Collaboration: Until May 31, you may not collaborate. After that, you may collaborate with others in the class or ask the TA or M. Duits leading questions. You may not discuss on chat sites.

Note: You may use any books, computer software, google.com, etc., but you may not collaborate on the Internet, including posting about the problems on chat sites.

Note: This homework is worth 180 points and so more than Homework 1–3, as in the past quarters.

1. Let $\mathcal{H}$ be a separable Hilbert space. The purpose of this problem is to prove that the only operator norm closed two-sided ideals in $\mathcal{L}(\mathcal{H})$ are $\{0\}$, $\mathcal{I}$ (compact operators), and $\mathcal{L}(\mathcal{H})$. Let $I$ be such an ideal.

(a) [20 points] Suppose for some $A \in I$ there is $\varphi$ with $A\varphi \neq 0$. Prove that $I$ contains a rank one operator, then all finite rank operators, then all of $I$.

(b) [15 points] Suppose $A \in I$ is not compact. Use the polar decomposition and then the spectral theorem to find an infinite-dimensional selfadjoint projection in $I$ and then prove that $1 \in I$. Conclude that $I = \mathcal{L}(\mathcal{H})$.

(c) [5 points] Put (a) and (b) together to prove that $I$ is one of $\{0\}$, $\mathcal{I}$, or $\mathcal{L}(\mathcal{H})$.

2. [30 points] Fix $1 \leq p < \infty$. You are asked to prove a noncommutative dominated convergence theorem as follows. Let $A_m, A, B$ be bounded operators with $B \geq 0$. Suppose that for all $m$,

$$|A_m| \leq B \quad |A_m^*| \leq B$$

$$|A| \leq B \quad |A^*| \leq B$$

and $A_m \to A$ weakly. If $B \in \mathcal{I}_p$ (so $A_m, A \in \mathcal{I}_p$), prove $\|A - A_m\|_p \to 0$. 

(Hint: Find a finite-dimensional projection $P$ with $Q = (1 - P)$ so $\|QBQ\|_p \leq \varepsilon$. Use this and the assumed operator inequalities to prove $QA_mP$, $QA_mQ$, and $PA_mQ$ all have small $p$ norms uniformly in $m$.)
3. [30 points] Prove Fan’s inequality for two compact operators $A, B$:

$$
\mu_{n+m+1}(A + B) \leq \mu_{n+1}(A) + \mu_{m+1}(B)
$$

(Hint: Use the min-max principle for singular values.)

4. [30 points] Let $A$ be trace class. Prove the following are equivalent

(i) $\sigma(A) = \{0\}$

(ii) $\det(1 + \mu A) = 1$ for all $\mu$.

(iii) $\text{Tr}(A^k) = 0$ for all $k$.

(iv) For some $K$, $\text{Tr}(A^k) = 0$ for all $k \geq K$.

Hint: In proving (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), the last is the hardest. You will need to prove that if $\omega_1, \omega_2, \ldots, \omega_n$ is a finite set in $\partial \mathbb{D}$, then it cannot be that $\sum_{j=1}^{n} \omega_j^k \to 0$ as $k \to \infty$. You can prove this lemma using

$$
\frac{1}{n} \sum_{k=1}^{n} \omega^k \to 0 \text{ (if } \omega \neq 1) \text{ or } 1 \text{ (if } \omega = 1)
$$

5. [20 points] Let $A$ be a selfadjoint operator on a finite-dimensional Hilbert space. Prove that $A$ has a cyclic vector if and only if every eigenvalue has multiplicity 1.

6. [30 points] Let $A$ be a commutative Banach algebra with unit. The radical of $A$ is defined to be the intersection of all maximal ideals of $A$. Prove that the following three statement about an element $x \in A$ are equivalent:

(a) $x$ is in the radical of $A$.

(b) $\lim_{n \to \infty} \|x^n\|^{1/n} = 0$

(c) $h(x) = 0$ for every complex homomorphism of $A$. 