1. For a pair of vertices $u, v$ in a graph $G$, the distance between $u$ and $v$, denoted $d(u, v)$, is the number of edges in the path from $u$ to $v$ using as few edges as possible. The diameter of $G$ is 

$$\max_{u,v \in V(G)} d(u, v).$$

Let $d, k$ be positive integers. Let $G$ be a graph with maximum degree $d$ and diameter $k$. Show that 

$$|V(G)| \leq 1 + d \sum_{i=0}^{k-1} (d - 1)^i.$$

Give an example of an $G$ with $(d, k) = (3, 2)$ where equality holds.

2. A cycle in a graph is Hamiltonian if it goes through all vertices. Suppose $G$ is a graph on $n$ vertices with the property that for every pair of vertices $u, v$ that are not adjacent, $\deg(u) + \deg(v) \geq n$. Prove that $G$ has a Hamiltonian cycle. Can we replace $n$ in the previous inequality by any value smaller than $n$ and guarantee $G$ is Hamiltonian?

3. (No Collaboration) Determine, with proof, the value of $ex(n, K_{1,r})$ for every pair of positive integers $n, r$.

4. (a) Suppose $G$ is a graph not containing $C_4$ (a cycle on four vertices) as a subgraph. Prove that 

$$\sum_{v \in V(G)} \left( \frac{\deg(v)}{2} \right) \leq \left( \binom{n}{2} \right),$$

and use this to prove there is a constant $C > 0$ such that 

$$ex(n, C_4) \leq C \cdot n \sqrt{n}$$

for all $n \geq 4$.

(b) Let $q$ be any prime and $\mathbb{F}_q$ be the finite field of order $q$. Consider the bipartite graph $G_q$ with bipartition $(\mathcal{P}_q, \mathcal{V}_q)$ where $\mathcal{P}_q$ is the set of 2-dimensional subspaces of $\mathbb{F}_q^3$ and $\mathcal{V}_q$ is the set of 1-dimensional subspaces of $\mathbb{F}_q^3$, with $p \in \mathcal{P}_q$ adjacent to $v \in \mathcal{V}_q$ if and only if $v$ lies in $p$. Show that for any prime $q$, $G_q$ does not contain $C_4$ as a subgraph, and 

$$|E(G_q)| \geq \frac{n \sqrt{n}}{2 \sqrt{2}},$$

where $n = |V(G_q)|$. Hence there is a constant $C > 0$ such that for infinitely many $n$, 

$$ex(n, C_4) \geq C \cdot n \sqrt{n}.$$
Recall that for any graphs $G_1, G_2, \ldots, G_k$, we define $R(G_1, G_2, \ldots, G_k)$ to be the smallest positive integer $n$ so that for any $k$-coloring of the edges of $K_n$, say with colors $c_1, c_2, \ldots, c_k$, there must exist some $i$ for which $G_i$ is a subgraph, all of whose edges are colored with color $c_i$. For simplicity, we denote $R(K_{r_1}, K_{r_2}, \ldots, K_{r_k})$ by $R(r_1, r_2, \ldots, r_k)$.

5. (No Collaboration) Let $m, n$ be positive integers, and assume $(m - 1)|(n - 1)$. Determine, with justification, a function $f(m, n)$ for which $R(T, K_{1,n}) = f(m, n)$ for every tree $T$ on $m$ vertices.

6. In this problem, we shall prove $R(3, 4) = 9$.

   (a) Consider the graph $G$ whose vertex set is $\{1, 2, 3, 4, 5, 6, 7, 8\}$, with $i$ and $j$ adjacent if and only if $i - j = \pm 1$ or $\pm 4 \text{ mod } 8$. Show that $G$ does not have $K_3$ and a subgraph, and $\overline{G}$ does not contain $K_4$ as a subgraph.

   (b) Show that for any graph $G$ on 9 vertices, either $G$ has $K_3$ as a subgraph, or $\overline{G}$ has $K_4$ as a subgraph. (Hint: Consider three cases: The existence of a vertex with degree at least 4, the existence of a vertex of degree at most 2, and $G$ being 3-regular).

7. (a) Show that $R(3, 3, \ldots, 3) \leq \lfloor e \cdot r! \rfloor + 1$.

   (b) Suppose the integers $\{1, 2, \ldots, n\}$ are partitioned into $r$ disjoint sets $A_1, A_2, \ldots, A_r$. Show that if $n \geq \lfloor e \cdot r! \rfloor + 1$, then one of the sets $A_i$ contains three elements $x, y, z$ such that $x + y = z$. 