For all problems except “No Collaboration” problems, you are allowed to use the textbook, class notes, and other book references. For “No Collaboration” problems, you are only allowed to use class notes.

1. (a) For $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, prove

$$\{x \in \mathbb{R}^n : a^T x \leq b\}$$

is convex.

(b) If $P \subset \mathbb{R}^n$ is an $n$-dimensional polytope and $F \subset P$ is a proper face, prove $\dim(F) < \dim(P)$.

(c) If $P \subset \mathbb{R}^n$ is bounded $n$-dimensional polytope, and the intersection of a finite number of half spaces, prove $P$ is the convex hull of finitely many points. (Hint: Use induction on $n$).

(d) A face of a convex set $S \subset \mathbb{R}^n$ is a convex subset $F \subset S$ such that If $x \in F$ and $x = \lambda x^{(1)} + (1 - \lambda)x^{(2)}$ with $\lambda \in (0, 1)$ and $x^{(1)}, x^{(2)} \in S$, then $x^{(1)}, x^{(2)} \in F$. Give an example of a non-empty convex set $S$ and a non-empty face $F$ of $S$ such that $F$ is not the intersection $F \cap H$ of $F$ with a hyperplane $H$, where $S$ lies on one side of $H$.

2. The Traveling Salesman Problem is a classical combinatorial optimization problem described as follows. A salesperson has to visit each of a set of cities exactly once, and return to the first one. Between any two cities $i$ and $j$, there is a travel cost $c_{ij}$. The salesperson wants to do this while minimizing the total travel cost. Let $G = (V, E)$ whose vertices are the cities and edges are the travel routes between the cities. Show that the following is an integer linear programming formulation for the traveling salesman problem:

$$\text{minimize } c^T x$$

$$\sum_{e \text{ incident to } v} x_e = 2, \text{ for all } v \in V(G)$$

$$\sum_{\text{just one end of } e \text{ is in } B} x_e \geq 2, \text{ for each } B \subseteq V, 3 \leq |B| \leq |V| - 3$$

$$0 \leq x_e \leq 1 \text{ for each } e \in E$$

$$x_e \in \mathbb{Z} \text{ for each } e \in E$$
3. Let $D = (V, A)$ be a directed graph with vertex set $V$ and directed edges $A$, with capacities $c_e$ for each $e \in A$. Let $s, t \in V$ be fixed vertices. Prove that the optimal value of the following linear program is precisely the maximum $st$-flow in $D$:

$$\max \sum_{e \text{ leaving } s} x_e - \sum_{e \text{ entering } s} x_e$$

$$\sum_{e \text{ leaving } v} x_e - \sum_{e \text{ entering } v} x_e = 0, \quad \text{for all } v \in V - \{s, t\},$$

$$0 \leq x_e \leq c_e, \quad \text{for all } e \in A,$$

Note: We have not restricted $x_e$ to be integers apriori!

4. Recall the following proposed linear programming formulation for the stable set problem we gave in class:

$$\max \sum_{v \in V(G)} x_v$$

$$x_u + x_v \leq 1, \quad \text{for all } uv \in E(G),$$

$$\sum_{v \in V(C)} x_v \leq \frac{|C| - 1}{2}, \quad \text{for all odd cycles } C \text{ in } G,$$

$$0 \leq x_v, \quad \text{for all } v \in V(G),$$

$$x_v \in \mathbb{R}, \quad \text{for all } v \in V(G).$$

(a) Give an example of a graph for which this linear program has a solution that exceeds the maximum stable set in the graph.

(b) Give an example of a graph for which this linear program has a solution that exceeds the maximum stable set in the graph, if the following set of inequalities is added:

$$\sum_{v \in V(Q)} x_v \leq 1 \quad \text{for all complete subgraphs } Q \text{ in } G.$$

5. Use linear programming duality and Problem 3 to prove the Max-Flow Min-Cut Theorem.