large 3-partite graphs. But since the Peterson graph can be 3-colored, it is the subgraph of a large enough complete 3-partite graph, so it will be a subgraph of $G$. This generalizes easily:
Theorem (Erdos, Stone, Simonovits). If $\chi(G) = r + 1$, then $ex(n; G) = (1 - 1/r)\left(\frac{n}{2}\right)^{r} + o(n^{2})$, or equivalently, $\lim_{n \to \infty} \frac{ex(n; G)}{\binom{n}{2}} = 1 - \frac{1}{r}$.

This result is remarkable. It calculates the limiting behavior of the extremal function for pretty much every graph. Why pretty much every graph? Well, we can’t do much with the bipartite case. All this result tells us is that the order of $ex(n; G)$ is less than $n^{2}$.

In the bipartite case, most things are open. Erdos et al. showed that for the cube, $ex(n; G) \sim n^{8/5}$. In general, we wonder what $\lim_{n \to \infty} \frac{\log(ex(n; G))}{\log n}$.

17.3 Szemerédi’s Regularity Lemma

This technical result is only called a lemma for historical reasons. We’ll see two applications of this, the first that motivated it, and the second a newer one with consequences in computer science.

Definition (Erdos). A set $A \subseteq N$ has positive upper density if $\limsup_{n \to \infty} \frac{|A \cap [n]|}{n} > 0$.

Erdos conjectured: Suppose $A$ has positive upper density. Then $A$ has arbitrarily long arithmetic progressions, i.e. given any $k$, there exist infinitely many arithmetic progressions of length $k$ in $A$.

This was a problem Erdos offered money for, and of those, the most expensive, $2000.

Bollobas’s treatment of this subject is very messy. There are typos, and it generally takes a bad approach. Instead, read


We’ll just state the theorem today.

Definition. Let $A$ and $B$ be disjoint subsets of vertices of a graph $G$. By the density $d(A, B) = \frac{e(A, B)}{|A||B|}$, the fraction of the number of edges between them.

Definition. Let $\epsilon > 0$. A pair $(A, B)$ is called $\epsilon$-regular if whenever $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \epsilon |A|$ and $|Y| \geq \epsilon |B|$, $|d(A, B) - d(X, Y)| \leq \epsilon$.

Theorem (Szemerédi’s Regularity Lemma). Given $\epsilon > 0$ and $t \in \mathbb{N}$, $\exists T(\epsilon, t)$ such that the following holds: For any graph $G$ on $n \geq T(\epsilon, t)$, there exists a partition of the vertices $(V_{0}; V_{1}, V_{2}, \ldots, V_{k})$ with $t \leq k \leq T(\epsilon, t)$, $|V_{i}| = |V_{j}|$ for $1 \leq i, j \leq k$, and $|V_{0}| \leq \epsilon n$, and with the exception of at most $\epsilon k^{2}$ pairs $(i, j)$, all the remaining pairs $(V_{i}, V_{j})$ are $\epsilon$-regular.

So we’re splitting up the vertices into a predetermined maximum number of parts that doesn’t depend on $n$. That’s the power, because it means that the $V_{i}$ get large.

18 Tuesday, November 23, 2010

Next term, Fokko van de Bult will be teaching the course, and talking about topics like generating functions. Third term, Professor Balachandran will be back to talk about probabilistic interpretations.

18.1 Szemerédi’s Regularity Lemma

Recall the definition:

Definition. Let $\epsilon > 0$. A pair $(U, W)$ of disjoint subsets of $V$ is called $\epsilon$-regular if whenever $X \subseteq U$ and $Y \subseteq W$ with $|X| \geq \epsilon |U|$ and $|Y| \geq \epsilon |W|$, $|d(U, W) - d(X, Y)| \leq \epsilon$, where $d(U, W) = \frac{e(U, W)}{|U||W|}$.
Consider as a useful example a random graph $G = G(n, p)$ with $n$ vertices and each of the $\binom{n}{2}$ edges chosen with probability $p$. The expectation of the density will always be $p$.

Now the statement of the lemma:

**Theorem (Szemerédi’s Regularity Lemma).** Given $\epsilon > 0$ and $t \in \mathbb{N}$, $\exists T(\epsilon, t)$ such that the following holds: For any graph $G$ on $n \geq T(\epsilon, t)$ vertices, there exists a partition of the vertices $(V_0; V_1, V_2, \ldots, V_k)$ with

(i) $|V_0| \leq \epsilon n$ (this is the special set).

(ii) $|V_1| = |V_2| = \cdots = |V_k|$

(iii) $t \leq k \leq T(\epsilon, t)$,

(iv) and with the exception of at most $ck^2$ pairs $(i, j)$, all the remaining pairs $(V_i, V_j)$ are $\epsilon$-regular.

First let’s focus on applications of the SRL. Write $G$ can be written in this form. Delete

1. ...all edges inside any of the parts.

2. ...all edges incident with $V_0$.

3. ...all edges between irregular pairs.

4. ...all edges between pairs of density less than some threshold $d > \epsilon$.

Let’s bound the number of edges we removed. Let $t = \lceil \frac{1}{\epsilon^2} \rceil$, so there are at least $1/\epsilon$ parts. Then part 1 removes at most $\left( \epsilon^2 \right) k \leq \frac{n^2}{2\epsilon^2} \leq \frac{\epsilon n^2}{2}$ edges. Part 2 removes less than $\epsilon n^2$, part 3, at most $(\epsilon k^2) \left( \frac{n}{k} \right)^2 = \epsilon n^2$, and part 4, at most $\epsilon n^2$ edges, so in all, at most $\frac{7}{2} \epsilon n^2$ edges are removed. Thus, if $e(G) \geq \delta n^2$, taking $\epsilon < \delta/4$ means that this ‘purified’ version has at least $\frac{1}{2} \delta n^2$ edges. This new version is a $t$-partite graph.

Suppose we’re looking at only three parts and it is $d$-regular with $d = 2\epsilon$. Consider $u \in V_1$. We call such $u$ good if $d(u, V_2) \geq \epsilon |V_2|$ and $d(u, V_3) \geq \epsilon |V_3|$. Consider $\{w \in V_1 : d(u, V_2) < \epsilon |V_2|\} = W_1$. This set cannot be too big: In fact, we claim $|W_1| < \epsilon |V_1|$. If not, $|W_1| \geq \epsilon |V_1|$, and $e(W_1, V_2) = \epsilon |V_2| |W_1| < \epsilon$. Thus doing the same thing on $V_3$, we have $|\text{GOOD}| \geq (1 - 2\epsilon)|V_1|$. Our goal is to get triangles out of this graph. Consider a good vertex. Then let its neighbors be $N_2(u)$ and $N_3(u)$ in each of $V_2$ and $V_3$, respectively. Then these have $|N_2(u)| \geq \epsilon |V_2|$ and $|N_3(u)| \geq \epsilon |V_3|$. In particular, since $(V_2, V_3)$ is $\epsilon$-regular, $e(N_2(v), N_3(v)) \geq \epsilon |N_2(v)||N_3(v)| \geq \epsilon^3 |V_1||V_2|= \epsilon^3 |V_1||V_2|$. Therefore, this produces at least $(1 - 2\epsilon)^3 |V_1||V_2| |V_3|$ triangles in $G$. This is a constant number times the number of potential triples!

This leads to a graph property tester with applications in theoretical computer science. Consider testing for the property of being triangle-free.

We say that a graph is $\epsilon$-far from being triangle-free if one has to delete $\epsilon n^2$ edges to remove all triangles. We’ve shown:

**Proposition.** Suppose $G$ is $\epsilon$-far from being triangle free. Then $G$ has $c(\epsilon)n^3$ triangles, where $c(\epsilon)$ is some function of $\epsilon$.

Now suppose you pick $N$ triples at random. The probability that none of these is a triangle is at most $\left(1 - \frac{c(\epsilon)n^3}{n^\frac{3}{2}}\right)^N = (1 - 6c(\epsilon))^N \leq e^{-6c(\epsilon)N}$. To make this less than a fixed probability, we only have to test a constant number of triples. That is, if we want an arbitrarily small probability of a graph either being triangle-free or being $\epsilon$-far from being triangle-free, we can do it with a fixed number of edges, independent of $n$.

A more general result (due to Alon, Shapira, 2006): Any hereditary graph property (a property retained by subsets) has a property tester.

The original application:

**Theorem** (Erdős, Turán). Any set of positive upper density has an arithmetic progression of size $k$. 

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Theorem (Roth, later by Szemeredi). If $A$ has positive upper density then $A$ has a three-term arithmetic progression.

This was an easier result that Szemeredi started from, before going on to the general case. Let’s prove this simpler problem (Roth’s theorem). Suppose $A$ has no three-term arithmetic progressions, and we will show that it does not have positive upper density. Let $A_n = A \cap [n]$. Now consider the bipartite graph $G$ constructed from parts $[2n]$ and $[3n]$, with $(x + a) \leftrightarrow (x + 2a)$ for all $a \in A_n$. Then $e(G) = n |A_n|$. Furthermore, $G$ is the union of $|A|$ matchings.

Szemeredi’s observation: Each of these matchings is induced, i.e. there are no edges other than the edges of the matching between the vertices of each matching. Indeed, suppose we had $(x + a) \leftrightarrow (y + 2a)$ and $(y + a) \leftrightarrow (x + 2a)$. Then $x + a = z + b$ and $y + 2a = z + 2b$ for some $b \in A_n$. Then $x + a = z + b(2a - 2b) = z + (2a - b)$. Somehow (we can’t figure out how) he concluded that $2a - b \in A$. Then $(b, a, 2a - b)$ is a 3-term arithmetic progression.

Well, let’s assume we’ve taken care of that and each matching only has edges between it. Given $\epsilon > 0$, we claim that for $n$ large enough, every induced matching must be of size (number of edges) $\leq \epsilon n$.

Proof. Suppose otherwise that some induced matching has more than $\epsilon n$ edges, or $2\epsilon n$ vertices. Purify this graph as before into $V_1, V_2, \ldots, V_k$.

Now there are $\epsilon n^2$ edges originally, so there is one of your original matchings that now has $\epsilon n$ edges (maybe you need to adjust $\epsilon$). Now suppose that this splits into $M_1, M_2, \ldots, M_k$ with $M_i \subseteq V_i$. Then we claim that some pair $M_1, M_2$ with $|M_1| \geq \epsilon |V_1|$ and $|M_2| \geq \epsilon |V_2|$ has an edge between them. Indeed, those with small size will not contribute enough edges between them (check this). Once we know that there is an edge between them, they must be $\epsilon$-regular, so $e(M_1, M_2) \geq \epsilon |V_1||V_2|$. But this is an induced matching, so this is bounded above by $\max\{|M_1|, |M_2|\}$. If $|V_i|$ are large enough, this is a contradiction, as desired.

We don’t have time to prove Szemeredi’s lemma, but we can give a sketch of the idea. Basically we define the index of a partition $\pi$ by

$$q(\pi) = \sum_{(U, W) \in \pi} d^2(U, W) \frac{|U||W|}{n^2}.$$  

If there are more than $\epsilon k^2$ irregular pairs, a standard argument shows that you can change $\pi$ to $\pi'$ such that

$q(\pi') > q(\pi) + \frac{\epsilon}{2}$. This constant is independent of $n$! Since all of these are bounded from above by $\frac{1}{2}$, in a finite number of steps not depending on $n$, a good partition can be reached.

Here are the best sources for that proof:

2. Komlós, Simonovits, “SRL and its applications in graph theory.”
3. Tao, Vu, “Additive Combinatorics”