(1) Consider the complete bipartite graph $K_{4,4}$ with vertex partition $(A,B)$ where $A = \{a, b, d, f\}$, $B = \{g, h, i, j\}$. Weights are placed on edges as follows:

\[
\begin{align*}
    c_{ag} &= 2 & c_{ah} &= 7 & c_{ai} &= 1 & c_{aj} &= 2 \\
    c_{bg} &= 3 & c_{bh} &= 4 & c_{bi} &= 3 & c_{bj} &= 2 \\
    c_{dg} &= 6 & c_{dh} &= 5 & c_{di} &= 5 & c_{dj} &= 5 \\
    c_{fg} &= 2 & c_{fh} &= 6 & c_{fi} &= 2 & c_{fj} &= 3
\end{align*}
\]

The following vector $y$ is a feasible solution for the dual to the maximum weight perfect matching linear program for the given graph:

\[
y_g = y_h = 0, \quad y_i = y_j = -1, \quad y_a = 7, y_b = 4, y_d = 6, y_f = 6.
\]

Use this to find a maximum weight perfect matching in the graph, and an optimal solution for the dual problem.

**Solution:** We begin looking at the graph $G(y)$. Observe that the neighborhood of $\{a, f\}$ in $G(y)$ is $h$, so $G(y)$ can not have a perfect matching. The neighborhood of $\{a, f\}$ in $G$ is $\{g, h, i, j\}$, so the Hungarian algorithm tells us we must increase $y_a, y_f$ and decrease $y_h$ all by 2. This gives us a new $y$ with values:

\[
y_g = 0, y_h = 2, \quad y_i = y_j = -1, \quad y_a = 5, y_b = 4, y_d = 6, y_f = 4,
\]

and in $G(y)$ for this new value of $y$, we have a perfect matching (for instance, the edges $\{ah, bi, dg, fj\}$ form one. Thus the weight of a max weight perfect matching is $7 + 3 + 6 + 3 = 19$.

(2) Consider the following alternate definition of a matroid: A matroid $M$ consists of a pair $(E, B)$ where $E$ is a set and $B$ is a set of subsets of $E$ called bases such that:

- $B \neq \emptyset$.
- No set in $B$ is a proper subset of an other set in $B$.
- If $B_1, B_2 \in B$, then for any $e \in B_1$ there exists $f \in B_2$ such that $(B_1 \setminus \{e\}) \cup \{f\} \in B$.

(a) Show that if $M = (E, I)$ is a matroid, and $B$ is the set of maximal independent sets (with respect to set inclusion) in $I$, then $(E, B)$ is a matroid under the alternate definition given above.

**Solution:** Let $B$ be the collection of maximal independent sets in $M$. First, since $M$ has at least one independent set (namely the empty set), there is at least one maximal independent set, so $B$ is not empty. Now suppose $B_1, B_2 \in B$. We claim that $|B_1| = |B_2|$. Suppose otherwise, and that without loss of generality that $|B_1| > |B_2|$. Since both $B_1$ and $B_2$ are independent, by Matroid axioms, there exists $e \in B_1 \setminus B_2$ such that $B_2 \cup \{e\}$ is independent. This contradicts the maximality of $B_2$ with respect to being independent. We conclude all
bases have the same size. Now suppose \( e \in B_1 \). The \( B_1 \setminus \{e\} \) is independent and has smaller size than \( B_2 \), so again by Matroid axioms, there exists \( f \in B_2 \) such that \( (B_1 \setminus \{e\}) \cup \{f\} \) is independent. Now the fact that \( (B_1 \setminus \{e\}) \cup \{f\} \) is maximal with respect to independence can again be proven by the same arguments as above.

(b) Conclude the following statements:

- If \( V \) is a finite dimensional vector space, and \( U \) is a set of vectors in \( V \), then all bases for \( \text{span}(U) \) have the same size.
  
  **Solution:** The matroid given by the matrix whose columns are the vectors in \( U \) has bases being precisely the maximal linearly independent sets, hence linear algebraic bases. By the previous part, all bases have the same size, so the result follows.

- All spanning forests in a graph have the same number of edges.
  
  **Solution:** A basis of a graph \( G \) with respect to its forest matroid is a maximal acyclic subgraph. Thus, in each component, a basis must consist of a spanning tree (we proved in class last quarter that spanning trees are maximal acyclic subgraphs of connected graphs). Thus bases are spanning forests, and hence by the above, they all have the same size.

(3) An independence system is a pair \((E, \mathcal{I})\) where \( E \) is a set, and \( \mathcal{I} \) is a collection of subsets of \( E \) such that

- \( \emptyset \in \mathcal{I} \).
- If \( J' \subseteq J \in \mathcal{I} \), then \( J' \in \mathcal{I} \).

The sets in \( \mathcal{I} \) are called independent sets. Suppose that \((E, \mathcal{I})\) is an independence system, and that for any set of weights \( c \in \mathbb{R}^E \), the greedy algorithm finds an optimal independent set. Prove that \((E, \mathcal{I})\) is a matroid.

**Solution:** We prove the contrapositive. Suppose that \((E, \mathcal{I})\) is not a matroid. Let \( I \) and \( J \) be independent sets with \(|I| < |J|\) such that there does not exist \( e \in J \setminus I \) such that \( I \cup \{e\} \) is independent. Consider the weights assigned to \( E \) as follows: \( c(e) = 1 + \frac{1}{2|I|} \) if \( e \in I \), \( c(e) = 1 \) if \( e \in J \setminus I \), \( c(e) = 0 \) otherwise. The greedy algorithm will choose all of \( I \) first, and afterward can not collect any elements of weight 1 since adding such elements would not maintain independence. Thus the greedy algorithm will return a basis with total weight \(|I|(1 + \frac{1}{2|I|}) = |I| + \frac{1}{2} \). But \( J \) is independent and has total weight \(|I|(1 + \frac{1}{2|I|}) + |J \setminus I|\) which exceeds the weight of the basis given by the greedy algorithm.

(4) (a) Suppose \( M \) is matroid, representable over \( \mathbb{Q} \), given by the matrix \((I \mid P)\) where \( I \) is the \( n \times n \) identity matrix, and \( P \) is a \( n \times m \) matrix with rational entries. Show that the dual of \( M \) is represented by the matrix \((-P^T \mid I)\).

**Solution:** The columns of the matrix are indexed by the ground set of \( M \). Let \( B \) be a basis of \( M \) and rearrange the rows and columns of \( A \) so that \( B = X_2 \cup Y_1 \) where \( X_2 \) and \( Y_1 \) are the column indices for
the 2nd and 3rd blocks of columns in the block matrix below:

\[
\begin{pmatrix}
I_{(m-k) \times (m-k)} & 0 & P_1 & P_2 \\
0 & I_{k \times k} & P_3 & P_4
\end{pmatrix}.
\]

This can be done because row and column switching are invertible row and column operations and hence do not affect the matroid up to isomorphism. So the matrix whose columns correspond to \(B\) are

\[
\begin{pmatrix}
0 & P_1 \\ I_{k \times k} & P_3
\end{pmatrix}.
\]

Since \(B\) is a basis, \(P_1\) has full rank. Now if we consider the matrix in question for a candidate of the dual

\[
\begin{pmatrix}
-P_1^T & -P_3^T & I & 0 \\
-P_2^T & -P_4^T & 0 & I
\end{pmatrix}
\]

We know that since \(P_1\) has full rank, \(-P_1^T\) does, and so the columns in this matrix by the complement of those \(B\) is indexed by in the original matrix form an independent set of vectors. By a similar argument, it is a maximal independent set and hence a basis.

Thus the complement of any basis of the original matroid is a basis in the prospective dual. Exactly the same argument can be used to show that the complement of a basis in the prospective dual is a basis in the original matroid. The result follows.

(b) Consider the matroid given by the column vectors in the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

considered over \(\mathbb{F}_2\), the finite field with two elements. Is this matroid representable over \(\mathbb{Q}\)?

**Solution:** Notice in our original matroid the following sets are dependent:

\(\{1, 2, 4\}, \{1, 3, 5\}, \{2, 3, 6\}, \{4, 5, 6\}, \{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}\).

and none of them contain any dependent proper subsets. Now assume this matroid has a representation over \(\mathbb{Q}\). Since \(\{1, 2, 3\}\) is a basis, then this representation can be written as

\[
\begin{pmatrix}
1 & 0 & 0 & * & * & * & * \\
0 & 1 & 0 & * & * & * & * \\
0 & 0 & 1 & * & * & * & *
\end{pmatrix}
\]

where the stars are unknown entries. Now since \(\{1, 2, 4\}, \{1, 3, 5\}, \{2, 3, 6\}\) are dependent and have no dependent subsets, we must have that the matrix is (up to column scaling)

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & * \\
0 & 1 & 0 & a & 0 & 1 & * \\
0 & 0 & 1 & 0 & b & c & *
\end{pmatrix},
\]
where $a, b, c$ are non-zero. Now $\{4, 5, 6\}$ has to be dependent, so the determinant of the matrix
\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
has to be 0. Thus $b = -ac$, so our matroid has representation
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & * \\
0 & 1 & 0 & a & 0 & 1 & * \\
0 & 0 & 1 & 0 & -ac & c & *
\end{pmatrix},
\]
where $a, c$ are non-zero. Now successively perform the following row and column operations: Divide row 2 by $a$. Divide row 3 by $c$. Multiply columns 2 and 3 by $a$ and $c$ respectively, multiply column 6 by $a$, divide row 3 by $a$ and multiply column 3 by $a$. We then get the matrix representation
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & x \\
0 & 1 & 0 & 1 & 0 & 1 & y \\
0 & 0 & 1 & 0 & -1 & 1 & z
\end{pmatrix},
\]
where $x, y, z$ are unknown rational numbers. Now finally, $\{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}$ are dependent. Taking the determinant of the matrices given by these columns, we get $\{1, 6, 7\}$ implies $y = z$, $z = -x$ and $x = y$, which would imply $x = y = z = 0$, contradicting that $\{7\}$ is independent. Thus the matroid is not representable over $\mathbb{Q}$.

(5) A circuit in a matroid $(E, I)$ is a subset of $S \subset E$ that is minimally dependent; that is, it is minimal under set inclusion amongst subsets of $E$ that are not independent. Show that if $C_1, C_2$ are circuits, $C_1 \neq C_2$, and if $e \in C_1 \cup C_2$, then there exists a circuit $C$ such that $C \subseteq (C_1 \cup C_2) \{e\}$.

State consequences in both graph theory and linear algebra.

Solution: Suppose $(C_1 \cup C_2) \{e\}$ does not contain a circuit. Then $(C_1 \cup C_2) \{e\}$ is independent. Thus $C_1 \cup C_2$ is a circuit since it is dependent and minimally so. But $C_1$ and $C_2$ are two circuits in $C_1 \cup C_2$, contradicting minimality.

In graph theory, this states that if two subgraphs $G_1$ and $G_2$ of a graph $G$ each contain exactly one cycle, then the graph $G'$ taking edges of $G_1$ and $G_2$ together has the property that if any edge is chosen and deleted, the remaining graph still contains a cycle. In linear algebra this says that in the union of two minimally dependent sets in a vector space, if any element is removed, there is still a minimally dependent set among the remaining elements.