1. Let \( n \geq 1 \) be a positive integer. Prove
\[
1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}.
\]

**Solution:** When \( n = 1 \), the left side is 1 and the right side is \( \frac{1^2 \cdot 2^2}{4} = 1 \). Now assume that the equality holds from some positive integer \( n \). We claim the statement holds for \( n + 1 \). Indeed,
\[
1^3 + 2^3 + \cdots + n^3 + (n + 1)^3 = \frac{n^2(n + 1)^2}{4} + (n + 1)^3
\]
\[
= (n + 1)^2 \left[ \frac{n^2}{4} + n + 1 \right]
\]
\[
= (n + 1)^2 \left[ \frac{n^2 + 4n + 4}{4} \right]
\]
\[
= \frac{(n + 1)^2(n + 2)^2}{4}.
\]
Thus by Mathematical Induction, the statement holds for all positive integers \( n \).

2. Prove that for any integer \( n \geq 1 \),
\[
(cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta),
\]
where \( i^2 = -1 \).

**Solution:** Equality clearly holds for \( n = 1 \). Now suppose \( n \geq 2 \) is a positive integer, and \((cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)\). Then
\[
(cos(\theta) + i \sin(\theta))^{n+1} = (cos(\theta) + i \sin(\theta))^n (cos(\theta) + i \sin(\theta))
\]
\[
= (cos(n\theta) + i \sin(n\theta)) \cdot (cos(\theta) + i \sin(\theta))
\]
\[
= (cos(n\theta) \cos(\theta) - \sin(n\theta) \sin(\theta)) + i(sin(n\theta) \cos(\theta) + \sin(\theta) \cos(n\theta))
\]
\[
= \cos((n + 1)\theta) + i \sin((n + 1)\theta)
\]
Thus by Mathematical Induction, the statement holds for all positive integers \( n \).

3. For each of the following, prove or disprove the limit exists.

(a) \[
\lim_{x \to 3} (2x - 1).
\]
Solution: Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then if $x$ satisfies $0 < |x - 3| < \delta$, then

$$|(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2\delta = \epsilon.$$ 

Thus, by the definition of a limit,

$$\lim_{x \to 3} (2x - 1) = 5.$$ 

(b) 

$$\lim_{x \to 3} x^2.$$ 

Solution: We claim $\lim_{x \to 3} x^2 = 9$. Indeed, given $\epsilon > 0$, we choose $\delta > 0$ to be

$$\delta = \min \left\{ \frac{\epsilon}{7}, 1 \right\}.$$ 

Then if $0 < |x - 3| < \delta$, we have

$$|x^2 - 9| = |x - 3||x + 3| = |x - 3|(|x - 3 + 6|) \leq |x - 3||x - 3| + 6) < 7|x - 3| < \epsilon.$$ 

The last inequality comes from the fact that $|x - 3| < \epsilon/7$. The second last inequality comes from the fact that $|x - 3| < 1$.

(c) 

$$\lim_{x \to 0} \frac{1}{x}.$$ 

Solution: Suppose otherwise. That is, suppose $\lim_{x \to 0} \frac{1}{x} = L$ for some real number $L$. We’ll assume $L \geq 0$; the argument for when $L < 0$ is similar. Let $\epsilon = 1$. Then for any arbitrary $\delta > 0$ it suffices to find a value of $x$ with $|x| < \delta$ such that $f(x) > L + 1$. choose

$$x = \min \left\{ \frac{\delta}{2}, \frac{1}{2(L + 1)} \right\}.$$ 

Certainly $x$ satisfies $|x| < \delta$, and we see that

$$f(x) = \frac{1}{x} \geq \frac{1}{2(L+1)} = 2(L + 1) > L + 1.$$ 

We conclude

$$\lim_{x \to 0} \frac{1}{x}$$

does not exist.

4. Give examples to show that the following definitions of $\lim_{x \to a} f(x) = \ell$ are not correct.
(a) For all \( \delta > 0 \) there is an \( \epsilon > 0 \) such that if \( 0 < |x - a| < \delta \) then \( |f(x) - \ell| < \epsilon \).

**Solution:** This statement is true for any bounded function, regardless if it’s continuous or not. As an example, consider the function \( f(x) \) defined by \( f(x) = 1 \) if \( x \geq 0 \) and \( f(x) = -1 \) if \( x < 0 \). For this function, \( \lim_{{x \to 0}} f(x) \) does not exist. However, given any \( \delta > 0 \), if we pick \( \epsilon = 3 \) we have that if \( x \) is in the range of values for which \( 0 < |x - 0| < \delta \), then \( |f(x) - f(1)| \) is at most 2, and hence \( |f(x) - f(1)| < 3 \), so this definition would imply \( \lim_{{x \to 0}} f(x) = 1 \), which is false.

(b) For all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( |f(x) - \ell| < \epsilon \) then \( 0 < |x - a| < \delta \).

**Solution:** We will show that there are functions that have limits at a specific point, but violate the given definition. Consider the function \( f(x) = 2 \) for every \( x \). Certainly, \( \lim_{{x \to 1}} f(x) = 2 \). However, the given definition implies does not hold. Indeed, choose \( \epsilon > 0 \), and let \( \delta > 0 \) be any thing. Then, in fact, every single value of \( x \) satisfies \( |f(x) - 2| < \epsilon \), but certainly if we make \( x \) very large it will not satisfy \( |x - 0| < \delta \).

5. Define the function \( f(x) \) on the interval \([0, 1]\) by

\[
 f(x) = \begin{cases} 
 0 & \text{if } x \text{ is irrational} \\ 
 1/n & \text{if } x \in \mathbb{Q} \text{ and } x = m/n \text{ in reduced form} 
\end{cases}
\]

Determine, with proof, whether or not

\[
 \lim_{{x \to \frac{9}{2012}}} f(x) = f\left(\frac{9}{2012}\right).
\]

Also, determine, with proof, whether or not

\[
 \lim_{{x \to \frac{1}{\sqrt{2}}}} f(x) = f\left(\frac{1}{\sqrt{2}}\right).
\]

**Solution:** We first consider

\[
 \lim_{{x \to \frac{9}{2012}}} f(x) = f\left(\frac{9}{2012}\right).
\]

We claim this statement is not true. Observe that there are only finitely many rational numbers in the interval \([0, 1]\) whose denominators are greater than 2012. Let \( M \) be the minimum distance from \( \frac{9}{2012} \) to any of these rational numbers. Then for any \( x \) in the interval \( \frac{9}{2012} - \frac{M}{2} \leq x \leq \frac{9}{2012} + \frac{M}{2} \) we have \( f(x) \leq \frac{1}{2013} \).

Now let \( \epsilon = \frac{1}{2012} - \frac{1}{2013} \). Now given any \( \delta > 0 \), choose a real number \( x \) such that that is in the interval \( \frac{9}{2012} - \frac{M}{2} \leq x \leq \frac{9}{2012} + \frac{M}{2} \) and the interval \( \frac{9}{2012} - \delta \leq x \leq \frac{9}{2012} + \delta \). Then \( 0 \leq f(x) \leq \frac{1}{2013}, \) so

\[
 |f(x) - f\left(\frac{9}{2012}\right)| = \left| f(x) - \frac{1}{2012} \right| = \frac{1}{2012} - f(x) \geq \frac{1}{2012} - \frac{1}{2013} = \epsilon.
\]
Now consider
\[ \lim_{x \to \sqrt{2}} f(x) = f \left( \frac{1}{\sqrt{2}} \right) = 0. \]

We prove this statement is in fact true. Choose \( \epsilon > 0 \). Let \( N \) be any integer such that \( \frac{1}{N} < \epsilon \). There are only finitely many fractions whose denominators are less than or equal to \( N \). Let \( M \) be the minimum distance from any of these fractions to \( \frac{1}{\sqrt{2}} \). Let \( \delta = M \). Then for any \( x \) satisfying \( 0 < |x - \frac{1}{\sqrt{2}}| < M \), we have
\[ |f(x) - f \left( \frac{1}{\sqrt{2}} \right)| = |f(x) - 0| = |f(x)| < \frac{1}{N} < \epsilon. \]

6. Compute the following limits (you don’t need to use the definition of a limit at all):

(a) \( \lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x}) \).

**Solution:** Observe that
\[ \sqrt{x+1} - \sqrt{x} = \frac{(x+1) - x}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}. \]
Thus
\[ \lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x \to \infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0. \]
Now since \( 0 \leq \frac{1}{\sqrt{x+1} + \sqrt{x}} \leq \frac{1}{2\sqrt{x}} \), and both \( \lim_{x \to 0} 0 = \lim_{x \to 0} \frac{1}{2\sqrt{x}} = 0 \), we conclude by the Squeeze Theorem
\[ \lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x}) = 0. \]

(b) \( \lim_{x \to 0} \frac{\tan(3x)}{x} \).

**Solution:** The argument in the limit is \( \frac{\tan(3x)}{x} = \frac{\sin(3x)}{x \cos(3x)} \). Thus, our limit is
\[ 3 \cdot \lim_{x \to 0} \cos(3x) \cdot \lim_{x \to 0} \frac{\sin(3x)}{3x} = 3 \cdot 1 \cdot 1 = 3. \]

(c) \( \lim_{x \to \infty} (7^x + 2^x)^{\frac{1}{x}} \).

**Solution:** Notice that \( 7^x \leq 7^x + 2^x \leq 2(7^x) \), the last inequality being true because \( 2^x < 7^x \) for positive \( x \) (which is sufficient to look at because we want to know about the limit as \( x \) goes to infinity). Thus
\[ (7^x)^{\frac{1}{x}} \leq (7^x + 2^x)^{\frac{1}{x}} \leq [2(7^x)]^{\frac{1}{x}}. \]
Simplifying, we have
\[ 7 \leq (7^x + 2^x)^{\frac{1}{x}} \leq 7 \cdot 2^{\frac{1}{x}}. \]
Since \( \lim_{x \to 0} 2^{\frac{1}{x}} = 1 \) (because \( \lim_{x \to 0} \frac{1}{x} = 0 \)), we have that \( \lim_{x \to 0} 7 = \lim_{x \to 0} 7 \cdot 2^{\frac{1}{x}} = 7 \). Thus
\[ \lim_{x \to \infty} (7^x + 2^x)^{\frac{1}{x}} = 7. \]