For all problems on assignments, you are allowed to use the textbook, class notes, and other book references. All problems must be written on your own, though you may collaborate on problem solving for all but the “No collaboration” problems. The internet can be used for reference material on course subjects, but should not be used to search for or obtain solutions.

1. (No Collaboration)

(a) The game of Motzhee consists of rolling five 8-sided dice (each die of a differing color), each of whose faces are numbered with the integers \{1, 2, 3, 4, 5, 6, 7, 8\}. How many ways are there to roll a full house? A full house consists of five dice, three of which have the same number facing up, the other two of which have the same number facing up as well, but where the number facing up in the triple is different from that of the pair. You can leave your answer in terms of binomial coefficients.

(b) Give a brief combinatorial explanation why, for positive integers \( k, l, n \) with \( k + l \leq n \),
\[
\binom{n}{k} \binom{n-k}{l} = \binom{n}{l} \binom{n-l}{k}.
\]

(c) At a party, certain pairs of individuals have shaken hands. Prove that there exist two people who have shaken the same number of hands.

2. A teacup ride at a county fair has 10 seats around a circular table. As the ride progresses, the table spins circularly. How many ways are there to seat 5 couples on this ride if no person is seated next to his/her partner? Seatings are considered equivalent if the table can be rotated to obtain one from the other. (You may use a calculator for this question.)

3. Let \( m, n \) be positive integers, \( m < n \).

(a) Give an algebraic proof that
\[
\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m}.
\]

(b) Give a combinatorial proof that
\[
\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m}.
\]
4. Let $m, n$ be positive integers with $m > n$. Show that the number of functions $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ that are surjective (i.e. onto) is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)^m.$$ 

5. Let $T_n$ be the number of ways of tiling a $2 \times n$ checkerboard with $1 \times 1$ square tiles and L shaped tiles. L shaped tiles are pictured below:

Determine a recurrence satisfied by $T_n$, and use it to find an explicit formula for $T_n$ in terms of $n$. Check your formula works for $n = 3$. (You can use a computer or computational tool for this problem).