1. Solution:
a. The order of $(13)(486)(79)$ is lcm$(2, 3) = 6$, and its type in $S_9$ is $1^22^23^1$. Now consider the map

$$S_9 \to S_9 : \sigma \mapsto (\sigma(0)\sigma(1))(\sigma(2)\sigma(3))(\sigma(4)\sigma(5)\sigma(6)).$$

This clearly maps $S_9$ onto all elements of $S_9$ of type $1^22^23^1$. Moreover, each such $\sigma$ has $(2^23^1)^2!$ elements which map to it, since every $n$-cycle can be “written” in $n$ many ways, and the order of the two 2-cycles doesn’t matter. We conclude that there are

$$\frac{9!}{(2^23^1)^2!}$$

many elements of $S_9$ of the desired type.
b. The product of the transpositions is $(15)$.

2. Solution:
We need only show that every product $\tau = (ij)(rs)$ of two transpositions can be written as a product of 3-cycles.
There are three cases to consider:
- If $(ij) = (rs)$ then $\tau = 1$.
- If $j = r, i \neq s$ then $\tau = (jsi)$.
- If $i, j, r, s$ are pairwise distinct, then $\tau = (ris)(ijr)$.

3. Solution:
a. Let $B$ be result of switching the $r$th and $s$th rows of $A$, so $B_{i,j} = A_{\tau(i),j}$, where $\tau = (rs)$. Then

$$\det(B) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)B_{1,\sigma(1)}\cdots B_{n,\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma)A_{\tau(1),\sigma(1)}\cdots A_{\tau(n),\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma\tau^{-1})\text{sgn}(\tau)A_{1,\sigma(\tau^{-1}(1))}\cdots A_{n,\sigma(\tau^{-1}(n))}$$

$$= \text{sgn}(\tau)\det(A)$$

$$= -\det(A),$$

as desired.
b. $A^T_{i,j} = A_{j,i}$, so
\[
\det(A^T) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1,\sigma(1)}^T \cdots A_{n,\sigma(n)}^T \\
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(1),1} \cdots A_{\sigma(n),n} \\
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1,\sigma^{-1}(1)} \cdots A_{n,\sigma^{-1}(n)} \\
= \sum_{\sigma \in S_n} \text{sgn}(\sigma^{-1}) A_{1,\sigma^{-1}(1)} \cdots A_{n,\sigma^{-1}(n)} \\
= \det(A).
\]

4. Solution:
Fix two adjacent vertices. Note that they are adjacent corners of a “square” within the larger object. A rotation of the object corresponding to a ninety degree rotation of this square can then be used to move one vertex to the other. It follows that the orbit of any given vertex is the full set of vertices.

Now choose a vertex and consider the rotations of the object which fix this vertex. By observing that any such rotation must send triangles to triangles, and similarly for squares, we see that there is only one non-trivial such rotation. It follows that the stabilizer of any vertex has size two.

5. Solution:
We will make use of Burnside’s lemma.

The grid admits six rotations about its center: a rotation of 60n degrees for each \(n < 6\). For each \(n\), we will count the number of grid colorings which are invariant under rotation by 60n degrees. Clearly, there are \(2^{24}\) grid colorings which are fixed by the trivial rotation. Now observe that the action of the rotation group partitions the cells in the grid into four distinct orbits. If a grid coloring is fixed by a rotation of 60 or 300 degrees, then each orbit must be colored uniformly, and so there are \(2^4\) invariant grid colorings for each of these rotations. Rotations of 120 or 240 degrees split each orbit into two pieces, each of which must be colored uniformly, so there are \(2^8\) invariant grid-colorings for each. Similarly, there are \(2^{12}\) invariant grid-colorings for the 180 degree rotation.

Finally, we use Burnside’s lemma to compute that, up to rotational equivalence, there are
\[
\frac{1}{6} (2^{24} + 2 \cdot 2^4 + 2 \cdot 2^8 + 2^{12}) = 2796976
\]
grid colorings.