Definition. A graph $G$ is \textit{planar} if it can be drawn in the plane so that no edges cross.

Fundamental Question: How can we detect if a graph is planar?

A planar embedding partitions the plane into connected regions called \textit{faces}; one of these regions, called the outer face, is unbounded. The vertices and edges incident with a face $f$ are called the \textit{boundary} of $f$. We say two faces are \textit{adjacent} if they are incident to a common edge. As one walks around the entire perimeter of a face $f$, one encounters the vertices and edges of this boundary. This is the boundary walk. The \textit{degree} of a face is the number of edges in its boundary walk.

(Provide an example).

Proposition 1 If $G$ is a connected planar graph that is not a tree, then in any planar drawing of $G$, the boundary of every face contains a cycle.

Theorem 2 Suppose $f_1, f_2, \ldots, f_s$ are faces of a planar drawing of a graph $G$. Then

$$
\sum_{i=1}^{s} f_i = 2|E(G)|
$$

Proof. Every edge contributes 2 to the total degree counting of faces.

One of the first necessary conditions for planarity is Euler’s Formula:

Theorem 3 (Euler’s Formula) Suppose $G$ is a connected planar graph. Suppose that a planar embedding of $G$ has $f$ faces. Then

$$
|V(G)| - |E(G)| + f = 2.
$$

Proof. We will fix $p = |V(G)|$, assuming $p \geq 1$, and prove the result by induction on $q = |E(G)|$. For a fixed $p$, since $G$ is connected, the smallest possible value of $q$ is $p - 1$ when $G$ is a tree, so this is our base case. In this case, $f = 1$, there is only the outer face. Thus $|V(G)| - |E(G)| + f = p - (p - 1) + 1 = 2$.

Now assume the result holds for all integers less than some given $q$, and here $q > p - 1$. This means $G$ is not a tree, so it contains a cycle. Let $e$ be any edge of this cycle. Then $G \setminus e$ is connected, has $p$ vertices, and $q - 1$ edges. Moreover, $G \setminus e$ has $f - 1$ faces. Thus $p - (q - 1) + (f - 1) = 2$. This implies $p - q + f = 2$.

Proposition 4 If $G$ is a planar graph with $|V(G)| \geq 3$, then $|E(G)| \leq 3|V(G)| - 6$. 


Proof. It suffices to assume $G$ is connected.

First, suppose $G$ is a tree. Then $3|V(G)| - 6 = (|V(G)| - 1) + (2|V(G)| - 5) > |V(G)| - 1 = |E(G)|$.

Now consider a connected graph and suppose it has faces $F_1, F_2, \ldots, F_f$ in some planar embedding of it. Every face of $G$ has degree at least 3, so

$$2|E(G)| \geq 3f.$$

Thus

$$2|E(G)| \geq 3f = 3(|E(G)| - |V(G)| + 2) = 3|E(G)| + 6 - 3|V(G)|,$$

and hence the result.

Example. If $n \geq 5$, then $K_n$ is not planar.

We have an idea that $K_{3,3}$ should be planar, but this does not eliminate $K_{3,3}$. Indeed $|E(K_{3,3})| = 9$ and $|V(K_{3,3})| = 6$, and we have $9 \leq 3(6) - 6$. Our argument above used that all faces have degree at least 3, but maybe we can strengthen this argument if our faces have higher degree.

Proposition 5 If $G$ is a planar graph with at least one cycle, and all cycles in $G$ have length at least $k$, with $k \geq 3$, then

$$|E(G)| \leq \frac{k(|V(G)| - 2)}{k - 2}.$$

Proof. if $f$ is the number of faces, then $2|E(G)| \geq kf$. Apply Euler’s formula and the result follows.

Example. $K_{3,3}$ is not planar. Indeed, each cycle in $K_{3,3}$ has length at least 4, so if it is planar, $|E(K_{3,3})| \leq 2(|V(K_{3,3})| - 2)$, which is false.

Let $G$ be a planar graph. A celebrated theorem of Paul Seymour et al. is the following:

Theorem 6 (Four-Color Theorem) Every planar graph is 4-colorable.

This proof is over 200 pages long and involves some computer search. Surprisingly, the proofs of six-colorability and five-colorability are short!

Theorem 7 Every planar graph is 6-colorable.

Proof. Induction on $|V(G)|$. Clearly if $|V(G)| \leq 6$, then $G$ is 6-colorable. It suffices to show that there exists $v \in V(G)$ with $\deg(v) \leq 5$, because if such a $v$ exists, we can color $G \setminus v$ inductively, and then choose one of the colors not used for the neighbors of $v$ to color $v$ itself.

If such a $v$ did not exists, then $2|E(G)| \geq 6|V(G)|$ which contradicts $|E(G)| \leq 3|V(G)| - 6$.

Theorem 8 Every planar graph is 5-colorable.
Proof. Again, use induction on $|V(G)|$. Clearly true for $|V(G)| \leq 5$.

Case 1: If $G$ has a vertex $v$ with degree at most 4, then $G\setminus v$ is 5-colorable, and we get a 5-coloring of $G$.

Case 2: There is a vertex $v \in V(G)$ such that $\deg(v) = 5$. Then two of the neighbors of $v$, say $a, b$, are not adjacent. Contracting both edges $av$ and $bv$. The resulting graph is 5-colorable. Use the color of the new vertex identifying $v, a, b$ to color both $a$ and $b$ in $G$. Then the neighbors of $v$ in $G$ are colored with at most 4 different colors, so pick a 5-th color for $G$. 
