1. Let $A$ be a finite set with subsets $A_1, \ldots, A_n$ and let $d_1, d_2, \ldots, d_n$ be positive integers. Show that there are disjoint subsets $D_k \subset A_k$ with $|D_k| = d_k$ for all $k$, if and only if $|\bigcup_{i \in I} A_i| \geq \sum_{i \in I} d_i$ for every $I \subset \{1, 2, \ldots, n\}$.

2. Let $G$ be a 3-regular connected graph that has no bridge, with $|V(G)| \geq 3$. Prove $G$ has a perfect matching.

3. Find a maximum flow in the network below from the source $S$ to the sink $T$. Justify that your flow is a maximum flow.

4. Let $G$ be a graph and $s, t \in V(G)$. A set $K \subset E(G)$ is said to separate $s$ from $t$ if every path from $s$ to $t$ in $G$ uses some edge in $K$. Using network flow theory, prove that the minimum number of edges needed to separate $s$ from $t$ is equal to the maximum number of edge-disjoint paths from $s$ to $t$. (Hint: Create a directed multigraph. Depending on how you do this, you may or may not want edges going both in and out of both the source and sink.) Use this to establish another proof of Problem 3 on Assignment 7.

5. For any positive integer $n$, let $G_n$ be the graph whose vertex set is the set of $n$-element subsets of $\{1, 2, 3, \ldots, 2n + 1\}$, with two vertices adjacent if and only if they have no elements in common. For instance, below is the graph $G_2$. 
Determine all positive integers \( n \) for which \( G_n \) is planar.

6. Soccer balls typically consist of hexagonal faces and pentagonal faces, with all vertices where faces meet having degree exactly 3. By considering this as a planar graph, show that regardless of the number of hexagonal faces, there must always be exactly 12 pentagonal faces. (Look at a few soccer balls and observe this is indeed true!)